INDEPENDENT LICT DOMINATION IN GRAPHS

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ABSTRACT

For any graph G , the lict graph $n(G)$ of a graph G,is a graph whose vertex set is the union of the set of edges and set of cutvertices of G in which two vertices are adjacent if and only if the corresponding members are adjacent or incident. A dominating set $D$of a lict graph $n(G)$ is an independent dominating set if the induced subgraph$< D >$ has no edges. The minimum cardinality of an minimal independent dominating set is called the Independent Lict Domination Number and is denoted by $\boldsymbol{i_n(G)}$ **.**

In this paper, many bounds on $i_n(G)$ were obtained in terms of the vertices, edges and many other different parameters of G but not in terms of the elements of $n(G)$. Further its relation with other different parameters are also developed.

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1. INTRODUCTION

In this paper, all the graph considered here are simple, finite,nontrivial , undirected and connected . The vertex set and edge set of graph G are denoted by $V(G) = p$ and $E(G) = q$ respectively. Terms not defined here are used in the sense of Harary[2] .

The degree, neighbourhood and closed neighbourhood of a vertex v in a graph G are denoted by $deg(v), N(v)\ , and\ N(v) = N(v) \cup \{v\}$ respectively. For a subset S of V , the graph induced by $S \subseteq V$ is denoted by $\langle S \rangle$.

As usual, the maximum degree of a vertex (edge) in G is denoted by$\Delta(G)\big(\Delta'(G)\big)$. For any real number$x$, $\lceil x \rceil$denotes the smallest integer not less than x and $\lfloor x \rfloor$denotes the greater integer not greater than x .

A vertex cover in a graph G is a set of vertices that covers all the edges of G.The vertex covering number $\alpha_o(G)$is the minimum cardinality of a vertex cover in G. An edge cover of a graph G without isolated vertices is a set of edges of G that covers all the vertices of G. The edge covering number $\alpha_1(G)$of a graph G is the minimum cardinality of an edge cover of G. A set of vertices/edges in a graph G is called independent set if no two vertices/edges in the set are adjacent. The vertex independence number$\beta_0(G)$ is the maximum cardinality of an independent set of vertices. The edge independence number $\beta_1(G)$of a graph G is the maximum cardinality of an independent set of edges.

A set D of graph $G = (V, E)$ is called a dominating set if every vertex in $V - D$ is adjacent to some vertex in$D$. The domination number $\gamma(G)$of G is the minimum cardinality taken over all dominating set of G.

A dominating set D is called connected dominating set of G if the induced subgraph $\langle D \rangle$ is connected. The connected domination number $\gamma_c(G)$ of a graph G is the minimum cardinality of a connected dominating set in G.

A set F of edges in a graph G is called an edge dominating set of G if every edge in $E - F$ is adjacent to atleast one edge in F. The edge domination number $\gamma'(G)$ of a graph G is the minimum cardinality of an edge dominating set of G .The edge dominating set F is called connected edge dominating set of G, if the induced subgraph $\langle F \rangle$ is also connected. The connected edge dominating set of G is denoted by $\gamma_c'(G)$ and is the minimum cardinality of the connected edge dominating set. Edge domination number was studied by S.L.Mitchell and Hedetniemi [4].

A dominating set D of a graph G is a strong Split dominating set if the induced subgraph $\langle V - D$ is totally disconnected with only two vertices.The Strong Split domination number γssG of a graph G is the minimum cardinality of a strong split dominating set of G. See [3].

A dominating set D of a graph $G = (V, E)$ is an independent dominating set if the induced subgraph $\langle D \rangle$ has no edges. The independent domination number $i(G)$ of a graph G is the minimum cardinality of an independent dominating set.

A set $D' \subseteq V'$ is said to be dominating set of $n(G)$, if every vertex in $V' - D'$ is adjacent to some vertex in D'. The domination number of $n(G)$ is denoted by $\gamma_n(G)$ and is the minimum cardinality of dominating set in $n(G)$.

Analogously, we define Independent Lict Domination Number as follows..

A dominating set D' of a Lict graph $n(G)$ is an independent dominating set if the induced subgraph$\langle D' \rangle$ has no edges.

The independentlict domination number $i_n(G)$is the minimum cardinality of an independent dominating set of $n(G)$.

In this paper , many bounds on $i_n(G)$ were obtained and expressed in terms of the vertices , edges and other parameters of G but not in terms of members of $n(G)$. Also we establish independentlict domination number and express the results with other different domination parameters of G.

We need the following Theorems to prove our later results .

***Theorem A** [1]*: For any connected (p, q)graph ,$\gamma'(G) \leq \left\lfloor \frac{p}{2} \right\rfloor$.

***Theorem B** [3]* : For any graph G , $\gamma_{ss}(G) = \alpha_0(G) + p_0$ where p_0 is the number of isolated vertices of G .

***Theorem C** [3]*: For any connected graph ,$\gamma_c(G) \leq p - \Delta(G)$.

2. RESULTS

First we list out the exact values of $i_n(G)$ for some standard graphs.

Theorem 1:

a. For any path P_p with $p \leq 2$ vertices ,

$$i_n(P_p) = \frac{p}{2} - 1 \text{if} p \text{ is even}$$

$$i_n(P_p) = \left\lfloor \frac{p}{2} \right\rfloor \text{if} p \text{ is odd .}$$

b. For any cycle C_p ,

$i_n(C_p) = \left\lceil \frac{p}{3} \right\rceil$.

c. For any star $K_{1,p}$,

$i_n(K_{1,p}) = 1$.

d. For any Wheel W_p ,

$i_n(W_p) = \frac{p}{2}$; where p is even

$i_n(W_p) = \left\lfloor \frac{p}{2} \right\rfloor$; where p is odd .

e. For any complete graph $K_{1,p}$,

$i_n(K_{1,p}) = \left\lfloor \frac{p}{2} \right\rfloor$.

The following Theorem relates $i_n(G)$ *and* $\gamma_n(G)$ in terms of the edges of G.

Theorem 2:For any connected (p,q) graph G, $\gamma_n(G) + i_n(G) \leq q$.

Proof: Suppose $V(n(G)) = \{v_1, v_2, \dots\dots\dots v_n\}$ and let $D_1 \subseteq V(n(G)) = \{v_1, v_2, \dots\dots\dots v_i\}$ for each $i, 1 \leq i \leq n$, be the set of vertices in $n(G)$ such that $N[D_1] = V(n(G))$. Then $|D_1| = \gamma_n(G)$. Suppose for each $v_i \in D_1$, the induced subgraph $\langle D_1 \rangle$ contains the set of vertices such that $deg(v_i) = 0$. Then D_1 itself forms an independent dominating set of $n(G)$. Otherwise, let $S = D_2 \cup I$ where $D_2 \subseteq D_1$ and $I = V(n(G)) - D_1$, such that for all $v_i \in \langle D_2 \cup I \rangle$, $deg(v_i) = 0$. Thus $\langle D_2 \cup I \rangle$ forms a minimal independent dominating set of $n(G)$. Now since $V(n(G)) = E(G) \cup C(G)$ where $C(G) =$

$\{c_1, c_2, \dots\dots\dots\dots c_n\} \subseteq V(n(G))$is the set of cutvertices in G . It follows that$|D_1| \cup |S| \leq |E(G)|$, which gives$\gamma_n(G) + i_n(G) \leq q$.

The Theorem below relates $i_n(G)$ and $\gamma_{ss}(G)$.

Theorem 3 : For any connected non trivial graph G , $i_n(G) \leq \gamma_{ss}(G)$.

Proof : Suppose $F = \{e_1, e_2, \dots\dots\dots\dots e_j\} \subseteq E(G)$ be the edge dominating set of graph G and $C = \{c_1, c_2, \dots\dots\dots\dots\dots c_i\}$be the set of cutvertices in G . In $n(G)$, since $V(n(G)) = E(G) \cup C(G)$, then the set $D_1 \subseteq F \cup C$ such that $N[D_1] = V(n(G))$ is the minimal dominating set of $n(G)$ .Suppos$E(D_1) = \emptyset$ in the subgraph $\langle D_1 \rangle$, then D_1itself is an independent dominating set of $n(G)$. Otherwise , let $I = D_2 \cup D_2'$ where $D_2 \subseteq D_1$ and $D_2' \subseteq V(n(G)) - D_1$ such that no two vertices in $\langle D_2 \cup D_2' \rangle$ are adjacent. Hence the subgraph $\langle D_2 \cup D_2' \rangle$ forms an independent dominating set of $n(G)$. Further let $S \subseteq V(G)$ be the maximum independent set of vertices incident on F in G and $S' \subset S$ be the set of all isolated vertices in$G$, then $\langle (V - S) \cup S' \rangle$ forms a strong split domination in G. Thus $|D_2 \cup D_2'| \leq |V - S| \cup S'$ which gives$i_n(G) \leq \gamma_{ss}(G)$.

Theorem 4 :For any connected graph G , $i_n(G) \leq \alpha_0(G) + p_0$ where $p_0 = V(G) - \gamma_{ss}(G)$.

Proof : From Theorem B and Theorem 3 we have the required result .

Theorem 5 :For any connected non-trivial graph G , $i_n(G) \leq \beta_1(G)$.

Proof:Suppose$F \subseteq E(G) = \{e_1, e_2 \dots\dots\dots\dots\dots e_i\}$, $\forall\, 1 \leq i \leq n$ be an edge dominating set of G and $C = \{c_1, c_2, c_3 \dots\dots\dots\dots c_n\}$ be the set of cutverticesin G. Now suppose $J \subseteq F(G)$is a set of maximum edges in G,such that for any $e_i, e_j \in J$, $N[e_i] \cap N[e_j] = \emptyset$, $1 \leq i, j \leq n$, then J forms maximal edge independent set of G with $|J| = \beta_1(G)$. Since$V(n(G)) = E(G) \cup C(G)$, then there exist an independent set of vertices $D \subseteq J' \cup H$ where $J' \subseteq J$ and $H \subseteq V(n(G)) - J$ such that$H \notin N[J]$, which covers all the vertices in$n(G)$. Then clearly $\langle D \rangle$ forms a minimal independent dominating set in$n(G)$. It follows that $|D| \leq |J|$ which gives$i_n(G) \leq \beta_1(G)$.

Theorem 6 : For any connected (p, q) graph G , $i_n(G) \leq \left\lceil \frac{p}{2} \right\rceil$.

Proof : Let $S = \{e_1, e_2 \dots \dots \dots \dots \dots e_n\}$be the minimal edge dominating set in G and $C = \{c_1 , c_2 \dots \dots \dots \dots \dots c_n\}$ be the set of cutvertices in G .Then $S \cup C \subseteq V(n(G))$. Now, we consider a minimal set of vertices $D_1 \subseteq S \cup C$ in $n(G)$such that $N[D_1] = V(n(G))$.Then D_1 is the minimal dominating set in $n(G)$.Further if $E(D_1) = \emptyset$ in the subgraph$\langle D_1 \rangle$, then D_1itself is an independent dominating set of $n(G)$. Otherwise , let $I = D_2 \cup D_2'$ where $D_2 \subseteq D_1$ and $D_2' \subseteq V(n(G)) - D_1$ such that no two vertices in $\langle D_2 \cup D_2' \rangle$ are adjacent. Hence the subgraph $\langle D_2 \cup$ $D2'$ forms an independent dominating set of nG . Since each $ei \in S$ is incident on two vertices and also by Theorem A,we get $i_n(G) \leq \left\lceil \frac{p}{2} \right\rceil$.

The next Theorem gives the relation between $i_n(G)$ *and* $\gamma_c(G)$.

Theorem 7 : For any graph G , $i_n(G) \leq \gamma_c(G)$.

Proof : Suppose D_1 be the minimal dominating set in G such that $|D_1| = \gamma(G)$. Now if $\forall\, v_i \in D_1$forms a connected path in $\langle D_1 \rangle$then $|D_1| = \gamma_c(G)$. Otherwise let $D_2 \subseteq V(G) - D_1$ *and* $D_2 \in N(D_1)$. If $D_2' \subseteq D_2$ is such that the induced subgraph $S = \langle D_1 \cup D_2' \rangle$ forms a minimal connected path in G , then $\langle S \rangle$ is the connected dominating set of G with $|S| = |D_1 \cup D_2'| = \gamma_c(G)$. Without loss of generality let $F_1 = \{e_1 , e_2 , \dots \dots \dots \dots\, e_j\} \subseteq E(G)$ and $F_2 \subseteq F_1$ be the set of edges incident on the vertices of S. By the definition of$n(G)$, the set$F_2 \subseteq V(n(G))$. Clearly F_2gives a set $D_3 = \{v_1 , v_2 , \dots \dots \dots \dots \dots . v_j\}$in $n(G)$ such that $\langle D_3 \rangle$ is connected. Thus $|F_2| \leq |S|$ which gives$|D_3| \leq |S|$. Now consider the set $D_3' \subseteq D_3$ such that $N[D_3'] = V(n(G))$ and $deg(v_i) = 0$ for each $v_i \in D_3'$. Then the induced subgraph $\langle D_3' \rangle$ forms an independent dominating vertices in $n(G)$ .Since $|D_3| \leq |S|$ and $D_3' \subseteq D_3$, then $|D_3'| \leq |S|$ which gives $i_n(G) \leq \gamma_c(G)$.

Theorem 8 : For any connected non trivial graph G , $i_n(G) \leq p - \Delta(G)$.

Proof : From Theorem 7 and Theorem C the result follows .

Theorem 9 : For any tree T , in which every support vertex is adjacent to atleast one end edge , then $i_n(T) \leq \left\lceil \frac{q-m}{2} \right\rceil + 1$, where m is the number of end edges in T . Further equality holds for$K_{1,p-1}$.

Proof :Let $F_1 = \{e_1, e_2, \dots \dots \dots \dots . e_n\}$ be the set of all end edges in T such that $|F_1| = m$. Without loss of generality $V(n(T)) = E(T) \cup C_1(T)$ where $C_1(T) \in V(T)$ is the set of cutverticesin T. Let $F_2 \subseteq E(T)$ be the non-end edges in$T$. Then clearly $F_2 = C_2$, where C_2 is the set of cutvertices in $n(T)$. Consider $C_1' \subseteq C_1(T)$and $C_2' \subseteq C_2(n(T))$. If every vertex $v_j \in V(n(T)) - (C_1' \cup C_2')$are adjacent to atleast one vertex of $(C_1' \cup C_2')$then $\langle C_1' \cup C_2' \rangle$forms a dominating set of $n(T)$. Further if every vertex in$\langle C_1' \cup C_2' \rangle$ are non-adjacent , then $\langle C_1' \cup C_2' \rangle$is the independent dominating set of $n(T)$ .Hence $|C_1' \cup C_2'| \leq \left\lceil \frac{q-m}{2} \right\rceil + 1$, which gives $i_n(T) \leq \left\lceil \frac{q-m}{2} \right\rceil + 1$.

For equality ,suppose$T \cong K_{1,p-1}$,then $q = m$ .Hence $n(T) = K_p$and$i_n(T) = 1$, which gives $i_n(T) = \left\lceil \frac{q-m}{2} \right\rceil + 1$.

Theorem 10: For any connected graph$G$,with $p \geq 2$ vertices $i_n(S(G)) \leq p - 1$.

Proof : Let T be a spanning tree of G.Clearly for any tree ,$\beta_1(S(T)) = p - 1$.

Any set of $(p - 1)$ independent edges of $S(T)$is an independent dominating set

of$n(G)$. Hence$i_n(S(G)) \leq p - 1$.

Theorem 11 : For any connected (p, q) graph G , $i_n(G) \leq q - \Delta'(G)$.

Proof : Let $E(G) = \{e_1, e_2, \dots \dots \dots \dots . e_n\}$and $E_1(G) \subseteq E(G)$ such that $\forall\, e_i \in \{E(G) - E_1(G)\}$ are adjacent to atleast one edge of $E_1(G)$ and $N(e_i) \cap N(e_j) = \emptyset$, $\forall\, e_i, e_j \in E_1(G)$. Hence $E_1(G)$ is a set of non-adjacent edges in G.Since $V(n(G)) = E(G) \cup C(G)$, $E_1(G) \subseteq V(n(G))$ and is an independent set of edges . Thus $|E_1(G)| = i_n(G)$ in $n(G)$. Further let there exists a set $E_2 = \{e_1, e_2, \dots \dots \dots \dots \dots e_i\}$ with $|E_2| = \Delta'(G)$ .Now $E_1(G) \subseteq E(G) - E_2$ and $|E_1(G)| \leq |E(G) - E_2(G)|$ which gives$i_n(G) \leq q - \Delta'(G)$.

The following Theorem gives Northus-Gaddum type of result.

Theorem 12 : Let G be a graph such that both G and $\bar{G}$ have no isolated edges , then ,

$$i_n(G) + i_n(\bar{G}) \leq 2\left\lceil\frac{p}{2}\right\rceil$$

$i_n(G) \,.\, i_n(\bar{G}) \leq \left\lceil\frac{p}{2}\right\rceil^2$.

References:

[1] .S.Arumugam and S.Velammal .(1998) Edge domination in graphs.Taiwaness J. of Mathematics 2(2) (173-179).

[2] .F.Harary ,(1969) ,Graph Theory ,Adison Wesley , Reading Mass (61-62).

[3]. V.R.Kulli ,(2010),Theory Of Domination In Graphs. Vishwa International Publication,Gulbarga India.

[4] .S.Mitchell and S.T.Hedetniemi ,(1977),Edge Domination in Trees . In: Proc .8th S.E.Conf.OnCombinatorics,Graph Theory and Computing , utilitasMathematica ,Winnipeg, 19 (489-509) .