HOMOGENOUS BI-QUADRATIC WITH FIVE UNKNOWNS

$$x^4 - y^4 = 26(z^2 - w^2)T^2$$

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Abstract:

We obtain non-trivial integral solutions for the Homogeneous Bi-quadratic with five unknowns' $x^4 - y^4 = 26(z^2 - w^2)T^2$. A few interesting relations for each pattern among the solutions are presented.

Keywords: Bi-quadratic with five unknowns, Integral solutions.

MSClassification: 11D41

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INTRODUCTION:

Diophantine equations have an unlimited field for research by reason of their variety. In particular, the Bi-quadratic Diophantine equations, Homogenous and Non Homogenous have aroused the interest of numerous mathematicians since antiquity [1-5]. In this context one may refer [6-9] for problems on the bi-quadratic Diophantine equations with five variables. However often we come across homogenous bi-quadratic equations and as such one may require its integral solutions in its most general form. This paper concerns with the problems of determining non-trivial integral solutions of the equation with five unknowns given by $x^4 - y^4 = 26(z^2 - w^2)T^2$. Explicit integral solutions of the above equations are presented. A few interesting relations among the solutions are obtained.

Notations:

P_n^m - Pyramidal number of rank n with size m.

$T_{m,n}$ - Polygonal number of rank n with size m.

$\Pr_n$ - Pronic number of rank n.

SO_n - Stella Octangular number of rank m.

Pt_n -Pentalope number of rank n.

METHOD OF ANALYSIS:

The Bi-quadratic Diophantine equation with five unknowns to be solved is given by

$$x^4 - y^4 = 26(z^2 - w^2)T^2 \qquad (1)$$

The substitution of the linear transformations

$$x = u + v,\ y = u - v,\ z = 2u + v,\ w = 2u - v \qquad (2)$$

in (1) leads to

$$u^2 + v^2 = 26T^2 \qquad (3)$$

(3) is solved through different approaches and the different patterns of solutions of (1) obtained are presented below:

PATTERN: 1

Consider (3) as,

$$u^2+v^2=25T^2+T^2$$

and write it in the form of ratio as

$$\frac{u+T}{5T+v}=\frac{5T-v}{u-T}=\frac{a}{b},b\neq 0$$

The above equation is equivalent to the system of equations

$$bu-av+(b-5a)T=0 \qquad (4)$$

$$-au-bv+(a+5b)T=0 \qquad (5)$$

Solving (4) & (5) by the method of cross multiplication, we have

$$\left.\begin{aligned} u&=-(a^2-b^2)-10ab\\ v&=5(a^2-b^2)-2ab\\ T&=-(a^2+b^2)\end{aligned}\right\} \qquad (6)$$

Substituting (6) in (2) , we get

$$x=x(a,b)=4(a^2-b^2)-12ab$$
$$y=y(a,b)=-6(a^2-b^2)-8ab$$
$$z=z(a,b)=3(a^2-b^2)-22ab$$
$$w=w(a,b)=-7(a^2-b^2)-18ab$$
$$T=T(a,b)=-(a^2+b^2)$$

which represent the non-zero distinct integer solutions to (1).

PROPERTIES:

- $X(A,1)+Y(A,1)+Z(A,1)+W(A,1)+t_{14,A}\equiv 6(\mod 65)$
- $X(A^2,A+1)+Y(A^2,A+1)+Z(A^2,A+1)+W(A^2,A+1)+6t_{4,A}^2+120P_A^5-t_{10,a}-t_{6,a}\equiv 6(\mod 16)$
- $X(A,2A^2-1)+Y(A,2A^2-1)+Z(A,2A^2-1)+W(A,2A^2-1)+60(SO_A)-24t_{4,A}^2+t_{62,A}\equiv 6(\mod 29)$

- $X(A,2A^2+1)+Y(A,2A^2+1)+Z(A,2A^2+1)+W(A,2A^2+1)+180(OH_A)-24t^2_{4,A}+t_{38,A}\equiv 6(\bmod 17)$

- $X(A,1)+Y(A,1)+Z(A,1)+W(A,1)+T(A,1)+t_{12,A}\equiv 7(\bmod 65)$

- $10(X(A,A)+Y(A,A)+Z(A,A)+W(A,A))$ is a Nasty Number.

PATTERN: 2

Assume

$$26=(1+5i)(1-5i) \qquad (7)$$

Write T as

$$T=T(a,b)=a^2+b^2 \qquad (8)$$

Substituting (7) and (8) in (3) and employing the method of factorization, define

$$(u+iv)=(1+5i)(a+ib)^2$$

Equating real and imaginary parts in the above equation, we get

$$\left.\begin{aligned} u&=a^2-b^2-10ab\\ v&=5(a^2-b^2)+2ab\end{aligned}\right\} \qquad (9)$$

Substituting (9) in (2),the corresponding integer solutions to (1) are given by

$$x(a,b)=6(a^2-b^2)-8ab$$

$$y(a,b)=-4(a^2-b^2)-12ab$$

$$z(a,b)=7(a^2-b^2)-18ab$$

$$w(a,b)=-3(a^2-b^2)-22ab$$

$$T(a,b)=a^2+b^2$$

PROPERTIES:

- $X(A,1)+Y(A,1)+t_{6,A}\equiv -2(\bmod 19)$

- $Y(A, A^2-1) + Z(A,2A^2-1) + 30(SO_A) + 12t^2_{4,A} - t_{32,A} \equiv -3 \pmod{14}$
- $Z(A,2A^2+1) + W(A,2A^2+1) + 120(OH_A) + 16t^2_{4,A} + t_{26,A} \equiv -4 \pmod{12}$
- $X(A^2, A+1) + W(A^2, A+1) + 60P^5_A - 30t^2_{4,A} + t_{62,A} \equiv -30 \pmod{89}$
- $W(A, A+1) + T(A, A+1) + 22P_{rA} - t_{6,A} \equiv 4 \pmod 9$
- $3T(A, A)$ is a Nasty Number.

PATTERN: 3

Instead of (7), consider 26 as

$$26 = (5+i)(5-i) \tag{10}$$

Following the analysis similar to pattern-2, the corresponding integer solutions to (1) are given by,

$$x(a,b) = 6(a^2-b^2)+8ab$$

$$y(a,b) = 4(a^2-b^2)-12ab$$

$$z(a,b) = 11(a^2-b^2)+6ab$$

$$w(a,b) = 9(a^2-b^2)-14ab$$

$$T(a,b) = a^2+b^2$$

PROPERTIES:

- $X(A,2A^2-1) + Y(A,2A^2-1) + 4(SO_A) + 40t^2_{4,A} - t_{102,A} \equiv -10 \pmod{49}$
- $Y(A,2A^2+1) + Z(A,2A^2+1) + 18(OH_A) + 60t^2_{4,A} + t_{92,A} \equiv -15 \pmod{44}$
- $Z(A^2, A+1) + W(A^2, A+1) + 16P^5_A - 20t^2_{4,A} + t_{4,A} \equiv -1 \pmod 2$
- $X(A, A+1) + W(A, A+1) + P_{rA} \equiv -13 \pmod{26}$
- $T(A(A+1), (A+2)(A+3)) - 2t^2_{4,A} - 36P^4_A - t_{6,A} \equiv 36 \pmod{60}$
- $30(Z(A, A) + W(A, A))$ is a nasty number

PATTERN: 4

Consider (3) as

$$u^2 + v^2 = 26T^2 * 1 \tag{11}$$

Write 1 as

$$1 = \frac{(m^2 - n^2 + i2mn)(m^2 - n^2 - i2mn)}{(m^2 + n^2)^2} \tag{12}$$

Substituting (7) and (12) in (11) and employing the method of factorization, define

$$u + iv = (1 + 5i)(a + ib)^2 \left[\frac{m^2 - n^2 + i2mn}{m^2 + n^2} \right]$$

Equating the real and imaginary parts in the above equation, we get

$$u = \frac{1}{m^2 + n^2}\left[(m^2 - n^2)(a^2 - b^2 - 10ab) - 2mn(5(a^2 - b^2) + 2ab)\right]$$

$$v = \frac{1}{m^2 + n^2}\left[(m^2 - n^2)(5(a^2 - b^2) + 2ab) + 2mn(a^2 - b^2 - 10ab)\right]$$

Replacing 'a' by $(m^2 + n^2)A$ and 'b' by $(m^2 + n^2)B$ in the above equations, we have

$$\left.\begin{aligned} u &= (m^2 + n^2)\left[(m^2 - n^2)(A^2 - B^2 - 10AB) - 2mn(5(A^2 - B^2) + 2AB)\right] \\ v &= (m^2 + n^2)\left[(m^2 - n^2)(5(A^2 - B^2) + 2AB) + 2mn(A^2 - B^2 - 10AB)\right] \end{aligned}\right\} \tag{3}$$

Substituting (13) in (12), we've

$$x(m, n, A, B) = (m^2 + n^2)\left[(m^2 - n^2)(6(A^2 - B^2) - 8AB) - 8mn(A^2 - B^2 + 3AB)\right]$$

$$y(m, n, A, B) = (m^2 + n^2)\left[-4(m^2 - n^2)(A^2 - B^2 + 3AB) - 4mn(3(A^2 - B^2) - 4AB)\right]$$

$$z(m, n, A, B) = (m^2 + n^2)\left[(m^2 - n^2)(7(A^2 - B^2) - 18AB) - mn(18(A^2 - B^2) - 28AB)\right]$$

$$w(m, n, A, B) = (m^2 + n^2)\left[-(m^2 - n^2)(3(A^2 - B^2) + 22AB) - 2mn(11(A^2 - B^2) - 6AB)\right]$$

$$T(m, n, A, B) = (m^2 + n^2)^2 (A^2 + B^2)$$

Which represent the non- zero distinct integer solutions to (1).

PATTERN: 5

Substituting (10) and (12) in (11) and following the procedure as in pattern-4 , the corresponding integer solutions to (1) are given by,

$$x(m,n,A,B) = (m^2+n^2)[(m^2-n^2)(6(A^2-B^2)+8AB)+2mn(4(A^2-B^2)+8AB)]$$

$$y(m,n,A,B) = (m^2+n^2)[(m^2-n^2)(4(A^2-B^2)-12AB)-2mn(6(A^2-B^2)-12AB)]$$

$$z(m,n,A,B) = (m^2+n^2)[(m^2-n^2)(14(A^2-B^2)+4AB)+2mn(2(A^2-B^2)+28AB)]$$

$$w(m,n,A,B) = (m^2+n^2)[(m^2-n^2)(8(A^2-B^2)+28AB)+2mn(14(A^2-B^2)+4AB)]$$

$$T(m,n,A,B) = (m^2+n^2)^2(A^2+B^2)$$

PATTERN: 6

Consider (3) as

$$26T^2 - v^2 = u^2 * 1 \qquad (14)$$

Assume

$$u = 26a^2 - b^2 \qquad (15)$$

Write (1) as

$$1 = (\sqrt{26}+5)(\sqrt{26}-5) \qquad (16)$$

Substituting (15) and (16) in (14) and employing the method of factorization, define

$$(\sqrt{26}T + v) = (\sqrt{26}a + b)^2(\sqrt{26}+5)$$

Equating the rational and irrational parts in the above equation, we get

$$\left.\begin{aligned} T &= 26a^2 + b^2 + 10ab \\ v &= 5(26a^2+b^2) + 52ab \end{aligned}\right\} \qquad (17)$$

Using (15) and (17) in (2), we get

$$x(a,b) = 156a^2 + 4b^2 + 52ab$$

$y(a,b) = -104a^2 - 6b^2 + 52ab$

$z(a,b) = 182a^2 + 3b^2 + 52ab$

$w(a,b) = -78a^2 - 7b^2 - 52ab$

$T(a,b) = 26a^2 + b^2 + 10ab$

which represent the non-zero distinct integer solutions to (1).

CONCLUSION:

In this paper, six different patterns of integer solutions to (1) are presented. It is worth to note that in (2), the transformations of z and w may also be taken as $z = 2uv + \propto, w = 2uv - \propto$.

To conclude, one may search for other patterns of non-zero distinct integer solutions to the considered Bi-quadratic with 5 unknowns and their corresponding properties.

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