IN SOLVING THE MASS SPRING SYSTEM WITH INHOMOGENEOUS DIRAC DELTA FUNCTION USING THE LAPLACE TRANSFORM METHOD

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Abstract

The mass spring system is generally modelled by second order linear ordinary differential equations. The equations are usually analytically solved using elementary methods such as the method of undetermined coefficients, the method of the differential operator and the method of variation of parameters. These methods face stiff challenges when the inhomogeneous term is a discontinuous function such as the step or impulse function. In this paper we solve the mass spring system with an inhomogeneous Dirac delta function using the Laplace transform method. We conduct numerical experiments to illustrate nature of solutions obtained. The results obtained are consistent with the general theory of mass spring systems.

Key words: Dirac delta function, Laplace transform method, inhomogeneous, discontinuous

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1 Introduction

The general form of the ODE of the mass spring system is given by

$$x''(t)+2\alpha x'(t)+\omega_n^2 x(t)=f(t) \text{ with } x(0)=x_0,\ x'(0)=x_0' \qquad (1)$$

where ω_n and α are positive constants known as the natural frequency [rads/sec] and the neper frequency [nepers/sec] of the system, respectively, $f(t)$ is the external force [N] applied to the system at time t [sec] while x_0 and x_0' are the given initial conditions,Braun [1] and Pishgar *et al* [2]. The natural frequency represents the frequency of oscillations when no external force is applied to the system while the neper frequency determines the rate at which the system's response is damped. Some authors refer to the neper frequency as the damping factor, but in this paper as in Gavin and Dolbow [3] and Pishgar et al [2]we take the damping factor denoted by ε to be given by

$$\varepsilon=\frac{\alpha}{\omega_n}. \qquad (2)$$

The damping factor is a very important parameter when solving equation (1) as it determines the type of the solutions obtained.

Solving equation (1) analytically using elementary methods such as the method of undetermined coefficients, the differential operator method and the method of variation of parameters can become tedious and difficult when $f(t)$is a discontinuous function, Braun [1] and Abas [4]. Examples of discontinuous functions include step functions such as the Heaviside or the unit step functionand impulse functionssuch as the Dirac delta or unit impulse function. The Dirac delta function usually denoted by $\delta(t)$is also known as the unit impulse function represents a force that acts at a single instant of time and is zero elsewhere. It is defined as $\delta(t)=0$for $t\neq 0$ with $\int_{-\infty}^{\infty}\delta(t)dt=1$. In other words $\delta(t)$represents a force with a total impulse of 1 N acting at the instant time $t=0$, Hunt $et\ al$[5]. In general the Dirac delta function can be expressed as$F(t)=F_0\delta(t-t_0)$ where F_0 N is the total impulse at the instant time $t=t_0$ for F_0 any constant. The Dirac delta function applied to the mass spring system is like a sharp blow acting on the system.

The Laplace transform method is very much suitable for solving differential equations of the form (1) especially when $f(t)$ is a discontinuous function, Braun [1], Logan [6] and Hunt *et al* [4]. The method not only solves the differential equation, but also simplifies the problem. First the differential equation is transformed into an algebraic equation that is easy to solve. The solution of the differential equation is then obtained by taking the inverse Laplace transform of the obtained algebraic solution, Braun [1].

In this paper is we introduce the mass spring system with the Dirac Delta which is then solved using the Laplace transform. Numerical experiments are conducted to establish consistency with the existing theory on the mass spring system.

2 The mass spring system with Dirac delta function

In this paper we consider the mass spring system problem with $f(t)$ the Dirac delta function given as

$$f(t) = \frac{1}{m} F(t)\delta(t - t_0) \quad (3)$$

where $m > 0$ is the mass [kg] and $F(t)$ is the external force [N] applied to the system at the instant time $t = t_0$. The differential equation (1) for this problem is then given as

$$x''(t) + 2\varepsilon\omega_n x'(t) + \omega_n^2 x(t) = \frac{1}{m} F(t)\delta(t - t_0) \text{ with } x(0) = x_0,\ x'(0) = x_0' \quad (4)$$

where $x''(t)$ is the acceleration $[m/s^2]$, $x'(t)$ is the velocity $[m/s]$ and $x(t)$ is the displacement [m] of the mass from the equilibrium position at time t [sec]. The natural frequency of the system is

$$\omega_n = \sqrt{\frac{k}{m}} \quad (5)$$

where $k > 0$ is the spring constant [N/m]. In this problem we assume viscous damping.

The characteristic equation (CE) for equation (4) is

$$\lambda^2 + 2\varepsilon\omega_n\lambda + \omega_n^2 = 0 \quad (6)$$

and has the roots

$$\lambda_{1,2} = -\varepsilon\omega_n \pm \omega_n\sqrt{\varepsilon^2 - 1}\,. \quad (7)$$

Equation (7) shows that depending on the value of the damping factor ε, the CE has three types of roots, namely; either real distinct roots [$\varepsilon > 1$], real repeated roots [$\varepsilon = 1$] or complex roots [$\varepsilon < 1$]and [$\varepsilon = 0$]. If $\varepsilon > 1$ the real distinct roots are

$$\lambda_{1,2} = -\varepsilon\omega_n \pm \omega_n\sqrt{\varepsilon^2 - 1} \quad (8)$$

and in this case the system is said to be over-damped. When $\varepsilon = 1$, the real repeated roots are

$$\lambda_{1,2} = -\omega_n \quad (9)$$

and in this case the system is said to be critically damped. For $\varepsilon < 1$, we have no real roots, but instead complex roots given by

$$\lambda_{1,2} = -\varepsilon\omega_n \pm i\omega_n\sqrt{1 - \varepsilon^2} (10)$$

and in this case the system is said to be under-damped. In the case of no-damping [$\varepsilon = 0$], we also have complex rootsand are given by

$$\lambda_{1,2} = \pm i\omega_n \qquad (11)$$

and in this case the system is said to be un-damped [1,7]. Before we solve the differential equation (4) using the Laplace transform method, we first consider some important definitions and properties of the Laplace transform, the Dirac delta function and the inverse Laplace transform.

Some important properties of the Dirac delta function considered include

$$(\boldsymbol{i}) \int_a^b f(t)\delta(t-t_0)\,dt = \begin{cases} f(t_0) & , if\ a \le t < b \\ 0 & , otherwise \end{cases} \qquad (12)$$

and hence

$$(\boldsymbol{ii}) \int_0^\infty \delta(t)dt = 1 \qquad , t \ge 0. \qquad (13)$$

3 Some properties of the Laplace Transform

3.1Some Properties of the Laplace Transform

Definition 1.*If $f(t)$ is defined for $0 \le t < \infty$, then the Laplace transform of $f(t)$ denoted by $F(s)$ or $\pounds\{f(t)\}$ is given by*

$$F(s) = \pounds\{f(t)\} = \int_0^\infty e^{-st} f(t)dt \qquad (14)$$

where s is in general a complex quantity with a positive real part [1, 7]. The Laplace transform has the following properties [1,7, 8]

• Linearity Property

$$\pounds\{af(t) + bg(t)\} = \pounds\{af(t)\} + \pounds\{bg(t)\} = a\pounds\{f(t)\} + b\pounds\{g(t)\} = aF(s) + bG(s) \qquad (15)$$

for a and b any two constants.

• Laplace Transform of Derivatives

$$(i)\pounds\{x'(t)\} = s\pounds\{x(t)\} - x(0) = sX(s) - x_0, \qquad (16)$$

$$(ii)\pounds\{x''(t)\} = s\pounds\{x'(t)\} - x'(0) = s[sX(s) - x(0)] - x'(0) = s^2X(s) - sx_0 - x_0'. \qquad (17)$$

In general for any positive integer n, we have

$$(iii)\pounds\{x^n(t)\} = s^nX(s) - s^{(n-1)}x_0 - s^{(n-2)}x_0' - s^{(n-3)}x_0' - \cdots - sx_0^{(n-2)} - x_0^{(n-1)}. \qquad (18)$$

• Laplace Transform of the Dirac Delta Function

$$£\{\partial(t - t_0)\} = \int_0^\infty e^{-st}\partial(t - t_0)\, dt = e^{-st_0},\ t_0 \geq 0 \quad (19)$$

by property (12). Hence for $t_0 = 0$, we have $£\{\partial(t)\} = 1$ [8]. Similarly, by property (12) we have

$$£\{f(t)\partial(t - t_0)\} = e^{-st_0} f(t_0). \quad (20)$$

3.2 Properties of the inverse Laplace transform

Definition 2. *If $F(s)$ is the Laplace transform of $f(t)$ then the inverse Laplace transform of $F(s)$is given by$£^{-1}\{F(s)\} = f(t)$ and has the following properties [1, 7, 8]*

• Linearity Property

If the inverse Laplace transforms of two functions $F(s)$ and$G(s)$ exist, then for any two constants a and b we have

$$£^{-1}\{aF(s) + bG(s)\} = a£^{-1}\{F(s)\} + b£^{-1}\{G(s)\} = af(t) + bg(t). \quad (21)$$

• Shift or Translation Property

If $F(s) = £\{f(t)\}$, then

$$£\{u(t - t_0)f(t - t_0)\} = e^{-st_0}F(s) \quad (22)$$

and hence

$$£^{-1}\{e^{-st_0}F(s)\} = u(t - t_0)f(t - t_0) \quad (23)$$

where $u(t - t_0)$ is the unit step function given by

$$u(t - t_0) = \begin{cases} 0 & ,t < t_0 \\ 1 & ,t \geq t_0. \end{cases} \quad (24)$$

More explicitly

$$u(t - t_0)f(t - t_0) = \begin{cases} 0 & ,t < t_0 \\ f(t - t_0) & ,t \geq t_0. \end{cases} \quad (25)$$

4 Solving the system

Applying the Laplace transform to both sides of the differential equation (4) and using properties (15), (17) and (20) together with the initial conditions we obtain the algebraic equation

$$X(s) = \overbrace{\frac{x_0 s + 2\varepsilon\omega_n x_0 + x_0'}{s^2 + 2\varepsilon\omega_n s + \omega_n^2}}^{\textit{Free Response}} + \overbrace{\frac{Ce^{-st_0}}{s^2 + 2\varepsilon\omega_n s + \omega_n^2}}^{\textit{Forced Response}} \quad (26)$$

where C is a constant given by $C = \frac{1}{m}F(t_0)$. Free response refers to the homogeneous part or the case where no external force is applied to the system$[F(t) = 0]$. Solving the algebraic equation (26) mainly depends on the nature of its denominator. That is, whether it is reducible or irreducible [6]. Equating the denominator of equation (26) to zero and replacing s byλ, yields the characteristic equation of the differential equation (4). Hence the nature of the roots of the characteristic equation (6) determines whether the denominator of the algebraic equation is reducible or irreducible. The denominator is reducible when the characteristic equation has real roots [over-damped and critically damped cases] and irreducible when the roots are complex [under-damped and un-damped cases] [6]. Next we solve the algebraic equation for the four cases in the following order; over-damped, critically damped under-damped and un-damped case.

Case I $[\boldsymbol{\varepsilon} > 1: over\ damped]$

In this casethe characteristic equation (6) has two real distinct roots

$\lambda_1 = -\varepsilon\omega_n + \omega_n\sqrt{\varepsilon^2 - 1}$ and$\lambda_2 = -\varepsilon\omega_n - \omega_n\sqrt{\varepsilon^2 - 1}$. Thus the denominator of equation (26) is reducible and simplifies to

$$s^2 + 2\varepsilon\omega_n s + \omega_n^2 = (s - \lambda_1)(s - \lambda_2). \qquad (27)$$

The algebraic equation (26) becomes

$$X(s) = \overbrace{\frac{x_0 s + 2\varepsilon\omega_n x_0 + x_0'}{(s - \lambda_1)(s - \lambda_2)}}^{\textit{Free Response}} + \overbrace{\frac{Ce^{-st_0}}{(s - \lambda_1)(s - \lambda_2)}}^{\textit{Forced Response}}. \qquad (28)$$

Solving for $X(s)$ using the method of partial fractions decomposition we obtain

$$X(s) = \overbrace{\frac{A_1}{s - \lambda_1} + \frac{B_1}{s - \lambda_2}}^{\textit{Free Response}} + \overbrace{Ce^{-st_0}F(s)}^{\textit{Forced Response}} \qquad (29)$$

where A_1 and B_1 are constants given by $A_1 = -\frac{[(\lambda_1 + 2\varepsilon\omega_n)x_0 + x_0']}{\lambda_2 - \lambda_1}$ and $B_1 = \frac{(\lambda_2 + 2\varepsilon\omega_n)x_0 + x_0'}{\lambda_2 - \lambda_1}$ with

$$F(s) = \frac{D_1}{s - \lambda_1} + \frac{E_1}{s - \lambda_2} \qquad (30)$$

where D_1 and E_1 are constants given by $D_1 = \frac{-1}{\lambda_2 - \lambda_1}$ and $E_1 = \frac{1}{\lambda_2 - \lambda_1}$. Taking the inverse Laplace transform of equation (29) and using the shift property (23) for the forced response part we obtain

$$x(t) = \overbrace{A_1e^{\lambda_1 t} + B_1e^{\lambda_2 t}}^{\boldsymbol{x_c(t)}} + \overbrace{Cu(t - t_0)f(t - t_0)}^{\boldsymbol{x_p(t)}} \qquad (31)$$

where

$$f(t) = £^{-1}\left\{\frac{D_1}{s-\lambda_1} + \frac{E_1}{s-\lambda_2}\right\} = D_1 e^{\lambda_1 t} + E_1 e^{\lambda_2 t}. \qquad (32)$$

Here$x_c(t)$ is the homogeneous solution and $x_p(t)$ is the particular solution. The general solution is

$$x(t) = x_c(t) + x_p(t) \qquad (33)$$

and can be expressed more explicitly as

$$x(t) = \begin{cases} A_1 e^{\lambda_1 t} + B_1 e^{\lambda_2 t} & , if\ t < t_0 \\ A_1 e^{\lambda_1 t} + B_1 e^{\lambda_2 t} + Cf(t-t_0), & if\ t \geq t_0. \end{cases} \qquad (34)$$

Case II [$\boldsymbol{\varepsilon = 1}$: $\boldsymbol{critically\ damped}$]

In this case the characteristic equation (6) has real repeated roots $\lambda_{1,2} = -\omega_n$ and the denominator of equation (26) is reducible and simplifies to

$$s^2 + 2\omega_n s + \omega_n^2 = (s-\lambda)^2 \qquad (35)$$

where $\lambda = \lambda_{1,2} = -\omega_n$. Equation (26) becomes

$$X(s) = \overbrace{\frac{x_0 s + 2\omega_n x_0 + x_0'}{(s-\lambda)^2}}^{Free\ Response} + \overbrace{\frac{Ce^{-st_0}}{(s-\lambda)^2}}^{Forced\ Response}. \qquad (36)$$

Solving for $X(s)$ we obtain

$$X(s) = \overbrace{\frac{A_2}{s-\lambda} + \frac{B_2}{(s-\lambda)^2}}^{Free\ Response} + \overbrace{Ce^{-st_0}F(s)}^{Forced\ Response} \qquad (37)$$

where A_2 and B_2 arte constants given by $A_2 = x_0$ and $B_2 = (\lambda + 2\omega_n)x_0 + x_0'$ with

$$F(s) = \frac{1}{(s-\lambda)^2}. \qquad (38)$$

Taking the inverse Laplace transform of equation (37) we obtain

$$x(t) = \overbrace{A_2 e^{\lambda t} + B_2 t e^{\lambda t}}^{x_c(t)} + \overbrace{Cu(t-t_0)f(t-t_0)}^{x_p(t)} \qquad (39)$$

where

$$f(t) = £^{-1}\left\{\frac{1}{(s-\lambda)^2}\right\} = te^{\lambda t}. \qquad (40)$$

More explicitly the general solution is given by

$$x(t) = \begin{cases} A_2 e^{\lambda t} + B_2 e^{\lambda t} & ,if\ t < t_0 \\ A_2 e^{\lambda t} + B_2 t e^{\lambda t} + Cf(t - t_0) & ,if\ t \geq t_0. \end{cases} \quad (41)$$

Case III [$\boldsymbol{\varepsilon} < 1$: *under damped*]

In this case the characteristic equation (6) has no real roots, but complex roots instead. The denominator of equation (26) is irreducible and using the method of completing the square the denominator simplifies to

$$s^2 + 2\varepsilon\omega_n s + \omega_n^2 = (s + \varepsilon\omega_n)^2 + \omega_d^2 \quad (42)$$

where

$$\omega_d = \omega_n\sqrt{1 - \varepsilon^2} \quad (43)$$

is the damped frequency of the system[1, 6]. Equation (26) then becomes

$$X(s) = \overbrace{\frac{x_0 s + 2\varepsilon\omega_n x_0 + x_0'}{(s + \varepsilon\omega_n)^2 + \omega_d^2}}^{Free\ Response} + \overbrace{\frac{Ce^{-st_0}}{(s + \varepsilon\omega_n)^2 + \omega_d^2}}^{Forced\ Response}. \quad (44)$$

Solving for $X(s)$ we obtain

$$X(s) = \overbrace{A_3\left\{\frac{(s + \varepsilon\omega_n)}{(s + \varepsilon\omega_n)^2 + \omega_d^2}\right\} + B_3\left\{\frac{\omega_d}{(s + \varepsilon\omega_n)^2 + \omega_d^2}\right\}}^{Free\ Response} + \overbrace{Ce^{-st_0}F(s)}^{Forced\ Response} \quad (45)$$

where A_3 and B_3 are constants given by $A_3 = x_0$ and $B_3 = \frac{\varepsilon\omega_n x_0 + x_0'}{\omega_d}$ with

$$F(s) = \frac{1}{(s + \varepsilon\omega_n)^2 + \omega_d^2}. \quad (46)$$

Taking the inverse Laplace transform of (45) we obtain

$$x(t) = \overbrace{e^{-\varepsilon\omega_n t}[A_3\cos(\omega_d t) + B_3\sin(\omega_d t)]}^{x_c(t)} + \overbrace{Cu(t - t_0)f(t - t_0)}^{x_p(t)} \quad (47)$$

where

$$f(t) = \pounds^{-1}\left\{\frac{1}{(s + \varepsilon\omega_n)^2 + \omega_d^2}\right\} = \frac{1}{\omega_d}e^{-\varepsilon\omega_n t}\sin(\omega_d t). \quad (48)$$

More explicitly the general solution is given by

$$x(t) = \begin{cases} e^{-\varepsilon\omega_n t}[A_3\cos(\omega_d t) + B_3\sin(\omega_d t)] & ,if\ t < t_0 \\ e^{-\varepsilon\omega_n t}[A_3\cos(\omega_d t) + B_3\sin(\omega_d t)] + Cf(t - t_0), & if\ t \geq t_0. \end{cases} \quad (49)$$

Case IV $[\boldsymbol{\varepsilon = 0: undamped}]$

In this case there is no damping and the characteristic equation (6) has no real roots, but complex roots instead like for the under-damped case. The denominator of equation (26) is irreducible. However, it is already in its simplest form and equation (26) becomes

$$X(s) = \overbrace{\frac{x_0 s + x_0'}{s^2 + \omega_n^2}}^{Free\ Response} + \overbrace{\frac{Ce^{-st_0}}{s^2 + \omega_n^2}}^{Forced\ Response} . \quad (50)$$

Solving for $X(s)$ we obtain

$$X(s) = \overbrace{A_4 \left\{\frac{s}{s^2 + \omega_n^2}\right\} + B_4 \left\{\frac{\omega_n}{s^2 + \omega_n^2}\right\}}^{Free\ Response} + \overbrace{Ce^{-st_0} F(s)}^{Forced\ Response} \quad (51)$$

where A_4 and B_4 are constants given by $A_4 = x_0$ and $B_4 = \frac{x_0'}{\omega_n}$ with

$$F(s) = \frac{1}{s^2 + \omega_n^2} . \quad (52)$$

Taking the inverse Laplace transform of equation (51) we obtain

$$x(t) = \overbrace{A_4 \cos(\omega_n t) + B_4 \sin(\omega_n t)}^{x_c(t)} + \overbrace{Cu(t - t_0) f(t - t_0)}^{x_p(t)} \quad (53)$$

where

$$f(t) = \pounds^{-1}\left\{\frac{1}{s^2 + \omega_n^2}\right\} = \frac{1}{\omega_n} \sin(\omega_n t). \quad (54)$$

More explicitly the general solution is

$$x(t) = \begin{cases} A_4 \cos(\omega_n t) + B_4 \sin(\omega_n t) & , if\ t < t_0 \\ A_4 \cos(\omega_n t) + B_4 \sin(\omega_n t) + Cf(t - t_0), & if\ t \geq t_0. \end{cases} \quad (55)$$

5Numerical Experiments

Suppose we have a mass spring system having a mass of 2 kg attached to a spring with a spring constant of 10 N/m. The mass is released from rest at 3 m below the equilibrium position at the time $t = 12$ sec an external force with a total impulse of 40 N is applied to the system. In this paper we determine the displacement $x(t)$ of the mass for the four cases: over-damped, critically damped, under-damped and un-damped.

The parameter values for the system are: mass,$m = 2\ kg$, spring constant, $k = 10\ N/m$, natural frequency, $\omega_n = \sqrt{5}\ rads/sec$, time when impulse force is applied, $t_0 = 12\ sec$, external force, $F(t) = 40\partial(t - 12)N$, the constant, $C = 20\ N$ and initial the conditions are $x_0 = 3\ m$, $x_0' = 0\ m/s$. The differential equation for the system (4) becomes

$$x''(t) + 2\sqrt{5}\,\varepsilon\, x'(t) + 5x(t) = 40\delta(t-12) \text{ with } x_0 = 3, x_0' = 0. \quad (56)$$

The graph of the external force $F(t) = 40\partial(t-12)$ is given in Figure 1. The function is $40\ N$ at the instant time $t = 12$ seconds and zero elsewhere.

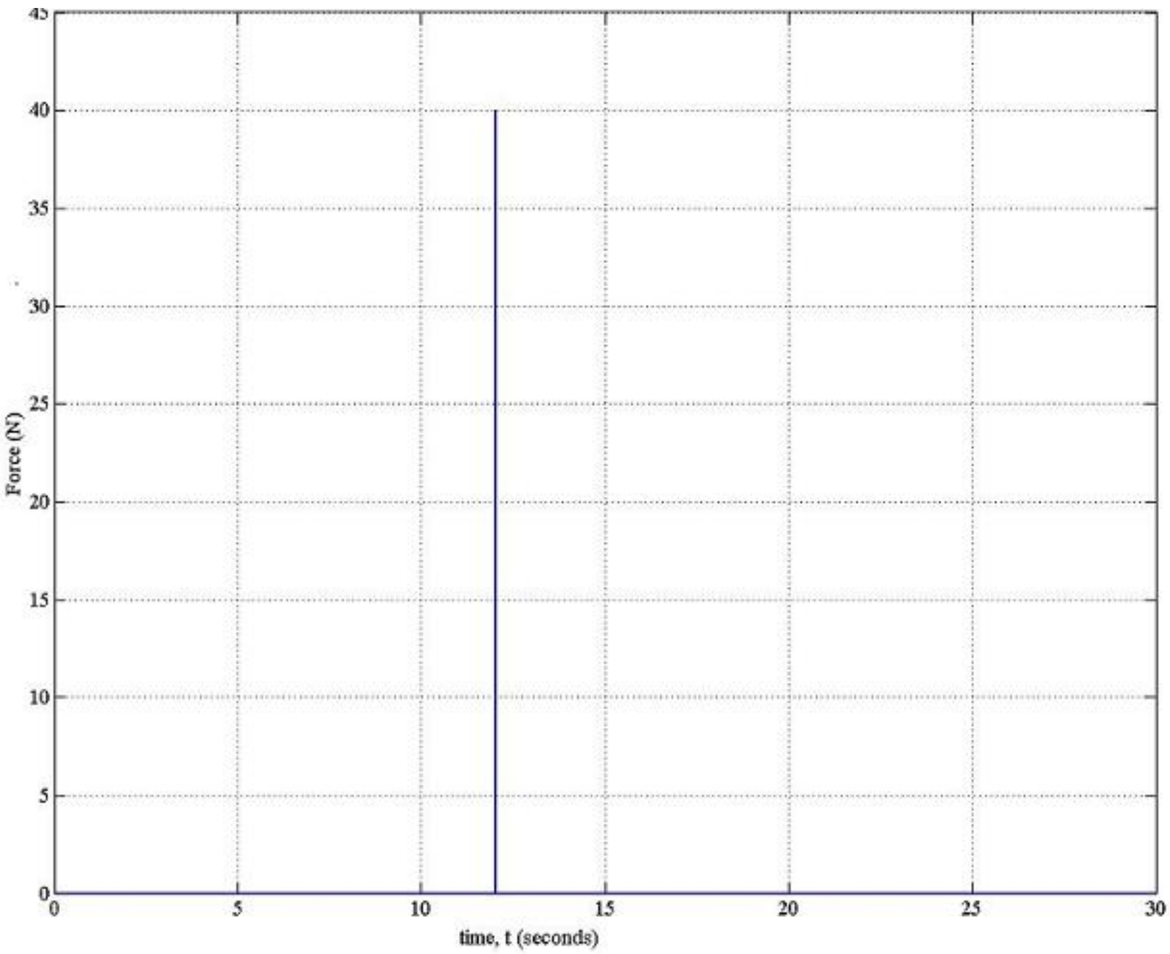

Figure 1: The Dirac delta function F(t) = 40δ(t -12)

6 Results and Analysis

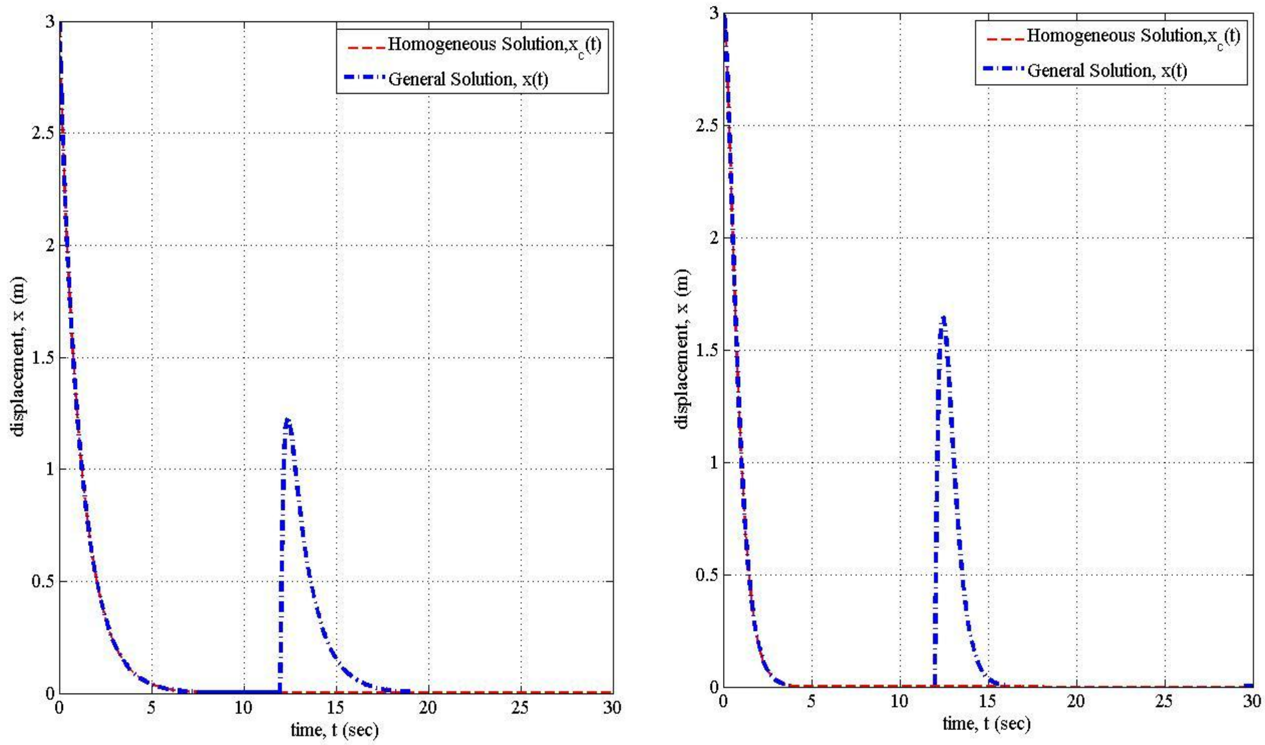

Figure 2: The homogeneous and the general solutions for the overdamped and critically damped cases

Results in Figure 2 show that in both the over-damped [part (a)] and critically damped [part (b)] cases, the mass eventually returns to the equilibrium position. The application of the impulse force at time $t = 12$ seconds displaces the mass a few meters below the equilibrium position. The displacement of the mass in the critically damped case is more than that for the over-damped case. The mass returns to the equilibrium position faster for the critically damped case as compared to the over-damped case.

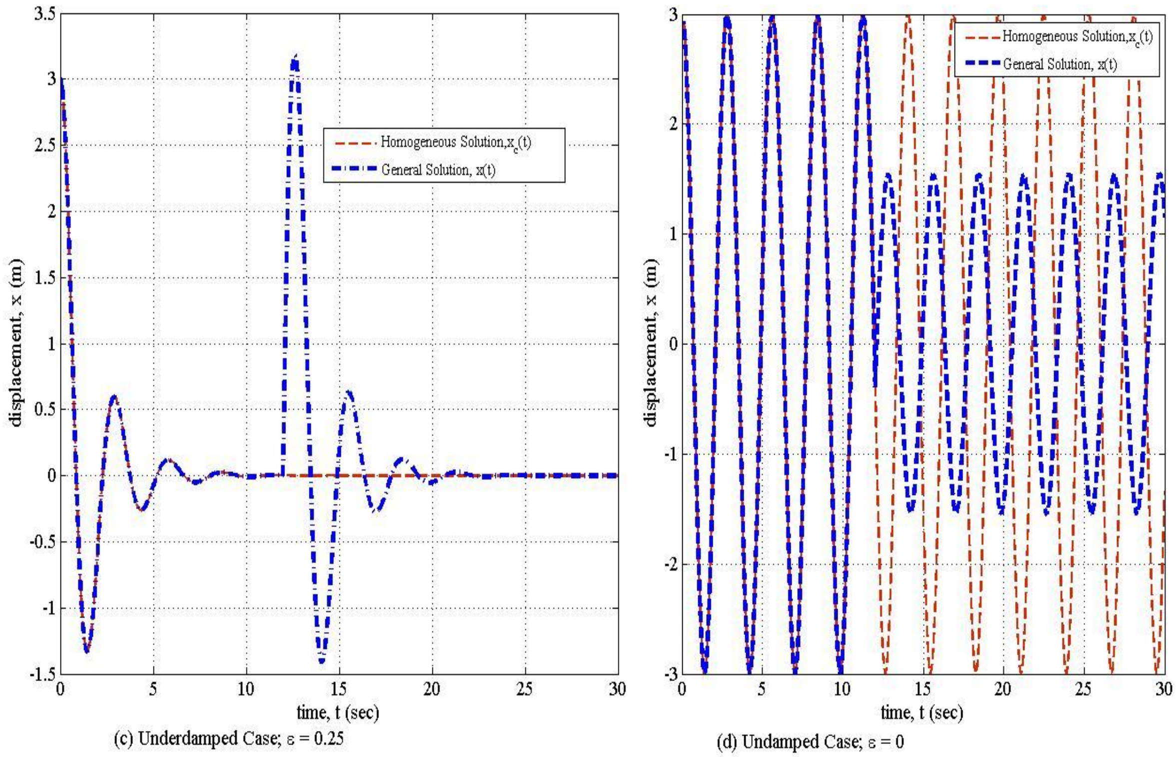

Figure 3: The homogeneous and the general solution for the underdamped and undamped cases

The results in Figure 3 show that for the under-damped case [part (c)], the mass oscillates and later returns to the equilibrium position. The application of the impulse force again displaces the mass below the equilibrium position as seen in Figure 2. In the un-damped case [part (d)], the mass oscillate continuously with the application of the impulse force leading to oscillations with a reduced amplitude as compared to the homogeneous solution.

7 CONCLUSIONS

In this paper, we have shown how the Laplace transform method easily solves the mass spring system. The nature of the roots of the characteristic equation of the differential equation leads to the four cases [over-damped, critically damped, under-damped and un-damped] and also plays a major role in solving the algebraic equation resulting after taking the Laplace transform of the differential equation. The application of the impulse force displaces the mass below the equilibrium and there are no oscillations for the over-damped and critically damped cases. On the other hand oscillations were obtained for the under-damped and un-damped case with continuous oscillations for the latter. Results obtained are consistent with the general theory of solving the mass spring system.

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