OBSERVATIONS ON THE HOMOGENEOUS CONE

$$z^2 = 53x^2 + y^2$$

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Abstract:

The ternary quadratic homogeneous equation representing homogeneous cone given by $z^2 = 53x^2 + y^2$ is analyzed for its non-zero distinct integer points on it . Three different patterns of integer points satisfying the cone under consideration are obtained. A few interesting relations between the solutions and special number patterns namely Polygonal number , Pyramidal number , Octahedral number, Pronic number Decagonal and Nasty number are presented. Also knowing an integer solution satisfying the given cone , three triples of integers generated from the given solution are exhibited.

Keywords: Ternary homogeneous quadratic, integral solutions.

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1. INTRODUCTION

The ternary quadratic Diophantine equations offer an unlimited field for research due to their variety [1,21]. For an extensive review of various problems, one may refer [2-20]. This communication concerns with yet another interesting ternary quadratic equation $z^2 = 53x^2 + y^2$ representing a cone for determining its infinitely many non-zero integral points. Also, a few interesting relations among the solutions are presented.

Notations:

P_n^m - Pyramidal number of rank n with size m.

$T_{m,n}$ - Polygonal number of rank n with size m.

$\Pr_n$ - Pronic number of rank n

OH_n - Octahedral number of rank n

$T_{10,n}$ -Decagonal number of rank n

2. METHOD OF ANALYSIS

The ternary quadratic equation under consideration is

$$z^2 = 53x^2 + y^2 \qquad (1)$$

To start with it is seen that the triples $(k, 26k, 27k), (2k+1, 2k^2+2k-26, 2k^2+2k+27)$ satisfy (1).

However, we have other choices of solutions to (1) which are illustrated below:

consider (1) as

$$53x^2 + y^2 = z^2 * 1 \qquad (2)$$

$$\text{Assume } z = a^2 + 53b^2 \qquad (3)$$

Write 1 as

$$1 = \frac{[(26+2n-2n^2)+i\sqrt{53}(2n-1)][(26+2n-2n^2)-i\sqrt{53}(2n-1)]}{(27-2n+2n^2)} \qquad (4)$$

Substituting (3) and (4) in (2) and employing the method of factorization define

$$y + i\sqrt{53}x = \frac{[(26+2n-2n^2)+i(2n-1)\sqrt{53}](a+i\sqrt{53}b)^2}{(27-2n+2n^2)}$$

Equating the real and imaginary parts in the above equation, we get

$$x = \frac{[2(26+2n-2n^2)ab+(a^2-53b^2)(2n-1)]}{(27-2n+2n^2)}$$

$$y = \frac{[(26+2n-2n^2)(a^2-53b^2)-106ab(2n-1)]}{(27-2n+2n^2)}$$

Replacing a by $(27-2n+2n^2)A$, b by $(27-2n+2n^2)B$ in the above equation corresponding integer solutions to (1) are given by

$$x = (27-2n+2n^2)[(A^2-53B^2)(2n-1)+\{2AB(26+2n-2n^2)\}]$$

$$y = (27-2n+2n^2)[(A^2-53B^2)(26+2n-2n^2)-\{106AB(2n-1)\}]$$

$$z = (27-2n+2n^2)^2(A^2+53B^2)$$

For simplicity and clear understanding, taking n=1 in the above equations, the corresponding integer solutions of (1) are given by

$$x = 27A^2 - 1431B^2 + 1404AB$$

$$y = 702A^2 - 37206B^2 - 2862AB$$

$$z = 27^2(A^2 + 53B^2)$$

Properties:

$$x(A,1) - t_{(56,A)} \equiv -1 \pmod{1430}$$

$$x(1,B) + t_{(2864,B)} \equiv 1 \pmod{26}$$

$$z(A,1) - t_{(1460,A)} \equiv 53 \pmod{728}$$

$$x(A+1, A^2) - t_{(56,A)} + 1431t_{(4,A^2)} - 2808p_A^5 \equiv 27 \pmod{80}$$

$$x(A, A+1) + t_{(2918,A)} - 1404pr_A \equiv -1431 \pmod{4319}$$

$$y(A,1) - t_{(1406,A)} \equiv -469 \pmod{2161}$$

$$x(A, 4A-3) + t_{(45740,A)} - 1404t_{(10,A)} \equiv -1403 \pmod{11476}$$

$$x(A, 2A^2+1) + 5724t_{(4,A^2)} + t_{(1136,A)} - 4212OH_A \equiv -1431 \pmod{5696}$$

Each of the following represents a nasty number

$$z(A,A) = 6(81A)^2$$

$$2(z(A,A) - y(A,A)) = 6(162A)^2$$

It is worth to note that 1 in (2) may also be represented as

$$1 = \frac{[(53-4n^2) + i(4n)\sqrt{53}][(53-4n^2) - i\sqrt{53}(4n)]}{(53+4n^2)^2}$$

Following the analysis presented above, the corresponding integer solutions to (1) are found to be

$$x = (53+4n^2)[(A^2-53B^2)(4n)+\{2AB(53-4n^2)\}]$$

$$y = (53+4n^2)[(A^2-53B^2)(53-4n^2)-\{424nAB\}]$$

$$z = (53+4n^2)^2(A^2+53B^2)$$

For the sake of simplicity, taking n=1 in the above equations, the corresponding integer solution of (1) are given by

$$x = 288A^2-12084B^2+5586AB$$

$$y = 2793A^2-148029B^2-24168AB$$

$$z = 57^2(A^2+53B^2)$$

Properties:

$$x(A,1)-t_{(458,A)} \equiv -458(\mathrm{mod}\,5813)$$

$$x(-1,A)+t_{(24170,A)} \equiv 228(\mathrm{mod}\,6497)$$

$$y(A^2,A+1)-t_{(296060,A)}-2793t_{(4,A^A)}+48336p_A^5 \equiv -148029(\mathrm{mod}\,444086)$$

$$y(A+1,A)+t_{(290474,A)}+24168pr_A \equiv 2793(\mathrm{mod}\,139649)$$

$$y(A,1)-t_{(5588,A)} \equiv -12573(\mathrm{mod}\,21376)$$

$$z(A,1)-t_{(6500,A)} \equiv 53(\mathrm{mod}\,3248)$$

$$[x(-1,A)+z(-1,A)]+t_{(24170,A)} \equiv 327(\mathrm{mod}\,450)$$

3.Generation of integer solutions

Let (x_0, y_0, z_0) be any given integer solution of (1)

Then, each of the following triples of integers satisfies (1):

Triple 1 : (x_{n1}, y_{n1}, z_{n1})

$$x_{n1} = 53^n x_0$$

$$y_{n1} = \frac{1}{6}[\{8(3)^n - 2(-3)^n\}y_0 + \{-4(3)^n + 4(-3)^n\}z_0]$$

$$z_{n1} = \frac{1}{6}[\{4(3)^n - 4(-3)^n\}y_0 + \{-2(3)^n + 8(-3)^n\}z_0]$$

Triple 2 : (x_{n2}, y_{n2}, z_{n2})

$$x_{n2} = \frac{1}{52}[\{-(26)^n + 53(-26)^n\}x_0 + \{(26)^n - (-26)^n\}z_0]$$

$$y_{n2} = 26^n y_0$$

$$z_{n2} = \frac{1}{52}[\{-53(26)^n + 53(-26)^n\}x_0 + \{53(26)^n\}z_0]$$

Triple 3: (x_{n3}, y_{n3}, z_{n3})

$$x_{n3} = \frac{1}{54}[\{(27)^n + 53(-27)^n\}x_0 + \{-(27)^n + (-27)^n\}y_0]$$

$$y_{n3} = \frac{1}{54}[\{-53(27)^n + 53(-27)^n\}x_0 + \{53(27)^n + (-27)^n\}y_0]$$

$$z_{n3} = 27^n z_0$$

4.CONCLUSION

In this paper, we have presented two different patterns of infinitely many non-zero distinct integer solutions of the homogeneous cone given by $53x^2 + y^2 = z^2$.To conclude, one may search for other patterns of solution and their corresponding properties.

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