VARIATIONAL ITERATION METHOD FOR SOLVING ONE-FACTOR COMMODITY PRICE MODEL EQUATION

R K Pavan Kumar Pannala*

Abstract:

In this work a one-factor model of stochastic behavior of commodity prices given by Eduardo S. Schwartz is solved usingVariational Iteration Method (VIM). Numerical example is studied to demonstrate the accuracy of the present method.

Key words: Commodity models, prices on commodities, series solution, variational iteration method, stochastic differential equation

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* Department of Mathematics, University of Petroleum & Energy Studies, Dehradun, India

1. Introduction

Commodities markets have been attracting greater attention of traders in the recent times. The existence of huge demand-supply gaps resulting in huge volatilities in the prices of different commodities has made the trading highly complex. Numerous researchers tried to evolve models that cover volatility in the spot and future trade contracts. In this process, Eduardo S. Schwartz [16] presented a set of models including one, two and three-factor models to analyze the behavior of commodity prices.

The variational iteration method was first proposed by He [5, 6] and was successfully applied to autonomous ordinary differential equations [7, 8], nonlinear systems of partial differential equations [13], construct solitary solutions and compacton-like solutions for partial differential equations [11], the Helmholtz equation [14], nonlinear differential equations of fractional order [15] and other fields [1, 2, 9, 10, 17].

Variational iteration method has been favorably applied to various kinds of nonlinear problems. The main property of the method is in its flexibility and ability to solve nonlinear equations accurately and conveniently [3, 4, 12].

In this paper a one-factor model of commodity prices is considered and solved using the Variational iteration method (VIM). The present paper is organized as follows. In section 2, the one-factorcommodity price model and its solution given by Schwartz are mentioned. In section 3, VIM is introduced for solving the equations of section 2, finally, numerical example is shown in section 4.

2. The one-factor model equation and its solution

2.1 The one-factor model equation:

$$\frac{\sigma^2}{2}x^2u_{xx} + k(\mu - \lambda - \ln(x))xu_x - u_t = 0 \qquad (2.1)$$

With terminal boundary condition $u(x,0) = x$ (2.2)

2.2 The solution in the literature

The closed form solution of the above equation (2.1) along with (2.2) found to be

$$u(x,t) = \exp\left[e^{-kt}\ln(x) + (1-e^{-kt})\left(\mu - \frac{\sigma^2}{2k} - \lambda\right) + \frac{\sigma^2}{4k}(1-e^{-2kt})\right] \quad (2.3)$$

3. Variational Iteration Method (VIM)

3.1 Basic Ideas of He's VIM

Consider the following differential equation

$$Lu + Nu = g(x,t) \quad (3.1)$$

Where L is linear operator, N is nonlinear operator and $g(x,t)$ is a known real function. According to VIM, a correction functional, $u(x,t)$ is as follows:

$$u_{n+1} = u_n(x,t) + \int_0^t \lambda\{Lu_n(x,\xi) + N\widetilde{u_n}(x,\xi) - g(x,\xi)\}d\xi \quad (3.2)$$

Where λ is the general Lagrange multiplier, u_0 is an initial approximation, $\widetilde{u_n}(x,t)$ is the restricted variation, i.e. $\delta\widetilde{u_n} = 0$. The optimal value of the general Lagrange multipliers λ can be identified by using the stationary conditions of the variational theory.

For sufficiently large values of n we can consider u_n as an approximation of the exact solution.

3.2 VIM for the equations of section 2

To obtain the approximate solution to equation (2.1) along with (2.2), according to VIM, it can be written as follows

$$u_{n+1} = u_n(x,t) + \int_0^t \phi\left\{\frac{\partial u_n(x,\xi)}{\partial \xi} - \frac{\sigma^2}{2}x^2\frac{\partial^2 u_n(x,\xi)}{\partial x^2} - k(\mu - \lambda - \ln(x))x\frac{\partial u_n(x,\xi)}{\partial x}\right\}d\xi \quad (3.3)$$

$$\delta u_{n+1} = \delta u_n(x,t) + \delta\int_0^t \phi\left\{\frac{\partial u_n(x,\xi)}{\partial \xi} - \frac{\sigma^2}{2}x^2\frac{\partial^2 \widetilde{u_n}(x,\xi)}{\partial x^2} - k(\mu - \lambda - \ln(x))x\frac{\partial \widetilde{u_n}(x,\xi)}{\partial x}\right\}d\xi$$

(3.4)

$$\delta u_{n+1} = \delta u_n(x,t) + \delta\int_0^t \phi\left\{\frac{\partial u_n(x,\xi)}{\partial \xi}\right\}d\xi \quad (3.5)$$

$$\delta u_{n+1} = \delta u_n(x,t)(1+\phi) - \delta\int_0^t \phi'\delta u_n(x,\xi)d\xi \quad (3.6)$$

This yields the stationary conditions

$$1+\phi=0,\ \phi'=0 \Longrightarrow \phi=-1 \qquad (3.7)$$

Substituting the value of $\phi=-1$ into the functional (3.3) give the iteration formulas

$$u_{n+1}=u_n(x,t)-\int_0^t\left\{\frac{\partial u_n(x,\xi)}{\partial\xi}-\frac{\sigma^2}{2}x^2\frac{\partial^2u_n(x,\xi)}{\partial x^2}-k(\mu-\lambda-\ln(x))x\frac{\partial u_n(x,\xi)}{\partial x}\right\}d\xi \qquad (3.8)$$

Convergence of VIM for one-factor Model:

Let the terms, $F_1\equiv x^2u_{xx}(x,t)$, $F_2\equiv xu_x(x,t)$, and $F_3\equiv x\ln x\, u_x(x,t)$ are Lipschitz continuous with $|F_1(u)-F_1(u^*)|\le L_1|u-u^*|$, $|F_2(u)-F_2(u^*)|\le L_2|u-u^*|$, and

$|F_3(u)-F_3(u^*)|\le L_3|u-u^*|$ where $x>0$, and $J=[0,\ T]$ $(T\in\mathbb{R})$.

Let $\beta_1=\{|a|L_1+|b|L_2+|k|L_3\}$, and $\beta_2=\{1-T(1-\beta_1)\}$ where $a=\frac{\sigma^2}{2}, b=k(\mu-\lambda)$

Theorem: The solution $u_n(x,t)$ obtained from (3.8) converges to the solution of problem (2.1) when $0<\beta_1<1$ and $0<\beta_2<1$.

Proof: $u_{n+1}=u_n(x,t)-\int_0^t\left\{\frac{\partial u_n(x,\xi)}{\partial\xi}-\frac{\sigma^2}{2}x^2\frac{\partial^2u_n(x,\xi)}{\partial x^2}-k(\mu-\lambda-\ln(x))x\frac{\partial u_n(x,\xi)}{\partial x}\right\}d\xi$

$$u_{n+1}(x,t)=u_n(x,t)-\int_0^t\left\{\frac{\partial u_n(x,\xi)}{\partial\xi}-aF_1(u_n)-bF_2(u_n)+kF_3(u_n)\right\}d\xi \quad (3.9)$$

$$u(x,t)=u(x,t)-\int_0^t\left\{\frac{\partial u(x,\xi)}{\partial\xi}-aF_1(u)-bF_2(u)+kF_3(u)\right\}d\xi \quad (3.10)$$

Let $e_{n+1}(x,t)=u_{n+1}(x,t)-u_n(x,t),\ e_n(x,t)=u_n(x,t)-u(x,t),$

$|e_n(x,t^*)|=\max_t|e_n(x,t)|$. Since e_n is a decreasing function with respect to 't' from (3.9), (3.10) and mean value theorem,

$$e_{n+1}(x,t)=e_n(x,t)+\int_0^t\left[\frac{\partial(-e_n)}{\partial\xi}-a\{F_1(u_n)-F_1(u)\}-b\{F_2(u_n)-F_2(u)\}+k\,\{F_3(u_n)-F_3(u)\}\right]dt$$

$$e_{n+1}(x,t) \le e_n(x,t) + \int_0^t (-e_n)dt + \{|a|L_1 + |b|L_2 + |k|L_3\} \int_0^t |e_n|dt$$

$$e_{n+1}(x,t) \le e_n(x,t) - Te_n(x,\omega) + \{|a|L_1 + |b|L_2 + |k|L_3\} \int_0^t |e_n|dt$$

$$e_{n+1}(x,t) \le e_n(x,t) - Te_n(x,\omega) + \{|a|L_1 + |b|L_2 + |k|L_3\}T\, |e_n(x,t)|$$

$$e_{n+1}(x,t) \le \{1 - T(1-\beta_1)\}|e_n(x,t^*)|$$

Where $0 \le \omega \le t$, hence$e_{n+1}(x,t) \le \beta_2 |e_n(x,t^*)|$, therefore,

$$\|e_{n+1}\| = \max_{\forall t \in J} |e_{n+1}| \le \beta_2 \max_{\forall t \in J} |e_n| \le \beta_2 \|e_n\|$$

Since,$0 < \beta_2 < 1$, then $\|e_n\| \to 0$.

4. Numerical Example

Example-1: Consider $\sigma = 1,\ k = 1,\ \mu = 1.2\ and\ \lambda = 1$ in the equation (2.1)

Then using (3.8), the following approximant for four iterations is obtained

$u_4(x,t)$ =((t^2*(50*log(x)^2 + 30*log(x) - 33))/100 - t*(log(x) - 1/5) + (t^4*(2500*log(x)^4 + 13000*log(x)^3 + 1600*log(x)^2 - 22280*log(x) - 2591))/60000 + (t^3*(5*log(x) + 14)*(20*log(x) - 50*log(x)^2 + 23))/1500 + 1)*x

Table-1:

x	t	absolute error for 4 iterations	absolute error for 10 iterations	absolute error for 15 iterations
0.1	0.1	0.000000269080663	0.000000000000003	0.000000000000000008327
0.2	0.3	0.000071598813756	0.000000001100168	0.000000000000011080026
0.3	0.2	0.000003136807479	0.000000000007988	0.000000000000000016653
0.3	0.4	0.000066286566480	0.000000017608175	0.000000000000787780952
0.4	0.2	0.000004915574328	0.000000000024079	0.000000000000000016653
0.4	0.5	0.000549149700013	0.000000525154193	0.000000000034576719266

0.5	0.3	0.000088570372058	0.000000002131796	0.00000000000023525626
0.5	0.5	0.001108030209802	0.000000526953829	0.00000000078254935865
0.6	0.4	0.000494478120082	0.000000025511016	0.00000000002379452191
0.6	0.5	0.001458464839562	0.000000267268399	0.00000000080269990654
0.7	0.1	0.000000615411354	0.000000000000001	0.00000000000000011102
0.7	0.4	0.000546779724672	0.000000011332593	0.00000000001300515251
0.7	0.5	0.001590621843618	0.000000150586089	0.00000000041950032337
0.8	0.5	0.001516166897892	0.000000625021322	0.00000000023764668011
0.9	0.5	0.001256016129682	0.000001072574928	0.00000000100792740820
1	0.5	0.000834698267464	0.000001431025665	0.00000000174120984298

Table-1, represents absolute error obtained using VIM with four, ten and fifteen iterations

Conclusion:

The results obtained in this paper provide the efficacy of Variational iteration method for finding the solution of one factor model of the stochastic behavior of commodity prices given by Eduardo S. Schwartz. In this work, the MATLAB package has been used to calculate the approximate series. The numerical results from table: 1, indicate improvement in solutions obtained by the variational iteration method at larger number of iterations as compared to the exact solution.

References

[1] Abdou M. A., and Soliman A. A., 2005, Variational iteration method for solving Burgers' and coupled Burgers' equations, J. Comput. Appl. Math., 181(2), pp 245–251.

[2] Bildik N., and Konuralp A., 2006, The use of variational iteration method, differential transform method and Adomian decomposition method for solving different types of nonlinear partial differential equations, Int. J. Nonlinear Sci. Numer. Simul., 7(1), pp 65–70.

[3] Duangpithak S., and Torvattanabun M., 2012, Variational Iteration Method for Solving Nonlinear Reaction-Diffusion-Convection Problems, Applied Mathematical Sciences, 6(17), pp 843–849.

[4] Golbabai A., and Javidi M., 2007, A variational iteration method for solving parabolic partial differential equations, Computers and Mathematics with Applications, 54, pp 987–992.

[5] He J. H., 1998, Approximate analytical solution for seepage flow with fractional derivatives in porous media, Comput. Methods Appl. Mech. Eng., 167(12), pp 57–68.

[6] He J. H., 1998, Approximate solution of nonlinear differential equations with convolution product nonlinearities, Comput. Methods Appl. Mech.Eng., 167(12), pp 69–73.

[7] He J. H., 1999, Variational iteration method—a kind of non-linear analytical technique: some examples, Int. J. Non-Linear Mech., 34(4), pp 699–708.

[8] He J. H., 2000, Variational iteration method for autonomous ordinary differential systems, Appl. Math. Comput., 114(2–3), pp 115–123.

[9] He J. H., 2001, Variational theory for linear magneto-electro-elasticity, Int. J. Nonlinear Sci. Numer. Simulation, 2(4), pp 309–316.

[10] He J. H., 2006, Some asymptotic methods for strongly nonlinear equations, Int. J. Modern Phys. B, 20(10), pp 1141–1199.

[11] He J. H., and Wu X. H., 2006, Construction of solitary solution and compacton-like solution by variational iteration method, Chaos Solitons Fractals, 29(1), pp 108–113.

[12] He J. H., and Wu X. H., 2007, Variational iteration method: New development and applications, Computers and Mathematics with Applications, 54, pp 881–894.

[13] Javidi M., and Golbabai A., 2008, Exact and numerical solitary wave solutions of generalized Zakharov equation by the variational iteration method, Chaos Solitons Fractals, 36(2), pp 309–313.

[14] Momani S., and Abuasad S., 2006, Application of He's variational iteration method to Helmholtz equation, Chaos Solitons Fractals, 27(5), pp 1119–1123.

[15] Odibat Z. M., and Momani S., 2006, Application of variational iteration method to nonlinear differential equations of fractional order, Int. J. Nonlinear Sci. Numer. Simul., 7(1), pp 27–36.

[16] Schwartz E. S., 1997, The Stochastic Behavior of Commodity Prices: Implications for Valuation and Hedging, The Journal of Finance, LII(3), pp 923-973.

[17] Soliman A. A., 2005, Numerical simulation of the generalized regularized long wave equation by He's variational iteration method, Math. Comput. Simulation, 70(2), pp 119–124.