STRONG SPLIT BLOCK DOMINATION IN GRAPHS

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Abstract:

For any graph $G = (V, E)$, the block graph $B(G)$ is a graph whose set of vertices is the union of the set of blocks of G in which two vertices are adjacent if and only if the corresponding blocks of G are adjacent. A dominating set D of a graph $B(G)$ is a strong split block dominating set if the induced sub graph $\langle V[B(G)] - D \rangle$ is totally disconnected with at least two vertices. The strong split block domination number $\gamma_{ssb}(G)$ of G is the minimum cardinality of strong split block dominating set of G. In this paper, we study graph theoretic properties of $\gamma_{ssb}(G)$ and many bounds were obtain in terms of elements of G and its relationship with other domination parameters were found.

Keywords: Dominating set/ independent domination/Block graph /strong split block domination.

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1. Introduction: In this paper, all the graphs consider here are simple and finite. For any undefined terms or notations can be found in Harary [2]. In general, we use $< X >$ to denote the subgraph induced by the set of vertices X and $N(v)(N[v])$ denote open (closed) neighborhoods of a vertex v .

The notation $\alpha_o(G)(\alpha_1(G))$ is the minimum number of vertices (edges) in a vertex (edge) cover of G. The notation $\beta_o(G)(\beta_1(G))$ is the maximum cardinality of a vertex (edge) independent set in G. Let $deg(v)$ is the degree of vertex v and as usual $\delta(G)(\Delta(G))$ is the minimum (maximum) degree. A block graph $B(G)$ is the graph whose vertices corresponds to the blocks of G and two vertices in $B(G)$ are adjacent if and only if the corresponding blocks in G are adjacent.

We begin by recalling some standard definitions from domination theory. A dominating set D of a graph $G = (V, E)$ is an independent dominating set if the induced subgraph $< D >$ has no edges. The independent domination number $i(G)$ of a graph G is the minimum cardinality of an independent dominating set.

The concept of Roman domination function (RDF) was introduced by E.J. Cockayne, P.A.Dreyer, S.M.Hedetiniemi and S.T.Hedetiniemi in [1]. A Roman dominating function on a graph $G = (V, E)$ is a function $f: V \rightarrow \{0,1,2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex of v for which $f(v) = 2$. The weight of a Roman dominating function is the value $f(V) = \sum_{v\epsilon V} f(v)$. The Roman domination number of a graph G , denoted by $\gamma_R(G)$,equals the minimum weight of a Roman dominating function on G. A dominating set D of a graph $B(G)$ is a strong split block dominating set if the induced subgraph $\langle V[B(G)] - D\rangle$ is totally disconnected. The strong split block domination number $\gamma_{ssb}(G)$ of G is the minimum cardinality of strong split block dominating set of G. In this paper, many bounds on $\gamma_{ssb}(G)$ were obtained in terms of elements of G but not the elements of $B(G)$. Also its relation with other domination parameters were established.

We need the following theorems for our further results.

Theorem A [3]: For any graph G, $\gamma(G) \geq \left\lceil \frac{p}{1+\Delta(G)} \right\rceil$.

2. Results:

Theorem 1: For any (p,q)graph G with n-blocks and $B(G) \neq K_P$, then

$$\gamma_{ssb}(G) + \gamma(G) \leq n(G) .$$

Proof: Suppose $B = \{b_1, b_2, b_3, \dots \dots \dots, b_n\}$ is the set of blocks in G. Then $\{B\} = V[B(G)]$. Let $A = \{b_1, b_2, b_3, \dots \dots \dots, b_i\}$,$1 \leq i \leq n$ such that $A \subseteq B$ and $\forall b_i \epsilon A$ are the non- end blocks in G which gives cut vertices in $B(G)$. Also $C = \{b_1, b_2, b_3, \dots \dots \dots, b_j\}$,$1 \leq j \leq n$ be the set of end blocks in G and $C \subseteq B$. Let $\{v_1, v_2, v_3, \dots \dots \dots, v_p\}$ be the set of vertices of G and D= $\{v_1, v_2, v_3, \dots \dots \dots, v_m\}$ where $m \leq p$ be a dominating set of G such that $\gamma(G) = |D|$. Now we consider $A^1 \subset A$ and $C^1 \subset C$. Then $V[B(G)] - \{A^1\} \cup \{C^1\} = \{K\}\ \forall v \epsilon K$ is an isolates. Hence $|A^1| \cup |C^1| = \gamma_{ssb}(G)$. Since $\{A\} \cup \{C\} = V[B(G)]$. Clearly $|A^1| \cup |C^1| + |D| \leq |A| \cup |C|$ which gives $\gamma_{ssb}(G) + \gamma(G) \leq n(G)$.

Theorem 2: For any (p,q)graph G and $B(G) \neq K_P$, then $\gamma_{ssb}(G) \leq \beta_o(G) - 1$. Where $\beta_o(G)$ is the maximal vertex independence number of G .

Proof: Suppose $B = \{B_1, B_2, B_3, \dots \dots \dots, B_n\}$ be the set of blocks in G and let $H = \{b_1, b_2, b_3, \dots \dots \dots, b_n\}$ be the set of vertices which corresponds to the blocks of B such that $V[B(G)] = |H|$.

Now we consider the following cases.

Case 1: Suppose G is a tree with at least 3-blocks. For at most two blocks $B(G)$ is complete hence $\gamma_{ssb}(G)$ set does not exists. For this we consider a tree with at least 3-blocks. Suppose $V = \{v_1, v_2, v_3, \dots \dots \dots, v_p\}$ be the set of vertices of G and $D = \{v_1, v_2, v_3, \dots \dots \dots, v_i\}$ for $i \leq p$ be the maximal independence set of vertices of G , such that $|D| = \beta_o(G)$.

Let $C = \{b_1, b_2, b_3, \dots \dots \dots, b_i\}$ be the set of cut vertices in $B(G)$. Since each block in $B(G)$ is complete and each cut vertex is incident with at least two blocks. Let $C^1 = V[B(G)] - C$ and consider a set ${C_1}^1 \subseteq C^1$ such that $V[B(G)] - \{C^1 \cup {C_1}^1\} = S$ where $\forall b_i \epsilon S$ is an isolates. Hence $|C^1 \cup {C_1}^1| = \gamma_{ssb}(G)$. Also $|C^1 \cup {C_1}^1| < |D| - 1$ which gives $\gamma_{ssb}(G) \leq \beta_o(G) - 1$.

Case 2: Suppose G is a not tree then there exists at least a block which is not an edge. Let B_i be the number of blocks which are not edges $V[B_i] \subset V[B(G)]$. Let $\{b_1, b_2, b_3, \dots \dots \dots, b_n\}$ be the set of vertices of $B(G)$ corresponding to the blocks $\{B_1, B_2, B_3, \dots \dots \dots, B_n\}$ respectively in G. Suppose D^1 is a dominating set of $B(G)$ such that $|D^1| = \gamma_{ssb}(G)$. Since $|D^1| \leq |D|, then\ \gamma_{ssb}(G) \leq \beta_o(G) - 1$.

Theorem 3: For any non-trivial tree T and $B(T) \neq K_P$, then $\gamma_{ssb}(T) \geq \gamma(T)$. Equality holds for a path P_p *with* P≥ 5.

Proof: Suppose $V = \{v_1, v_2, v_3, \dots \dots \dots, v_p\}$ be the set of vertices of T. Let D= $\{v_1, v_2, v_3, \dots \dots \dots, v_k\}$ $1 \leq k \leq p$ be a minimal dominating set of T such that $|D| = \gamma(T)$. Further $B = \{B_1, B_2, B_3, \dots \dots \dots, B_n\}$ be the number of blocks in T. In $B(T), V[B(T)] = \{b_1, b_2, b_3, \dots \dots \dots, b_n\}$ be the set of vertices corresponding to the blocks $\{B_1, B_2, B_3, \dots \dots \dots, B_n\}$ of T. In $B(T)$ each blocks is complete. Let $\{{B_1}^1, {B_2}^1, {B_3}^1, \dots \dots \dots, {B_p}^1\}$ be the set of blocks in $B(T)$ with the property such that $\forall B_i, 1 \leq i \leq p$ has at least two vertices. From each block in $B(T), p-1$ numbers of vertices forms a dominating set D^1 such that $|D^1| = \gamma_{ssb}(T)$. Hence $|D| \leq |D^1|$, which gives $\gamma_{ssb}(T) \geq \gamma(T)$. For equality, suppose $T = P_p$ with $P \leq 4$. If $B(T) = k_{1,2,}$ which gives $\gamma_{ssb}(T) \ngeq \gamma(T)$ for $P = 2,3$, $\gamma_{ssb}(T)$ does not exists. Hence we consider $T = P_p$ with $P \geq 5$. Suppose $T = P_p$ with $P \geq 5$.

Let $T = P_p: \{v_1, v_2, v_3, \dots \dots \dots, v_p\}$ be a path with $P \geq 5$ then we consider a set D $= \{v_2, v_5, v_8, \dots \dots \dots, v_{p-n}\}$ such that $N(v_{p-n}) \cap N(v_{p-n-1}) = \emptyset$. Hence D be a $\gamma - set$ of P_p. In $B(P_p), V[B(P_p)] = P - 1$, then we consider a set $K \subset V[B(P_p)]$ such that $V[B(P_p)] - K = M$ where each element in M is an isolate. Clearly $|M| = |D|$ which gives $\gamma_{ssb}(P_p) = \gamma(P_p)$.

We have the following proposition.

Proposition 1: If $B(G)$ is a star, then $\gamma_{ssb}(G) = 1$.

Theorem 4: For any connected (p, q)graph G and $B(G) \neq K_P$, then $\gamma_{ssb}(G) + \gamma(G) \leq P - 1$.

Proof: Let G be a connected graph with P −vertices and n- blocks. Let $\{b_1, b_2, b_3, \ldots\ldots\ldots, b_n\}$ be the number of vertices in $B(G)$ corresponding to the blocks $\{B_1, B_2, B_3, \ldots\ldots\ldots, B_n\}$ in G. Let $H = \{v_1, v_2, v_3, \ldots\ldots\ldots, v_n\}$ be the set vertices in G, $J = \{v_1, v_2, v_3, \ldots\ldots\ldots, v_i\}$ where $1 \leq i \leq n$ such that $J \subset H, v_i \epsilon J$ which covers all the vertices of G and there does not exists any proper sub set $J^1$of J such that $v_k \epsilon H - J^1$ for which $N(v_k) \cap J^1 \neq u$ where $u \epsilon J^1$. Hence J is a minimal dominating set of G and $|J| = \gamma(G)$.

Let $S_1 = \{B_i\}$ where $1 \leq i \leq n, S_1 \subset S$ and $\forall B_i \epsilon S_1$ are non end blocks in G. The we have $M_1 \subset M$ which corresponding to the elements of S_1 such that M_1 forms a minimal dominating set of $B(G)$. Since each element of $H - M_1$ is an isolates then $|M_1| = \gamma_{ssb}(G)$. Further $M_1 \cup J \leq P - 1$, which gives $\gamma_{ssb}(G) + \gamma(G) \leq P - 1$.

Theorem 5: For any non-trivial tree $T$and $B(T) \neq K_P$,then $\gamma_{ssb}(T) \leq 2\alpha_o(T) - 1$. Where $\alpha_o(T)$ is the vertex covering number of G.

Proof: Suppose $B(T) = K_P$. Then $\gamma_{ssb} - set$ does not exists. We consider a non-trivial tree T with $V(T) = \{v_1, v_2, v_3, \ldots\ldots\ldots, v_p\}$. Let $V_1 = \{v_1, v_2, v_3, \ldots\ldots\ldots, v_i\}, 1 \leq i \leq p$ be the set of cut vertices which are adjacent to end vertices and $V_2 = \{v_1, v_2, v_3, \ldots\ldots\ldots, v_l\}, 1 \leq l \leq p$ be the set of cut vertices such that $\forall v_k \epsilon N(v_l)$ are non-end vertices $1 \leq l \leq p$. Suppose a set $v_j \subseteq V_1\ or\ V_2$. Then we consider another subset $V_2{}^1 = \{v_1, v_2, v_3, \ldots\ldots\ldots, v_n\}, 1 \leq n \leq l$ which are at a odd distance from the vertices of T with $\deg(v_p) \geq 3$. Then every vertex belongs to $V_1 \cup V_2 \cup V_j \cup V_2{}^1$ which covers all the edges of T. Hence $|V_1| \cup |V_2| \cup |V_j| \cup |V_2{}^1| = \alpha_o(T)$. In $B(T)$, each block is complete. And to get $\gamma_{ssb}(T)$. We consider the following cases.

Case 1: Suppose each block of $B(T)$ is an edge. Then $H = \{v_1, v_2, v_3, \ldots\ldots\ldots, v_n\} \subset V[B(T)]$ are in alternate sequence such that $\forall v \epsilon V[B(T)] - H$ is an isolates. Hence H is a $\gamma_{ssb} - set$ and $\alpha_o - set$. Clearly $|H| = \gamma_{ssb}(T) = \alpha_o(T)$. Which gives the equality of the result.

Case 2: Suppose there exist at least one block of $B(T)$ which is not an edge. Now assume each block of $B(T)$ is a complete graph with $P \geq 2$ vertices. Let H be a $\gamma_{ssb} - set$ of $B(T)$ which contains $P - 1$ vertices from each block of $B(T)$. Since $B(T)$ has n number of blocks, then

$n[P-1]\epsilon V[B(T)\,] = H$.Hence $2\{|V_1| \cup |V_2| \cup |V_j| \cup |{V_2}^1|\} - 1 = |H|$, which gives $\gamma_{ssb}(T) \le 2\alpha_o(T) - 1$.

Theorem 6 : For any (p,q) graph G with $n - blocks\ and\ B(G) \neq K_p,\ then$

$$\gamma_{ssb}(G) \le n + \gamma(G) - 4.$$

Proof: Suppose $S = \{B_1, B_2, B_3, \dots \dots \dots, B_n\}$ be the blocks of G. Then $M = \{b_1, b_2, b_3 \dots \dots \dots, b_n\}$ be the corresponding block vertices in $B(G)$ with respect to the set S Let $H = \{v_1, v_2, v_3, \dots \dots \dots, v_n\}$ be the set of vertices in G, such that $V(G) = H$. If $J = \{v_1, v_2, v_3, \dots \dots \dots, v_m\}$ where $1 \le m \le n$ and $J \subset H$ such that $N(J) = V(G) - J$ gives a minimal domination set in$G$.Hence $|J| = \gamma(G)$.

Suppose $M^1 = \{b_1, b_2, b_3, \dots \dots \dots, b_j\}\ where\ 1 \le j \le n$ such that $M^1 \subset M\ then$ $\forall\, b_i \in M^1$ are cut vertices in $B(G\,)$. Further $M^{11} \subset M$ be a set of vertices in $B(G)$ such that $V[B(G)\,] - \{M^1 \cup M^{11}\} = N$ where $\forall\, v_i \in N$ is an isolates. Hence $|N| = \gamma_{ssb}(G)$. In $B(G\,)$ each block is complete with $P \ge 2$ vertices. Then $|N| \le n + |J| - 4$ which gives $\gamma_{ssb}(G) \le n + \gamma(G) - 4$.

Theorem 7: For any non-trivial tree T and $B(T) \neq K_P$, then $\gamma_{ssb}(T) \le \gamma_R(T)$.

Proof: Let $f = (V_0, V_1, V_2)$ be any γ_R –function of T. Then V_2 is a $\gamma - set$ of $H = G[V_0 \cup V_2]$ such that $|H| = \gamma_R(T)$.

Next we consider $\{b_1, b_2, b_3, \dots \dots \dots, b_n\}$ be the set of vertices of $B(T)$ corresponding to the blocks $\{B_1, B_2, B_3, \dots \dots \dots, B_n\}$ of T. Let $D^1 = \{b_1, b_2, b_3, \dots \dots \dots, b_m\}$ where $m < n$ is a minimal dominating set of $B(T)$ such that $V[B(T)\,] - D^1 = N, \forall v_i \epsilon N$ is a isolates, then $|\,D^1| = \gamma_{ssb}(T)$. Hence $\gamma_{ssb}(T) = |\,D^1| \le |H| = \gamma_R(T)$ which gives $\gamma_{ssb}(T) \le \gamma_R(T)$.

Theorem 8: For any non-trivial tree $T \neq P_4$ and $B(T) \neq K_P$, then $\gamma_{ssb}(T) \ge \gamma(T)$. Equality holds if $T = P_n$ with $n = 1,2, \dots \dots \dots, 7$ and H where H is $k_{1,3}$ together with an end edge adjoined at most three end vertices.

Proof: Suppose $T = P_4$. Then $B(T) = K_{1,2,}\, \gamma_{ssb} = 1 \neq \gamma(T)$. Now we consider a tree $T \neq P_4$. Let $V = \{v_1, v_2, v_3, \dots \dots \dots, v_n\}$ be the set of vertices in T. Let

$V_1 = \{v_1, v_2, v_3, \dots \dots \dots, v_k\}$, $1 \le k \le n$ be the set of cut vertices of G $V_2 = \{v_1, v_2, v_3, \dots \dots \dots, v_j\}, 1 \le j \le k$ such that $V_2 \subset V_1$ and $|V_2| = \gamma(T)$. Let $E = \{e_1, e_2, e_3, \dots \dots \dots, e_i\}$ be the set of non-end edges in T. In $B(T)$, $V[B(T)] = E[T]$. Since each edge is a block, then $\{e_1, e_2, e_3, \dots \dots \dots, e_i\} \in V[B(T)]$ are cut vertices and each e_i lie an exactly two blocks of $B(T)$. Now $V[B(T)] - E, E \in V[B(T)]$ gives e_i number of components and each component is again a complete graph. Since one less than P number of vertices from each block of $B(T)$ are removed, then we get a null graph. Hence E_1 and E_2 represent the cut vertex set and other vertices of P_i components. Hence $E_1 \cup E_2$ is a $\gamma_{ssb} - set$ which gives $|E_1 \cup E_2| = \gamma_{ssb}(T)$. Suppose $\Delta(T) \ge 2$. Then $|E_1 \cup E_2| \ge |V_2|$ which gives $\gamma_{ssb}(T) \ge \gamma(T)$.

Theorem 9: For any (p, q) graph $G, and\ B(G) \ne K_p, then\ \gamma_{ssb}(G) \le \left\lceil \frac{P.\Delta(G)}{2+\Delta(G)} \right\rceil$.

Proof: We consider only those graphs which are not $B(G) = K_p$. let D be a $\gamma_{ss} - set\ of\ B(G)$ it follows that for each vertex $v \in D$ there exist a vertex $u \in V[B(G)] - D$. such that v is adjacent to u. Since each block in $B(G)$ is a complete, this implies that $V[B(G)] - D$ is a dominating set of $B(G)$ such that $\forall\, v_i \epsilon V[B(G)] - D$ is an isolates. By Theorem A, we have $\gamma_{ssb}(G) \le \left\lceil \frac{P.\Delta(G)}{2+\Delta(G)} \right\rceil$.

Theorem 10: For any (p, q) non-trivial tree T and $B(T) \ne K_P$, then $\gamma_{ssb}(T) \le \left\lceil \frac{q+M(T)}{2} \right\rceil - 1$. where $M(T)$ is the number of end vertices in T .

Proof: We consider a tree $T \ne K_{1,n}$ $n \ge 1$. Let T be a tree with $q \ge 3$ edges, $E(T) = \{e_1, e_2, e_3, \dots \dots \dots, e_n\}$. Now $H = \{v_1, v_2, v_3, \dots \dots \dots, v_n\}$ be the set of vertices which corresponds to the set of edges in $E(T)$. Let $D = \{v_1, v_2, v_3, \dots \dots \dots, v_i\}$,$1 \le i \le n$,and $\forall v_j \epsilon \{H - D\}$ is adjacent to at least one $v_i \epsilon D$. Since each edge is a block in T,then D is a dominating set of $B(T)$. Suppose $B(T)$ is a path with even number of vertices. Then $v \epsilon D$ is an end vertex in $B(T)$. Suppose $B(T)$ is a path with odd number of vertices. Then $v \notin D$. Hence $|D| \le \left\lceil \frac{q+M(T)}{2} \right\rceil - 1$.

Suppose T is not a path. Then there exists at least one vertex v with $degv \ge 3$. Let $L = \{v_1, v_2, v_3, \dots \dots \dots, v_k\}$ $\forall v_k$, deg $[v_k] \ge 3$ and $E_1 = \{e_1, e_2, e_3, \dots \dots \dots, e_k\}$ be the edges in

T incident with $v_k \in L$. Suppose $E_2 \subset E_1$ which are non end edges in T. Then $|E_2| \cup |E_1| = M$. Suppose $E_3 = \{e_1, e_2, e_3, \dots\dots\dots, e_j\}$, $1 \le j \le k$, $E_3 \subset \{E_2 \cup E_1\}$ $\forall v_i \in E_3$ is an element of E_2 or E_1 and hence $\{E_3\} \subset V[B(T)]$ and $\{E_1 \cup E_2 \cup E_3\} = D$ and $|E_1 \cup E_2 \cup E_3| = q + M$ thus $\{E_1 \cup E_2 \cup E_3\} \le \frac{\{E_1 \cup E_2\} + \{E(T)\}}{2} - 1$ hence $\gamma_{ssb}(T) \le \left\lceil \frac{q+M(T)}{2} \right\rceil - 1$.

Theorem 11: For any (p, q) non-trivial tree T and $B(T) \ne K_P$, then

$$\gamma_{ssb}(T) + 2\gamma_c(T) \ge \gamma_t(T) + \Delta(T).$$

Proof: Suppose G is a tree , $F = \{v_1, v_2, v_3, \dots\dots\dots, v_n\}$ be the set of all end vertices of G and $V^1 = V - F$. Then $D^1 \subseteq V^1$ is a minimal connected dominating set of G. Further if $\{v_j\} \in N(D^1)$ and $\{v_j\} \subseteq V^1$, then $D^1 \cup v_j$ forms a minimal total dominating set of G. If $\{v_j\} = \emptyset$, then there exists at least one vertex $v \in F$ such that $D^1 \cup \{v\}$ forms a total dominating set of G. Let $D = \{u_1, u_2, u_3, \dots\dots\dots, u_k\}$ be the dominating set of $B(T)$. If the neighbors of each u_i $1 \le i \le k$ are at a distance at least two which generates D to be a minimal dominating set of $B(T)$ such that $V[B(T)] - D = X$ where $\forall v_i \epsilon X$ is an isolates. Hence $|D|$ a $\gamma_{ssb} - set$ of T . Suppose $V^1 = \{v_1, v_2, v_3, \dots\dots\dots, v_k\} \subseteq V(G)$ such that $\deg(v_i) \ge 2, 1 \le i \le k$. Then there exists at least one vertex $v \in V^1$ such that $\deg(v) = \Delta(T)$. Now we have $|D| + 2|D^1| \ge |D^1 \cup \{v_j\}| + \Delta(T)$ which gives $\gamma_{ssb}(T) + 2\gamma_c(T) \ge \gamma_t(T) + \Delta(T)$.

Theorem 12: For any tree T and $B(T) \ne K_P$, then $i(T) \ge \gamma_{ssb}(T)$ where $i(T)$ is a independent domination number.

Proof: Suppose D be a dominating set of T. Let v be an end vertex of T and root the tree T at v. Let A be the set of all vertices in $V(T) - D$ that are dominated only from above by a vertex in D. Thus the parent of each vertex in A belongs to D and no child vertex of A belongs to D. Possibly $A = \emptyset$. Let $B = V(T) - (A \cup D)$. Then every vertex of B is dominated from below by D that is every vertex B has a child that belongs to D. Let B_1 be those vertices in B adjacent to vertex in A. Then B_1 is an independent set of T and dominates A (from below). We now extend B_1 to an independent set B^* that dominates B by adding vertices in $B - B_1$. Then B^* dominates $A \cup B$. Let D_1 be the set of all vertices of D that are dominated by B^* and $D_1 = D \cap N(B^*)$. Since every vertex in B is dominated from below by at least one vertex of D,then $|B^*| \ge |D_1|$. Let

$D_2 = D - D_1$, and let D^* be a maximal independent set of vertices of D_2. Then D^* dominates D_2. Further more $|D^*| \geq |D_2|$. By construction, $B^* \cup D^*$ is an independent dominating set of T. Hence $|B^* \cup D^*| = i(T)$. Since every block in $B(T)$ is complete and every cut vertex of $B(T)$ lies on exactly two blocks of $B(T)$. Let $k_{n_1}k_{n_2}, \dots\dots\dots, k_{n_m}$ be the number of blocks which are complete. Then each block is complete with $\{v_1, v_2, \dots\dots\dots, v_{n_1}\} \in k_{n_1}, \dots\dots \{v_1, v_2, \dots\dots\dots, v_{n_2}\} \in k_{n_2}, \dots\dots\dots \{v_1, v_2, \dots\dots\dots, v_{n_m}\} \in k_{n_m}$, number of vertices.

Now assume S be a dominating set of $B(T)$ and

$S = \{v_1, v_2, \dots\dots\dots, v_i;\ v_1, v_2, \dots\dots\dots, v_j; v_1, v_2, \dots\dots\dots, v_k \dots\dots\dots, v_Z\}$ such that $1 \leq i \leq n_1; \dots\dots\dots 1 \leq j \leq n_2; \dots\dots\dots 1 \leq Z \leq n_m;\ \forall v_i \in k_{n_1}; \dots\dots\dots \forall v_j \in k_{n_2}; \dots\dots\dots \forall v_Z \in k_{n_m}$. Now $V[B(T)] - S = H$ where each vertex of H is an isolates which gives $|S| = \gamma_{ssb}(T)$. Hence $|B^* \cup D^*| \geq |S|$ and we have $i(T) \geq \gamma_{ssb}(T)$.

Theorem 13: For any (p, q) graph G and $B(G) \neq K_P$, then $\gamma_{ssb}(G) \leq 3q - 2p$.

Proof: suppose G has a block say B with maximum number of vertices and edges. Then $3q - 2p$ is always more with $\gamma_{ssb}(G)$. Hence we require to get the sharp bound. For this we consider the graph G is a non-trivial tree with at least 3-blocks.

We consider the following cases.

Case 1: Suppose G is a path $P_n, n \geq 4$ vertices. Then $B(G) = P_{n-1}$. Since the path P_n has $p-$ vertices and $q-$ edges, then $3q - 2p = 3(p-1) - 2p = p - 3$ for $P \geq 4$. One can easily verify that $\gamma_{ssb}(G) \leq p - 3 = 3q - 2p$.

Case 2: Suppose G is not a path. Then there exists at least one vertices $v, degv \geq 3$. Let $C = \{v_1, v_2, v_3, \dots\dots\dots, v_i\}$ be the number cut vertices and D be a dominating set of $B(G)$. Suppose each block of $B(G)$ complete with $P-$vertices. Then $D = \{v_1, v_2, v_3, \dots\dots\dots, v_{p-1}\}$ where D consists of $P - 1$vertices from each block $B(T)$ such that

$C \subseteq D$ and $V[B(T)] - D = H$,where $v_i \in H$ is an isolates, clearly $|D| = \gamma_{ssb}(T) \leq p - 3 = 3q - 2p$.

Theorem 14: For any (p, q) non-trivial tree T and $B(T) \neq K_n$, then

$$\gamma_{ssb}(T) + \gamma_t(T) \geq +P - \Delta(T).$$

Proof: Suppose $B(T) = K_n$. Then by definition $\gamma_{ssb}(T) - set$ does not exist. Hence and $B(T) \neq K_n$. Assume T is a tree. Then every block of T is an edge. Let $A = \{B_1, B_2, B_3, \dots\dots\dots, B_n\}$ be the blocks of T and $M = \{b_1, b_2, b_3, \dots\dots\dots, b_n\}$ be the block vertices in $B(T)$ corresponding to the blocks of A.

Let $\{B_i\} \subset A$ such that each B_i is an non end block of T. Then $\{b_i\} \subseteq V[B(T)]$ which are vertices corresponding to the set $\{B_i\}$ since each block is complete in $B(T)$. Again we consider a subset $\{{b_i}^1\}$ such that $\{{b_i}^1\} \subset V[B(T)] - \{b_i\}$. Suppose there consists at least one edge then $V[B(T)] - \{{b_i}^1 \cup b_i\} = \{b_k\}$ where each element of b_k is an isolates. Then $|\{{b_i}^1 \cup$ bi}=γssbT. If bi1=∅,then VBT−{bi } give at least two isolates such that bi=γssbT. Let $S \subseteq V[B(T)]$ is minimal total dominating set of T such that $\gamma_t(T) = |S|$. Now assume $\Delta(T) \leq 2$. Then $T = P_n, n \geq 4$. Hence $P - \Delta(T) \leq |{b_i}^1| + |S|$ which gives $\gamma_{ssb}(T) + \gamma_t(T) \geq +P - \Delta(T)$.

Further if $\Delta(T) \geq 3$. Then there exists a positive integer j such that $j \leq P - \Delta(T)$. Also $j \leq |{b_i}^1| + |S|$ which gives $P - \Delta(T) \leq \gamma_{ssb}(T) + \gamma_t(T)$.

Finally we obtained the Nordhous-Gaddum type results.

Theorem 15: For any (p, q) graph G , and $B(G) \neq K_P$, then

I. $\gamma_{ssb}(G) + \gamma_{ssb}(\bar{G}) \leq p$.

II. $\gamma_{ssb}(G).\gamma_{ssb}(\bar{G}) \leq 2p$.

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