LAPLACE TRANSFORMS AND STABILITY ANALYSIS OF THE WAGE FUNCTION

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Abstract:

In this paper, Laplace transforms has been used to solve the wage equation. The subsequent wage function is analyzed and interpreted for stability. The equation incorporates speculative parameters operating in free range. Restricting these parameters to negative has caused stability of the wage function as time approaches infinity. The function could initially stand off the equilibrium wage but in the long run, it asymptotically stabilizes in inter temporal sense. If initially the wage rate is at equilibrium then in the long run, it stabilizes there. The results obtained here compares with other results demonstrated in [6, 7, and 8] but with quite involving algebra.

Key words: wage equation, wage function, wage rate, equilibrium wage rate, stability, Laplace transforms.

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1. Introduction

The differential wage equation

$$\frac{d^2W}{dt^2}+a_1\frac{dW}{dt}+a_2W=b \tag{1.1}$$

with $a_1=\frac{\tau}{\psi},\ a_2=-\left(\frac{\sigma+\lambda}{\psi}\right),\ and\ b=-\left(\frac{\theta+\eta}{\psi}\right)$ was developed in [8] using linear demand and supply functions of labor

$$N_d=\eta-\sigma W+\tau\frac{dW}{dt}+\psi\frac{d^2W}{dt^2},\quad (\eta,\sigma>0) \tag{1.2}$$

and

$$N_s=-\theta+\lambda W+\vartheta\frac{dW}{dt}+\xi\frac{d^2W}{dt^2},\quad (\theta,\lambda>0) \tag{1.3}$$

respectively. In this development, the parameters $\tau,\psi,\vartheta\ and\ \xi$ introduced to dictate employers and laborers expectations had their signs operating in free range. For example, for $\tau>0$ then a rising wage rate caused the number of laborers demanded to increase. This suggested that employers expected rising wage rate to continue to rise and preferred to increase employment then, when the wage rate was still relatively low. On the other hand, for $\tau<0$, the wage trend was falling, and employers opted to cut employment at that time while they waited for wage rate to fall further. The inclusion of the parameter ψ made employers behavior to also depend on the rate of change of wage rate $\frac{dW}{dt}$. The introduction of new parameters $\tau\ and\ \psi$ injected wage rate speculation in the model. Similarly, for $\vartheta>0$, a rising wage rate caused the number of laborers supplied to fall. Laborers expected rising wage rate to continue rising and preferred to withhold their services while they waited for higher wage rate. On the other hand, for $\vartheta<0$, wage rate showed a falling trend and laborers preferred to offer their services while the wage rate was still relatively high hoping that any delay was to make wage rate fall even further. The introduction of the parameter ξ made the behavior of laborers to depend much on the wage rate. An implicit model was then developed by assuming that only labor demand function contains wage expectations. Specifically, both $\vartheta\ and\ \xi$ of function (1.3) were set equal to zero; while τ *and* ψ of function (1.2) were set as non zero. Further, it was assumed that labor market cleared at every point in time. Equation (1.1) was therefore solved in [6] by the method of undetermined coefficients, in [7] by differential operator method and in [8] by method of variation of parameters. The subsequent wage function was analyzed and interpreted for stability in both cases with similar behavioral patterns. The variation of speculative parameters, which were included in modeling, caused both stability and instability of the wage function depending on circumstances. Where the wage function was exponential, asymptotic stability towards the equilibrium wage rate was observed but where it consisted of both exponential and periodic factors, the time path showed periodic fluctuations with successive cycles giving smaller amplitudes until the ripples die naturally.

In [1], dynamics of market prices is studied. It was found out that if the initial price of the price function lies off the equilibrium point, then in the long run its stability is realized at equilibrium position. In [2], equilibrium solutions representing a special class of static solutions are discussed. The study found that if a system starts exactly at equilibrium condition, then it will remain there forever.

The resistance-inductance electric current circuit for constant electromotive force is modeled into a differential equation in [4].The stability of the solution was studied in the long run and found to be a constant, which is the ratio of constant electromotive force to resistance. The solution is an exponential function, which converges to zero in the long run. Also, in resistance-inductance-capacitance series circuit, a second order equation is developed. Its solution consists of an exponential homogeneous part and a periodic integral part. The study found out that the homogeneous part converges to zero as time approaches infinity, while the periodic part exhibits practically harmonic oscillations.

In [5] natural decay equation is developed. The equation describes a phenomenon where quantity gradually decreases to zero. In the work, it is emphasized that convergence depends on the sign of the change parameter. If the change parameter is negative, it turned into a growth equation and if it is positive it stabilizes in the long run. The study of slope fields for autonomous equations and qualitative properties of the decay equation are also demonstrated. It is found that the solution could be positive, negative or zero. In all the three cases, the solution approaches zero in limit as time approaches infinity. In the same work, models representing Newton's law of cooling, depreciation, population dynamics of diseases and water drainage are also presented with similar property in the long run.

In [9], a deterministic logistic first order differential price equation is considered. The equation is solved by an integrating factor. In the analysis of the solution, it is observed that in the long run asset price settles at a constant steady state point at which no further change can occur.

The current paper therefore proposes solving wage equation [1.1] by Laplace transforms demonstrated in [3; 4], analyzing it, and interpreting the results.

2. Solution of the wage equation

In this section, we use Laplace transforms to solve the differential equation

$$\frac{d^2W}{dt^2} + a_1\frac{dW}{dt} + a_2W = b, \qquad W(t)\big|_{t=0} = W_0 \; and \; \frac{dW}{dt}\bigg|_{t=0} = 0 \qquad (2.1)$$

Taking the Laplace transform of both sides of equation (2.1) we obtain

$$\left(s^2\overline{W} - sW\big|_{t=0} - W_1\right) + a_1\left(s\overline{W} - W\big|_{t=0}\right) + a_2\overline{W} = \frac{b}{s} \qquad (2.2)$$

Inserting the initial conditions in equation (2.2) and solving for $\overline{W}$ gives

$$\overline{W} = \frac{(s+a_1)W_0}{s^2+a_1s+a_2} + \frac{b}{s(s^2+a_1s+a_2)} \qquad (2.3)$$

The denominators of the fractions on the right hand side of (2.3) contain a quadratic factor $s^2+a_1s+a_2$, which must be factorized to enable the fractions to be expressed as partial fractions.

Let the quadratic factor take the form

$$s^2+a_1s+a_2 = (s-r_1)(s-r_2) \qquad (2.4)$$

where $r_1 = \frac{1}{2}\left(-\frac{\tau}{\psi} + \sqrt{\left(\frac{\tau}{\psi}\right)^2 + 4\left(\frac{\sigma+\lambda}{\psi}\right)}\right)$ and $r_2 = \frac{1}{2}\left(-\frac{\tau}{\psi} - \sqrt{\left(\frac{\tau}{\psi}\right)^2 + 4\left(\frac{\sigma+\lambda}{\psi}\right)}\right)$.

Equation (2.3) therefore takes the form

$$\overline{W} = \frac{(s + a_1)W_0}{(s - r_1)(s - r_2)} + \frac{b}{s(s - r_1)(s - r_2)} \qquad (2.5)$$

Each part of the right hand side of equation (2.5) is written as partial fraction so as to obtain

$$\overline{W} = \frac{(r_1 + a_1)W_0}{r_1 - r_2}\left(\frac{1}{s - r_1}\right) + \frac{(r_2 + a_1)W_0}{r_2 - r_1}\left(\frac{1}{s - r_2}\right) + \frac{b}{r_1 r_2}\left(\frac{1}{s}\right) + \frac{b}{r_1(r_1 - r_2)}\left(\frac{1}{s - r_1}\right) + \frac{b}{r_2(r_2 - r_1)}\left(\frac{1}{s - r_2}\right) \qquad (2.6)$$

Taking inverse Laplace transforms, equation (2.6) gives the solution

$$W(t) = \frac{r_2}{r_1 - r_2}\left(\left(\frac{r_1 + a_1}{r_2}\right)W_0 + \frac{b}{r_1 r_2}\right)\exp r_1 t + \frac{r_1}{r_2 - r_1}\left(\left(\frac{r_2 + a_1}{r_1}\right)W_0 + \frac{b}{r_1 r_2}\right)\exp r_2 t + \frac{b}{r_1 r_2} \qquad (2.7)$$

$But\ since\ r_1 = \frac{1}{2}\left(-\frac{\tau}{\psi} + \sqrt{\left(\frac{\tau}{\psi}\right)^2 + 4\left(\frac{\sigma + \lambda}{\psi}\right)}\right),\ r_2 = \frac{1}{2}\left(-\frac{\tau}{\psi} - \sqrt{\left(\frac{\tau}{\psi}\right)^2 + 4\left(\frac{\sigma + \lambda}{\psi}\right)}\right),\ a_1 = \frac{\tau}{\psi}\ and$

$b = -\frac{\theta + \eta}{\psi}$, *we proceed to simplify the coefficients and the constant terms of solution* (2.7)

as follows:

$$r_1 r_2 = \frac{1}{4}\left(-\frac{\tau}{\psi} + \sqrt{\left(\frac{\tau}{\psi}\right)^2 + 4\left(\frac{\sigma + \lambda}{\psi}\right)}\right) \times \left(-\frac{\tau}{\psi} - \sqrt{\left(\frac{\tau}{\psi}\right)^2 + 4\left(\frac{\sigma + \lambda}{\psi}\right)}\right)$$

$$= -\frac{\sigma + \lambda}{\psi}$$

$$\therefore \frac{b}{r_1 r_2} = -\frac{\theta + \eta}{\psi} \times -\frac{\psi}{\sigma + \lambda}$$

$$= \frac{\theta + \eta}{\sigma + \lambda}, \quad \sigma \neq -\lambda \qquad (2.8)$$

$$r_1 + a_1 = \frac{\tau}{\psi} + \frac{1}{2}\left(-\frac{\tau}{\psi} + \sqrt{\left(\frac{\tau}{\psi}\right)^2 + 4\left(\frac{\sigma+\lambda}{\psi}\right)}\right)$$

$$= \frac{1}{2}\left(\frac{\tau}{\psi} + \sqrt{\left(\frac{\tau}{\psi}\right)^2 + 4\left(\frac{\sigma+\lambda}{\psi}\right)}\right)$$

$$\therefore \ \frac{r_1 + a_1}{r_2} = \frac{\frac{1}{2}\left(\frac{\tau}{\psi} + \sqrt{\left(\frac{\tau}{\psi}\right)^2 + 4\left(\frac{\sigma+\lambda}{\psi}\right)}\right)}{\frac{1}{2}\left(-\frac{\tau}{\psi} - \sqrt{\left(\frac{\tau}{\psi}\right)^2 + 4\left(\frac{\sigma+\lambda}{\psi}\right)}\right)} = \frac{-r_2}{r_2} = -1 \qquad (2.9)$$

$$r_2 + a_1 = \frac{\tau}{\psi} + \frac{1}{2}\left(-\frac{\tau}{\psi} - \sqrt{\left(\frac{\tau}{\psi}\right)^2 + 4\left(\frac{\sigma+\lambda}{\psi}\right)}\right)$$

$$= -\frac{1}{2}\left(-\frac{\tau}{\psi} + \sqrt{\left(\frac{\tau}{\psi}\right)^2 + 4\left(\frac{\sigma+\lambda}{\psi}\right)}\right)$$

$$\therefore \ \frac{r_2 + a_1}{r_1} = \frac{-\frac{1}{2}\left(-\frac{\tau}{\psi} + \sqrt{\left(\frac{\tau}{\psi}\right)^2 + 4\left(\frac{\sigma+\lambda}{\psi}\right)}\right)}{\frac{1}{2}\left(-\frac{\tau}{\psi} + \sqrt{\left(\frac{\tau}{\psi}\right)^2 + 4\left(\frac{\sigma+\lambda}{\psi}\right)}\right)} = \frac{-r_1}{r_1} = -1 \qquad (2.10)$$

$$r_2 - r_1 = \left(-\frac{\tau}{2\psi} - \frac{1}{2}\sqrt{\left(\frac{\tau}{\psi}\right)^2 + 4\left(\frac{\sigma+\lambda}{\psi}\right)}\right) - \left(-\frac{\tau}{2\psi} + \frac{1}{2}\sqrt{\left(\frac{\tau}{\psi}\right)^2 + 4\left(\frac{\sigma+\lambda}{\psi}\right)}\right)$$

$$= -\sqrt{\left(\frac{\tau}{\psi}\right)^2 + 4\left(\frac{\sigma+\lambda}{\psi}\right)}$$

$$\therefore \frac{r_1}{r_2-r_1}=\frac{1}{2}\left(\frac{-\frac{\tau}{\psi}+\sqrt{\left(\frac{\tau}{\psi}\right)^2+4\left(\frac{\sigma+\lambda}{\psi}\right)}}{-\sqrt{\left(\frac{\tau}{\psi}\right)^2+4\left(\frac{\sigma+\lambda}{\psi}\right)}}\right)$$

$$=\frac{\tau}{2\psi\sqrt{\left(\frac{\tau}{\psi}\right)^2+4\left(\frac{\sigma+\lambda}{\psi}\right)}}-\frac{1}{2} \tag{2.11}$$

$$r_1-r_2=\left(-\frac{\tau}{2\psi}+\frac{1}{2}\sqrt{\left(\frac{\tau}{\psi}\right)^2+4\left(\frac{\sigma+\lambda}{\psi}\right)}\right)-\left(-\frac{\tau}{2\psi}-\frac{1}{2}\sqrt{\left(\frac{\tau}{\psi}\right)^2+4\left(\frac{\sigma+\lambda}{\psi}\right)}\right)$$

$$=\sqrt{\left(\frac{\tau}{\psi}\right)^2+4\left(\frac{\sigma+\lambda}{\psi}\right)}$$

$$\therefore \frac{r_2}{r_1-r_2}=\frac{1}{2}\left(\frac{-\frac{\tau}{\psi}-\sqrt{\left(\frac{\tau}{\psi}\right)^2+4\left(\frac{\sigma+\lambda}{\psi}\right)}}{\sqrt{\left(\frac{\tau}{\psi}\right)^2+4\left(\frac{\sigma+\lambda}{\psi}\right)}}\right)$$

$$=-\frac{\tau}{2\psi\sqrt{\left(\frac{\tau}{\psi}\right)^2+4\left(\frac{\sigma+\lambda}{\psi}\right)}}-\frac{1}{2} \tag{2.12}$$

We now substitute the results (2.8), (2.9), (2.10), (2.11) and (2.12) in solution (2.7) and letting $\hat{W}=\frac{\theta+\eta}{\sigma+\lambda}, \sigma\neq\lambda$ we obtain

$$W(t)=\frac{r_2}{r_2-r_1}\left(W_0-\hat{W}\right)\exp r_1 t-\frac{r_1}{r_2-r_1}\left(W_0-\hat{W}\right)\exp r_2 t+\hat{W} \tag{2.13}$$

3. Results, analysis and interpretation

Equation (2.1) has been solved by Laplace transforms for the first time to obtain

$$W(t) = \frac{r_2}{r_2 - r_1}\left(W_0 - \hat{W}\right)\exp r_1 t - \frac{r_1}{r_2 - r_1}\left(W_0 - \hat{W}\right)\exp r_2 t + \hat{W} \qquad (3.1)$$

The solution (3.1) is therefore investigated for stability by finding its time path. Suppose we let $\psi < 0, \tau < 0$ *with* $\sigma, \lambda > 0$, the expression $\left(\frac{\tau}{\psi}\right)^2 + 4\left(\frac{\sigma + \lambda}{\psi}\right)$ *in* r_1 *and* r_2 is less than $\left(\frac{\tau}{\psi}\right)^2$ and the square root must be less than $\frac{\tau}{\psi}$; therefore r_1 *and* r_2 are both negative. Thus, the limiting value of function (3.1) in the long run can be found. In this case, $\left(\frac{r_2}{r_2 - r_1}\right)\left(W_0 - \hat{W}\right)$ and $\left(\frac{r_1}{r_2 - r_1}\right)\left(W_0 - \hat{W}\right)$ of function (3.1) are constants and the value of the limit function depends on the exponential factors $\exp r_1 t$ and $\exp r_2 t$. In view of the fact that $r_1, r_2 < 0$ the limits of the first and second term of function (3.1) both tends to zero; thus

$$W(t) = \hat{W} \qquad (3.2)$$

This means that the wage function (3.1) consequently moves towards the equilibrium position in the long run and it is therefore dynamically stable so long as $\psi < 0$ *and* $\tau < 0$.

Further analyses of the limit function are possible by considering the relative positions of W_0 *and* $\hat{W}$; that is, by comparing the relative positions of the initial wage rate and the equilibrium wage rate. This is discussed in three different cases.

CASEI: In this case, we consider both the limiting functions and let $W_0 = \hat{W}$ .This means the in the function (3.1) becomes $W(t) = \hat{W}$ at an infinite time, which is a constant path and is parallel to the time axis. The wage function becomes stable at equilibrium wage rate in the long run.

CASE II: In this case, we let $W_0 > \hat{W}$. The first term on the right hand side of function (3.1) is positive if $r_1 > r_2$ and the second term is only positive if $r_1 < r_2$.Therefore, they will decrease since as $t \to \infty$ they are lowered by the values of the exponential factors $\exp r_1 t$ and $\exp r_2 t$

respectively. The function thus has its time path asymptotically approaching the equilibrium wage rate $\hat{W}$ from above, and in the long run, become stable.

CASE III: In this case, we let, $W_0 < \hat{W}$ i.e. the initial wage rate is taken to be less than the equilibrium wage rate. The first term on the right hand side of the function is negative if $r_1 > r_2$ and the second term is only negative if $r_1 < r_2$, making W_0 to rise asymptotically towards the equilibrium wage rate $\hat{W}$ $as\, t \to \infty$. These three cases are illustrated in figure 3.1.

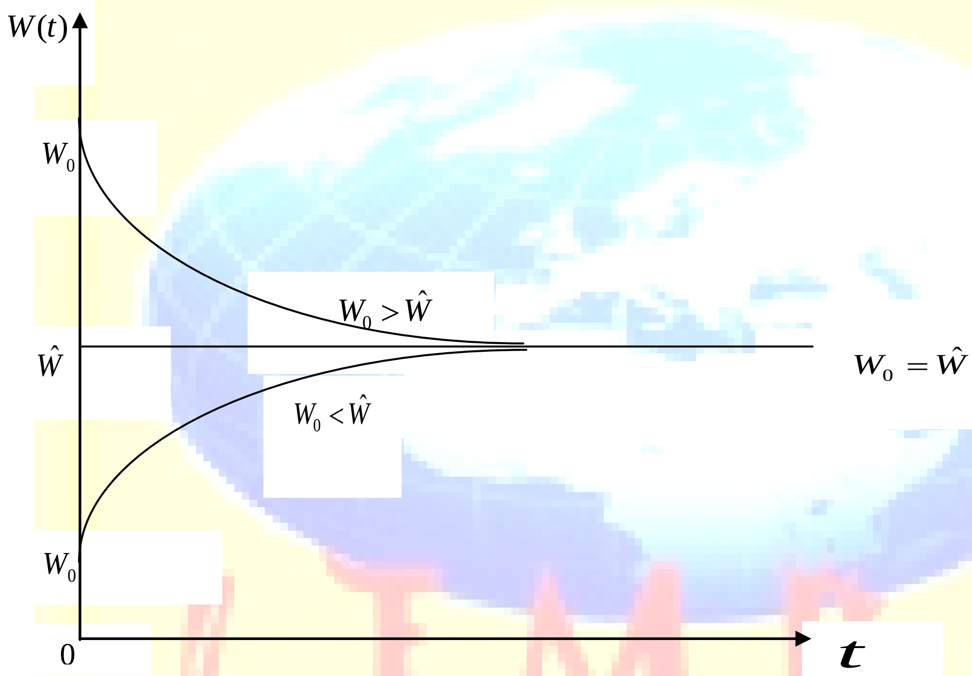

Figure 3.1: Stability Analysis of the Wage Function

Figure 3.1 shows that when $W_0 = \hat{W}, then\, W(t) = \hat{W}$, which is a constant function. If $W_0 > \hat{W}, then\, W(t)$ decreases asymptotically towards $\hat{W}$, *and if* $W_0 < \hat{W}$ *then* $W(t)$ increases asymptotically towards the equilibrium wage rate $\hat{W}$. The results therefore shows that as $t \to \infty$, the function (3.1) approach the equilibrium wage rate and becomes stable so long as $\psi < 0$ *and* $\tau < 0$.

4. Conclusion

In this paper, Laplace transforms has been used to solve wage equation. The subsequent wage function is analyzed and interpreted for stability. The equation incorporates speculative parameters operating in free range. Restricting these parameters to negative has caused stability of the wage function as time approaches infinity. Also, if the initial wage is set either above or below the equilibrium wage, it asymptotically approaches the equilibrium wage in the long run. If the initial wage rate is set at equilibrium wage then in the long run, it remains there; that is, it becomes stable at equilibrium wage. The results obtained here compares with other results demonstrated in [6, 7, and 8] but with quite involving algebra.

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