FINITE DIFFERENCE SOLUTION OF THE THIRD ORDER VISCOUS WAVE EQUATION

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***Abstract*—**

A wave motion is the transmission of energy from one place to another through a material or a vacuum .Wave motion may occur in many forms such as water waves, sound waves, radio waves, light waves among other forms. In this paper, we use finite difference method to solve the third order viscous wave equation

$u_{tt} = u_{xx} + u_{xxt}\ , \qquad 0 < x < \infty, t > 0$

where x is the distance along the axis of propagation and t denotes time,

subject to boundary conditions

$u(0,t) = f(t)\ , u(\infty,t) = 0 : t > 0$

and initial conditions

$u(x,0) = 0\ , u_t(x,0) = 0$

The results we have obtained agree with the reality that sound wave propagation in viscous fluids is damped.

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I. INTRODUCTION

A wave can be described as a disturbance that travels through a medium, transporting energy from one location to another. The medium is the material that facilitates propagation of wave energy; it can be thought of as a series of interconnected and interacting particles. The media can be gases, liquids, solids and plasmas. There are two categories of waves; mechanical waves and electromagnetic waves. Electromagnetic waves have an electric and magnetic nature and are capable of being transmitted through a vacuum while mechanical waves require a material medium for propagation because they rely on particle interaction for transmission. Sound waves are mechanical waves which cannot travel through a vacuum.

The science of sound including its production, transmission and effects is called **acoustics.**

Acoustic waves are pressure disturbances in the form of vibrational waves that propagate through a compressible medium. These vibrational waves displace the molecules of the medium from their quiescent point, after which a restoring elastic force pulls the molecules back. This elastic force along with inertia causes the molecules to oscillate, allowing acoustic waves to propagate [7]. According to Keith [3], the velocity of sound is greater in solids and liquids than in gases. Serway and Faughn [5] asserted that a sound wave propagates through an elastic medium as a longitudinal wave as exhibited by compressions and rarefactions in the material medium. According to Evans [2], the wave equation is a simplified model for a vibrating string $(n = 1)$, membrane $(n = 2)$ or elastic solid $(n = 3)$ where n is the dimension of the wave. In these physical interpretations, $u(x, t)$, for $(n = 1)$, represents the displacement in some direction of the point x at time $t \geq 0$.

Here, we have considered sound waves. We have addressed the propagation of sound waves in viscous fluid.

There are several attempts that have been made to solve different kinds of sound wave equations as discussed below.

The lossy linear wave equation is given as

$$p_{xx} - \frac{1}{c_0^2} p_{tt} + \tau_s p_{xxt} = 0 \quad (1)$$

where $\tau_s = \frac{0.75\eta + \eta_\beta}{p_0 c_0^2}$ [7].

This equation is linear, dissipative wave equation for the propagation of sound in fluids. This equation is identical to the lossless wave equation except for the added third term on the left hand side which can be considered the viscous dissipative term.

For a plane wave travelling in the positive x direction, the solution to this equation is

$$p(x,t) = p_0 e^{-\alpha x} \cos(\omega_0 t - kx)$$

where α is the attenuation coefficient in Np/m (Nepers per metre) which causes the amplitude to decay in an exponential form.

Here $p(x,t)$ is the pressure field.

Verwer [6] solved the viscous wave equation

$$u_{tt} = u_{xxt} + u_{xx} + s(x,t) t > 0 \,, \; 0 < x < 1 \quad (2)$$

where $s(x,t)$ is the source term,

subject to Dirichlet boundary conditions and got the solution

$$u(x,t) = x^2 \sin(t)$$

Dean [1] solved the nondimensionalized viscous sound wave equation

$$u_{tt} = u_{xx} + u_{xxt} \quad , \quad 0 < x < \infty, t > 0 \quad (3)$$

subject to boundary conditions

$u(0,t) = f(t) \,, u(\infty,t) = 0 : t > 0$ and initial conditions

$$u(x,0) = 0 \,, u_t(x,0) = 0$$

using Laplace Transforms and obtained the solution

$$u(x,t) = \frac{2x}{\pi t}\int_0^1 exp(-2t\omega^2) cos\left[2t\omega(1-\omega^2)^{\frac{1}{2}} - 2x\omega\right] d\omega + \frac{e^{-2t}}{\pi}\int_0^\infty e^{-t\omega}\left[sin\left(\frac{2+\omega}{\sqrt{1+\omega}}x\right) - \sin(2x)\right] d\omega$$

This result has no physical significance. We have therefore solved this equation numerically using the finite difference method.

II. FORWARD DIFFERENCE SCHEME

The finite difference approximations to the partial derivatives with respect to time are given as follows

$$u_t = \frac{U_{m,n+1} - U_{m,n}}{k} + O(k)$$

$$= \frac{U_{m,n}-U_{m,n-1}}{k} + O(k)$$

$$= \frac{U_{m,n+1}-U_{m,n-1}}{2k} + O(k^2)$$

$$u_{tt} = \frac{U_{m,n+1}-2U_{m,n}+U_{m,n-1}}{k^2} + \quad O(k^2)$$

where $k = \Delta t$

and with respect to x as

$$u_x = \frac{U_{m+1,n}-U_{m,n}}{h} + O(h)$$

$$u_x = \frac{U_{m,n}-U_{m-1,n}}{h} + O(h)$$

$$u_x = \frac{U_{m+1,n}-U_{m-1,n}}{2h} + O(h^2) \qquad u_{xx} = \frac{U_{m+1,n}-2U_{m,n}+U_{m-1,n}}{h^2} + O(h^2)$$

The forward-central approximation for u_{xxt} is given by

$$u_{xxt} = \frac{\partial}{\partial t}(u_{xx})$$

$$= u_{xxt} = \frac{1}{kh^2}[U_{m+1,n+1} + U_{m-1,n+1} - (U_{m+1,n} + U_{m-1,n}) + 2(U_{m,n} - U_{m,n+1})] + O(k) + O(h^2) \qquad (4)$$

In the forward difference scheme, we replace u_{tt} by the central difference approximation, u_{xx} by the central difference approximation and u_{xxt} by the forward-central approximation. This yield

$$\frac{u_{m,n+1} - 2u_{m,n} + u_{m,n-1}}{k^2} = \frac{u_{m+1,n} - 2u_{m,n} + u_{m-1,n}}{h^2}$$

$$+ \frac{1}{kh^2}[u_{m+1,n+1} + u_{m-1,n+1} - (u_{m+1,n} + u_{m-1,n}) - 2(u_{m,n+1} - u_{m,n})]$$

Let $r = \frac{k^2}{h^2}$ and $\beta = \frac{k}{h^2}$, then the scheme above becomes

$$U_{m,n+1} - 2U_{m,n} + U_{m,n-1} = rU_{m+1,n} - 2rU_{m,n} + rU_{m-1,n} + \beta[U_{m+1,n+1} - 2U_{m,n+1} + U_{m-1,n+1} - U_{m+1,n} + 2U_{m,n} - U_{m-1,n}]$$

which becomes

$$-\beta U_{m-1,n+1} + (2\beta + 1)U_{m,n+1} - \beta U_{m+1,n+1}$$

$= (r-\beta)U_{m-1,n} + (2\beta + 2 - 2r)U_{m,n} +$
$(r-\beta)U_{m+1,n} - U_{m,n-1}$

(5)

This is for $m = 1,2,3, \dots \dots \dots \dots \dots \dots \dots, (E-1), E$ where E is number of divisions along the x-axis.

III. STABILITY ANALYSIS

(5)

The stability of numerical schemes is closely associated with numerical error. A finite difference scheme is stable if the errors made at one time step of the calculation do not cause the errors to increase as the computations are continued. A *neutrally stable scheme* is one in which errors remain constant as the computations are carried forward. If the errors decay and eventually damp out, the numerical scheme is said to be stable. If, on the contrary, the errors grow with time the numerical scheme is said to be unstable. For time-dependent problems, stability guarantees that the numerical method produces a bounded solution whenever the solution of the exact differential equation is bounded.

Here, we have used von Neumann stability analysis

Expanding the scheme (5) by taking $m = 1,2,3, \dots \dots \dots (E-1)$, we get

$$\begin{bmatrix} (2\beta+1) & -\beta & \cdots & 0 & 0 \\ -\beta & (2\beta+1) & -\beta & 0 & 0 \\ \vdots & -\beta & \ddots & -\beta & \vdots \\ 0 & 0 & -\beta & (2\,\beta+1) & -\beta \\ 0 & 0 & \cdots & -\beta & (2\,\beta+1) \end{bmatrix} \begin{bmatrix} U_{1,n+1} \\ U_{2,n+1} \\ \vdots \\ U_{E-2,j+1} \\ U_{E-1,j+1} \end{bmatrix}$$

$$= \begin{bmatrix} (2\beta+2-2r) & (r-\beta) & \cdots & 0 & 0 \\ (r-\beta) & (2\beta+2-2r) & (r-\beta) & 0 & 0 \\ \vdots & (r-\beta) & \ddots & (r-\beta) & \vdots \\ 0 & 0 & (r-\beta) & (2\beta+2-2r) & (r-\beta) \\ 0 & 0 & \cdots & (r-\beta) & (2\beta+2-2r) \end{bmatrix} \begin{bmatrix} U_{1,n} \\ U_{2,n} \\ \vdots \\ U_{E-2,n} \\ U_{E-1,n} \end{bmatrix}$$

$$+ \begin{bmatrix} \beta U_{0,n+1} + (r-\beta)U_{0,n} - U_{1,n-1} \\ -U_{2,n-1} \\ \vdots \\ -U_{E,n-1} \\ \mu U_{E,n+1} + (r-\beta)U_{E,n} - U_{E+1,n-1} \end{bmatrix}$$

which can also be written as

$$(\mathbf{I} - \beta \mathbf{A}_{E-1})\mathbf{U}_{n+1} = (2\mathbf{I} + (r - \beta)\mathbf{A}_{E-1})\mathbf{U}_n + \mathbf{b}$$

where $$\mathbf{A}_{E-1} = \begin{bmatrix} -2 & 1 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 1 & -2 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$, **I** is an identity matrix

and

$$\boldsymbol{b} = \begin{bmatrix} \beta U_{0,n+1} + (r - \beta)U_{0,n} - U_{1,n-1} \\ -U_{2,n-1} \\ \vdots \\ -U_{E,n-1} \\ \mu U_{E,n+1} + (r - \beta)U_{E,n} - U_{E+1,n-1} \end{bmatrix}$$

Therefore

$$\mathbf{U}_{n+1} = [(2\mathbf{I} + (r - \beta)\mathbf{A}_{E-1})(\mathbf{I} - \beta\mathbf{A}_{E-1})^{-1}]\mathbf{U}_n + \boldsymbol{c} = \boldsymbol{B}\boldsymbol{U}_n + \boldsymbol{c}$$

B is the amplification matrix and

$$\boldsymbol{c} = (\mathbf{I} - \beta\mathbf{A}_{E-1})^{-1}\boldsymbol{b}$$

The eigenvalues of B are given by $\frac{2-4(r-\beta)\sin^2\left(\frac{m\pi}{m+1}\right)}{1+4\beta\sin^2\left(\frac{m\pi}{m+1}\right)}$

For stability, $\left|\frac{2-4(r-\beta)\sin^2\left(\frac{m\pi}{m+1}\right)}{1+4\beta\sin^2\left(\frac{m\pi}{m+1}\right)}\right| \le 1.$

Thus

$r > 0, \beta > 0, r \ge \frac{1}{4}, r < \beta$ and $0 \le x \le 5$ yields the stability criterion.

IV. RESULTS

We considered two different mesh sizes as given below. We have tabular results and also the graphical outputs of the results.

Case one

For $h = \frac{1}{2}$, $k = \frac{1}{2}$, $\beta = 2$, $r = 1$,the scheme becomes

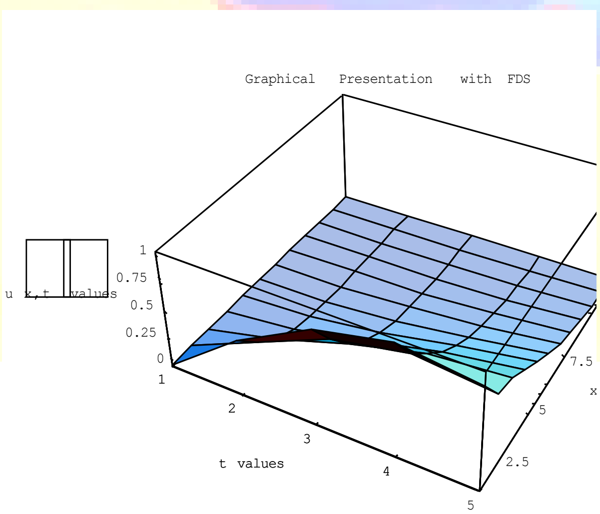
$$-2U_{m-1,n+1} + 5U_{m,n+1} - 2U_{m+1,n+1} = -U_{m-1,n} + 4U_{m,n} - U_{m+1,n} - U_{m,n-1} \qquad (6)$$

For $0 \leq x \leq 5$, we obtain the results (using Mathematica) as tabulated below

	t_0=0	t_1=0.5	t_2=1.0	t_3=1.5	t_4=2.0
X_0=0	0	0.4794255386	0.8414709848	0.9974949866	0.9092974268
X_1=0.5	0	0.300879	0.589256	0.719299	0.643848
X_2=1	0	0.150439	0.369843	0.544545	0.594099
X_3=1.5	0	0.0752186	0.222522	0.383488	0.488624
X_4=2	0	0.0376076	0.130048	0.256684	0.372362
X_5=2.5	0	0.0188004	0.074393	0.165388	0.268373
X_6=3	0	0.00939329	0.0418339	0.103342	0.185006
X_7=3.5	0	0.00468287	0.0231464	0.0627647	0.12245
X_8=4	0	0.00231389	0.0125201	0.0367957	0.0772317
X_9=4.5	0	0.00110185	0.00641843	0.0201236	0.0447421
X_{10}=5	0	0.000440741	0.00269959	0.00883728	0.020402
X_{11}=5.5	0	0	0	0	0

Table 1: Values of $U_{m,n}$ for forward difference scheme (Case one)

The graphical output of the results is as shown below.



Out[1]:□Surface Graphics□ Figure 1: Graphical Presentation Using FDS with $\beta = 2$ and $r = 1$

Case two

For $h = \frac{1}{5}$, $k = \frac{1}{5}$, $\beta = 5$, $r = 1$,the schemes become

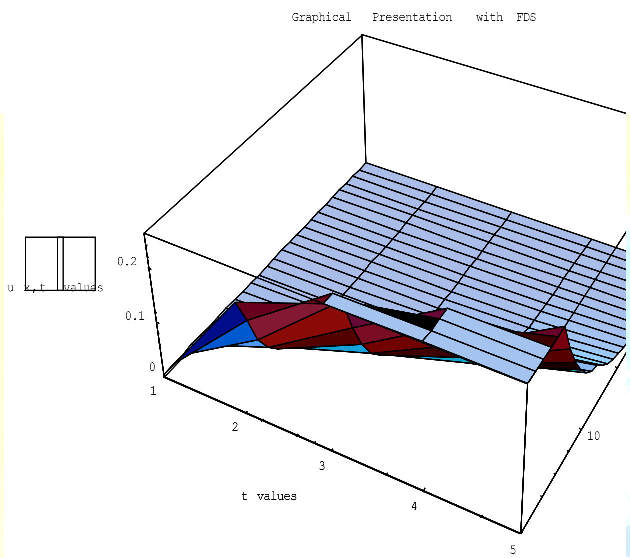
$$-5U_{m-1,n+1} + 11U_{m,n+1} - 5U_{m+1,n+1} = 4 - U_{m-1,n} + 10U_{m,n} - 4U_{m+1,n} - U_{m,n-1} \quad (7)$$

and using Mathematica for $0 \leq x \leq 5$, we obtain the results as tabulated below

	t_0=0	t_1=0.2	t_2=0.4	t_3=0.6	t_4=0.8
X_0=0	0	0.1986693308	0.3894183423	0.5646424734	0.7173560909
X_1=0.2	0	0.147911	0.319491	0.48469	0.630683
X_2=0.4	0	0.0949205	0.229887	0.375187	0.514584
X_3=0.6	0	0.0609145	0.163479	0.286306	0.413802
X_4=0.8	0	0.0390914	0.115148	0.21594	0.328767
X_5=1.0	0	0.0250866	0.0804646	0.161439	0.25883
X_6=1.2	0	0.0160992	0.0558534	0.120115	0.202718
X_7=1.4	0	0.0103315	0.0385489	0.0895383	0.158872
X_8=1.6	0	0.00663017	0.0264746	0.0676999	0.125685
X_9=1.8	0	0.00425486	0.0181041	0.0531007	0.101642
X_{10}=2.0	0	0.00273052	0.0123331	0.0448106	0.0853554
X_{11}=2.2	0	0.00175229	0.00837354	0.0425446	0.0755013
X_{12}=2.4	0	0.00112452	0.0056681	0.0288309	0.0562462
X_{13}=2.6	0	0.000721646	0.00382638	0.019532	0.0413734
X_{14}=2.8	0	0.000463104	0.00257675	0.013227	0.0301229
X_{15}=3.0	0	0.000297184	0.00173132	0.00895258	0.0217459
X_{16}=3.2	0	0.0001907	0.00116083	0.00605559	0.0155854
X_{17}=3.4	0	0.000122355	0.000776737	0.00409265	0.011099
X_{18}=3.6	0	0.0000784825	0.000518628	0.0027628	0.00785728
X_{19}=3.8	0	0.000050306	0.000345408	0.00186166	0.00552855
X_{20}=4.0	0	0.0000321906	0.000229197	0.00125036	0.00386182
X_{21}=4.2	0	0.0000205135	0.000151099	0.000834386	0.00266941
X_{22}=4.4	0	0.0000129389	0.000098298	0.000549187	0.00181197
X_{23}=4.6	0	0.00000795223	0.0000620512	0.000350337	0.00118598
X_{24}=4.8	0	0.00000455597	0.000036306	0.000206727	0.000713659
X_{25}=5.0	0	0.00000207089	0.0000167287	0.0000957841	0.000334773

Table 2: Values of $U_{m,n}$ for forward difference scheme (Case Two)

The graphical output of the results is as shown below



Out[2]:Surface Graphics Figure 2: Graphical Presentation Using FDS with $\beta = 5$ and $r = 1$

V. DISCUSSION AND CONCLUSION

From the graphical presentations of the solutions, it can be observed that, for a given value of x, $u(x,t) \approx U_{m,n}$ increases to nearly one as t tends to infinity and for a given value of t, $u(x,t) \approx U_{m,n}$ decreases to nearly zero as x tends to infinity. Sound propagation in viscous fluid is damped i.e. the amplitude of the pressure of the sound wave decreases with increasing distance from the sound source [4]. Our results are confirming this since the displacement of the particles given by $u(x,t)$ is decreasing with an increase in the distance from the source (in this case at $t = 0$). We wish to recommend that further research can be undertaken to explore other solution to this problem using other methods apart from Laplace Transforms and the finite difference forward-central approach.

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