ALGORITHM OF UNDETERMINED COEFFICIENTS AND STABILITY ANALYSIS OF THE WAGE FUNCTION

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Abstract: In this paper, algorithm of undetermined coefficients has been used to solve the wage equation. The subsequent wage function is analyzed and interpreted for stability. The wage function has speculative parameters operating in free range. The variations of these parameters causes stability and instability of the wage function in certain circumstances. Where the wage function is exponential, asymptotic stability towards the equilibrium wage rate is observed but where it consists of both exponential and periodic factors, the time path shows periodic fluctuations with successive cycles giving smaller amplitudes until the ripples dies naturally. It is also observed that though algorithm of undetermined coefficients is just as effective as differential operator shown in [6] and variation of parameters demonstrated in [7]; it is fast with limited algebra than variation of parameters but relatively has more algebra than operator method.

Key words: wage equation, wage function, wage rate, equilibrium wage rate, stability, undetermined coefficients.

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1. Introduction

The differential wage equation

$$\frac{d^2W}{dt^2} + a_1\frac{dW}{dt} + a_2W = b \qquad (1.1)$$

with $a_1 = \frac{\tau}{\psi}, a_2 = -\left(\frac{\sigma+\lambda}{\psi}\right), and\ b = -\left(\frac{\theta+\eta}{\psi}\right)$ was developed in [7] using linear demand and supply functions of labour

$$N_d = \eta - \sigma W + \tau\frac{dW}{dt} + \psi\frac{d^2W}{dt^2}, \quad (\eta, \sigma > 0) \qquad (1.2)$$

and

$$N_s = -\theta + \lambda W + \vartheta\frac{dW}{dt} + \xi\frac{d^2W}{dt^2}, \quad (\theta, \lambda > 0) \qquad (1.3)$$

respectively. In this development, the parameters τ, ψ, ϑ *and* ξ introduced to dictate employers and laborers expectations had their signs operating in free range. For example, if $\tau > 0$ then a rising wage rate caused the number of laborers demanded to increase. This suggested that employers expected rising wage rate to continue to rise and preferred to increase employment then, when the wage rate was still relatively low. On the other hand, for $\tau < 0$, the wage trend was falling, and employers opted to cut employment as at that time while they waited for wage rate to fall further. The inclusion of the parameter ψ made employers behavior to also depend on the rate of change of wage rate $\frac{dW}{dt}$. The introduction of new parameters τ *and* ψ injected wage rate speculation in the model. Similarly, for $\vartheta > 0$, a rising wage rate caused the number of laborers supplied to fall. Laborers expected rising wage rate to continue rising and preferred to withhold their services while they waited for higher wage rate. On the other hand, for $\vartheta < 0$, wage rate showed a falling trend and laborers preferred to offer their services while the wage rate was still relatively high hoping that any delay was to make wage rate fall even further. The introduction of the parameter ξ made the behavior of laborers to depend much on the wage rate. An implicit model was then developed by assuming that only labor demand function contains wage expectations. Specifically, both ϑ *and* ξ of function (1.3) were set equal to zero; while

τ *and* ψ of function (1.2) were set as non zero. Further, it was assumed that labor market cleared at every point in time. Equation (1.1) was therefore solved in [6] by the method of variation of parameters and in [7] by differential operator method. The subsequent wage function was analyzed and interpreted for stability in both cases with similar behavioral patterns. The variation of speculative parameters, which were included in modeling, caused both stability and instability of the wage function depending on circumstances. Where the wage function was exponential, asymptotic stability towards the equilibrium wage rate was observed but where it consisted of both exponential and periodic factors, the time path showed periodic fluctuations with successive cycles giving smaller amplitudes until the ripples die naturally.

In [1], dynamics of market prices were studied. It was found out that if the initial price of the price function lies off the equilibrium point, then in the long run its stability was realized at equilibrium position. In [2], equilibrium solutions representing a special class of static solutions are discussed. The study found that if a system starts exactly at equilibrium condition, then it will remain there forever. The study further found that in real systems, small disturbances often a rise which move a system away from the equilibrium state. Such disturbances, regardless of their origin gave rise to initial conditions which did not coincide with the equilibrium condition. If the system is not at equilibrium point, then some of its derivatives will be non zero and the system therefore exhibits a dynamic behavior, which can be monitored by watching orbits involved in the phase space.

The resistance-inductance electric current circuit for constant electromotive force is modeled into a differential equation in [4].The stability of the solution is studied in the long run and is found to be a constant, which is the ratio of constant electromotive force to resistance. It was also found that if electromotive force is periodic, in the long run, current executes harmonic oscillations. In this case, steady state solution is the periodic part of the solution. The resistance-capacitor electric current circuit equation was also discussed for constant electromotive force. The solution was an exponential function, which converges to zero in the long run. Also, in resistance-inductance-capacitance series circuit, a second order equation was developed. Its solution consisted of an exponential homogeneous part and a periodic integral part. The study found out that the homogeneous part converges to zero as time approaches infinity, while the periodic part exhibits practically harmonic oscillations.

In [5] a natural decay equation was developed. The equation describes a phenomenon where a quantity gradually decreases to zero. In the work, it is emphasized that convergence depends on the sign of the change parameter. If the change parameter is negative then it turns into a growth equation and if it is positive it stabilizes in the long run. The study of slope fields for autonomous equations and qualitative properties of the decay equation were also demonstrated. It was found that the solution could be positive, negative or zero. In all the three cases, the solution approaches zero in limit as time approaches infinity. In the same work, models representing Newton's law of cooling, depreciation, population dynamics of diseases and water drainage are also presented with similar property in the long run.

In [7], a deterministic logistic first order differential price equation is considered. The equation is solved by an integrating factor. In the analysis of the solution, it is observed that in the long run asset price settles at a constant steady state point at which no further change can occur.

The current paper therefore proposes solving wage equation [1.1] by an algorithm of undetermined coefficients demonstrated in [3; 4], analyzing it, and interpreting the results.

2. Solution of the differential equation

In this section, we use an algorithm of undetermined coefficients to solve second order differential wage equation

$$\frac{d^2W}{dt^2}+a_1\frac{dW}{dt}+a_2W=b \tag{2.1}$$

where $a_1=\frac{\tau}{\psi}, a_2=-\left(\frac{\sigma+\lambda}{\psi}\right)$ *and* $b=-\left(\frac{\theta+\eta}{\psi}\right)$. The equation is first investigated for a complementary solution by solving its homogeneous equation

$$\frac{d^2W}{dt^2}+a_1\frac{dW}{dt}+a_2W=0 \tag{2.2}$$

The characteristic equation of (2.2) is

$$r^2+a_1r+a_2=0 \tag{2.3}$$

This equation is quadratic in nature and is solved for r by the quadratic formula to obtain

$$r_1, r_2 = \frac{1}{2}\left(-a_1 \pm \sqrt{a_1^{\,2} - 4a_2}\right) \qquad (2.4)$$

If we substitute back the values of a_1 *and* a_2 as given in equation (2.1) the roots become

$$r_1, r_2 = \frac{1}{2}\left(-\frac{\tau}{\psi} \pm \sqrt{\left(\frac{\tau}{\psi}\right)^2 + 4\left(\frac{\sigma+\lambda}{\psi}\right)}\right). \qquad (2.5)$$

The roots (2.5) yields varied results depending on the value of the discriminant $\left(\frac{\tau}{\psi}\right)^2 + 4\left(\frac{\sigma+\lambda}{\psi}\right)$. This is discussed in three cases.

Case I: Suppose $\left(\frac{\tau}{\psi}\right)^2 + 4\left(\frac{\sigma+\lambda}{\psi}\right) > 0$ then roots r_1 *and* r_2 of the characteristic equation (2.3) are real and different. The complementary solution therefore becomes

$$W_c(t) = c_1 \exp r_1 t + c_2 \exp r_2 t \qquad (2.6)$$

Case II: Suppose $\left(\frac{\tau}{\psi}\right)^2 + 4\left(\frac{\sigma+\lambda}{\psi}\right) = 0$ then the roots r_1 *and* r_2 of the characteristic equation (2.3) are real and equal; that is,

$$r_1, r_2 = \alpha = -\frac{\tau}{2\psi} \qquad (2.7)$$

and the complementary solution then becomes

$$W_c(t) = (c_1 + c_2 t)\exp \alpha t. \qquad (2.8)$$

Case III: Suppose $\left(\frac{\tau}{\psi}\right)^2 + 4\left(\frac{\sigma+\lambda}{\psi}\right) < 0$ then the roots r_1 *and* r_2 of the characteristic equation (2.3) are complex; that is,

$$r_1, r_2 = \alpha \pm i\beta, \ with\ \alpha = -\frac{\tau}{2\psi} \text{ and } \beta = \frac{1}{2}\sqrt{-\left(\frac{\tau}{\psi}\right)^2 - 4\left(\frac{\sigma+\lambda}{\psi}\right)} \qquad (2.9)$$

The complementary solution therefore becomes

$$W_c(t) = \exp \alpha t\,(c_1 \cos \beta t + c_2 \sin \beta t) \qquad (2.10)$$

To solve equation (2.1) completely requires the demonstration of how to find an integral function $W_I(t)$. This is done in this paper by an algorithm of undetermined coefficients. We consider equation (2.1) and check at its right hand side to enable us decide on the shape of the integral function. Since the right hand side of the equation is a constant, the integral function should be such that

$$W_I(t) = k, \quad k = constant \tag{2.11}$$

The first and second derivatives of the integral function (2.11) are

$$\left.\begin{aligned} &\frac{dW_I}{dt} = 0 \\ &and \\ &\frac{d^2W_I}{dt^2} = 0 \end{aligned}\right\} \tag{2.12}$$

The integral function (2.11) and its derivatives (2.12) are substituted in the differential equation (2.1) in order to evaluate the constant. Thus,

$$\frac{d^2W}{dt^2} + a_1\frac{dW}{dt} + a_2W = b \Rightarrow \frac{d^2W_I}{dt^2} + a_1\frac{dW_I}{dt} + a_2W_I = b$$

$$\Rightarrow a_2k = b$$

$$\Rightarrow k = \frac{b}{a_2}$$

$$\therefore W_I(t) = \frac{b}{a_2}, \quad a_2 \neq 0 \tag{2.13}$$

Since $a_2 = -\left(\frac{\sigma+\lambda}{\psi}\right)$ *and* $b = -\left(\frac{\eta+\theta}{\psi}\right)$, then the integral function becomes

$$W_I(t) = \hat{W} = \frac{\eta+\theta}{\sigma+\lambda}, \quad \sigma \neq -\lambda \tag{2.14}$$

If we consider the complementary solution (2.6) together with the integral function (2.14), the general solution becomes

$$W(t) = c_1 \exp r_1 t + c_2 \exp r_2 t + \hat{W} \qquad (2.15)$$

This can further be solved for a particular solution if we make use of the initial conditions. Suppose $W(t)\big|_{t=0} = W_0 \; and \; \frac{dW}{dt}\Big|_{t=0} = 0$ are the initial conditions, then a particular solution is given as

$$W(t) = \frac{r_2}{r_2 - r_1}\left(W_0 - \frac{b}{a_2}\right)\exp r_1 t - \frac{r_1}{r_2 - r_1}\left(W_0 - \frac{b}{a_2}\right)\exp r_2 t + \hat{W} \qquad (2.16)$$

If we consider the complementary solution (2.8) together with the integral function (2.14), the general solution becomes

$$W(t) = (c_1 + c_2 t)\exp(\alpha t) + \hat{W} \qquad (2.17)$$

If we use initial conditions $W(t)\big|_{t=0} = W_0 \; and \; \frac{dW}{dt}\Big|_{t=0} = 0$, then the particular solution becomes

$$W(t) = (W_0 - \hat{W})(1 - \alpha t)\exp(\alpha t) + \hat{W} \qquad (2.18)$$

Considering the complementary function (2.10) together with the integral function (2.14) the general solution becomes

$$W(t) = \exp \alpha t\,(c_1 \cos \beta t + c_2 \sin \beta t) + \hat{W}. \qquad (2.19)$$

Solution (2.19) is solved for a particular solution if we use the initial conditions. Suppose we let $W(t)\big|_{t=0} = W_0 \; and \; \frac{dW}{dt}\Big|_{t=0} = 0$ the particular solution becomes

$$W(t) = \exp(\alpha t)\left[(W_0 - \hat{W})\cos \beta t - \frac{\alpha}{\beta}(W_0 - \hat{W})\sin \beta t\right] + \hat{W} \qquad (2.20)$$

3. Results, analysis and interpretation

Equation (2.1) has been solved in this paper by method of undetermined coefficients for the first time. For $\left(\frac{\tau}{\psi}\right)^2 + 4\left(\frac{\sigma + \lambda}{\psi}\right) > 0$, the general solution is found as

$$W(t) = c_1 \exp r_1 t + c_2 \exp r_2 t + \hat{W} \qquad (3.1)$$

with r_1 *and* r_2 described in solution (2.4), and $\hat{W}$ in solution (2.14). In this case, suppose $\psi > 0$, *then* $4\left(\frac{\sigma+\lambda}{\psi}\right) > 0$ and $\left(\frac{\tau}{\psi}\right)^2 + 4\left(\frac{\sigma+\lambda}{\psi}\right) > 0, \forall$ *values of* τ. The solution (3.1) is therefore valid. Moreover, with $\psi > 0$, since $\sigma, \lambda > 0$, $\left(\frac{\tau}{\psi}\right)^2 + 4\left(\frac{\sigma+\lambda}{\psi}\right)$ is positive and its square root exceeds $\left(\frac{\tau}{\psi}\right)^2$. Therefore solutions (2.4) produce one positive root r_1 and one negative root r_2. Consequently, inter temporal equilibrium is dynamically unstable. For the function (3.1) to be stable, we have to set constant c_1 to zero so that it becomes

$$W(t) = c_2 \exp r_2 t + \hat{W} \tag{3.2}$$

If we use the initial conditions $W(t)\big|_{t=0} = W_0$ *and* $\frac{dW}{dt}\Big|_{t=0} = 0$ then solution (3.2) becomes

$$W(t) = \left(W_0 - \hat{W}\right) \exp r_2 t + \hat{W} \tag{3.3}$$

The solution (3.3) is now investigated for stability by taking the limit as t tends to infinity, i.e.

$$W(t) = \lim_{t\to\infty} \left[\left(W_0 - \hat{W}\right) \exp r_2 t + \hat{W}\right] \tag{3.4}$$

In this case, $\left(W_0 - \hat{W}\right)$ is constant and the value of limit function (3.4) depends on the exponential factor $\exp r_2 t$. In view of the fact that $r_2 < 0$, $\left(W_0 - \hat{W}\right) \exp r_2 t \to 0$ *as* $t \to \infty$. The limit function (3.4) therefore becomes

$$W(t) = \hat{W}. \tag{3.5}$$

This means time path of the wage function (3.3) consequently moves towards equilibrium position in the long run.

Considering particular solution (2.16), suppose we let $\psi < 0$ *with* $\sigma, \lambda > 0$, the expression under the square root in solution (2.4) is less than $\left(\frac{\tau}{\psi}\right)^2$ and the square root must be less than $\frac{\tau}{\psi}$;

therefore for $\left(\frac{\tau}{\psi}\right)^2 + 4\left(\frac{\sigma+\lambda}{\psi}\right) > 0,$ *if we let* $\tau < 0$, then the roots of (2.4) would produce two negative roots. The solution (2.16) is therefore investigated for stability by finding its path, i.e.

$$W(t) = \lim_{t\to\infty}\left(\frac{r_2}{r_2 - r_1}\right)\left(W_0 - \hat{W}\right)\exp r_1 t - \lim_{t\to\infty}\left(\frac{r_1}{r_2 - r_1}\right)\left(W_0 - \hat{W}\right)\exp r_2 t + \lim_{t\to\infty}\hat{W}. \qquad (3.6)$$

In this case, $\left(\frac{r_2}{r_2 - r_1}\right)\left(W_0 - \hat{W}\right)$ and $\left(\frac{r_1}{r_2 - r_1}\right)\left(W_0 - \hat{W}\right)$ are constants and the value of the limit function (3.6) depends on the exponential factors $\exp r_1 t$ and $\exp r_2 t$. In view of the fact that $r_1, r_2 < 0$ the limits of the first and second term of function (3.6) both tends to zero; thus

$$W(t) = \hat{W}. \qquad (3.7)$$

This means that the wage function (2.16) consequently moves towards the equilibrium position in the long run and it is therefore dynamically stable so long as $\psi < 0$ *and* $\tau < 0$.

Interestingly, if we consider $\left(\frac{\tau}{\psi}\right)^2 + 4\left(\frac{\sigma+\lambda}{\psi}\right) = 0$, the general solution of equation (2.1) is as shown in (2.17). If we use the initial conditions, a particular solution (2.18) is obtained and it is investigated for stability, by taking its limit as time tends to infinity i.e.

$$W(t) = \lim_{t\to\infty}\left[\left(W_0 - \hat{W}\right)\left(-\alpha t\right)\exp\left(\alpha t\right) + \hat{W}\right] \qquad (3.8)$$

In this case, the first term of the limit function (3.8) consists of a linear factor $\left(W_0 - \hat{W}\right)\left(-\alpha t\right)$ *and an* exp*onential factor* $\exp\left(\alpha t\right)$. Its value therefore depends on the exponential factor $\exp\left(\alpha t\right)$. In view of the fact that $\alpha < 0$, $\left(W_0 - \hat{W}\right)\left(-\alpha t\right)\exp\left(\alpha t\right) \to 0$ *as* $t \to \infty$. The limit function (3.8) therefore becomes

$$W(t) = \hat{W}. \qquad (3.9)$$

This means the time path of the wage function (2.18) consequently moves towards the equilibrium wage rate as time tends to infinity.

Further analyses of the limit functions (3.4), (3.6) and (3.8) is possible by considering the relative positions of $W_0 \; and \, \hat{W}$; that is, by comparing the relative positions of the initial wage rate and the equilibrium wage rate. This is discussed in three different cases.

CASEI: In this case, we consider both the limiting functions and let $W_0 = \hat{W}$ .This means the limit functions (3.4), (3.6) and (3.8) becomes $W(t) = \hat{W}$ at an infinite time, which is a constant path and is parallel to the time axis. The wage function in both situations becomes stable at equilibrium wage rate in the long run.

CASE II: In this case, we let $W_0 > \hat{W}$. The first term on the right hand side of function (3.4) is positive but it decreases since as $t \to \infty$ it is lowered by the value of the exponential factor $\exp r_2 t$ for $r_2 < 0$. The first term on the right hand side of function (3.6) is positive if $r_1 > r_2$ and the second term is only positive if $r_1 < r_2$.Therefore, they will decrease since as $t \to \infty$ they are lowered by the values of the exponential factors $\exp r_1 t$ and $\exp r_2 t$ respectively. Finally, the first term on the right hand side of function (3.8) is positive but it decreases as $t \to \infty$ since it is lowered by the exponential factor $\exp(\alpha t)$. The limit functions (3.4), (3.6) and (3.8) thus have their time path asymptotically approaching the equilibrium wage rate $\hat{W}$ from above, and in the long run, become stable.

CASE III: In this case, we let, $W_0 < \hat{W}$ i.e. the initial wage rate is taken to be less than the equilibrium wage rate. The first term on the right hand side of the limit function (3.4) is negative and the exponential factor infinitely makes W_0 to rise asymptotically towards the equilibrium wage $\hat{W} \; as \, t \to \infty$. Similarly, the first term on the right hand side of function (3.6) is negative if $r_1 > r_2$ and the second term is only negative if $r_1 < r_2$, making W_0 to rise asymptotically towards the equilibrium wage rate $\hat{W} \; as \, t \to \infty$. Finally, the first term on the right hand side of function (3.8) is negative and it infinitely makes W_0 to rise asymptotically towards the equilibrium wage $\hat{W} \; as \, t \to \infty$. These three cases are illustrated in figure 3.1.

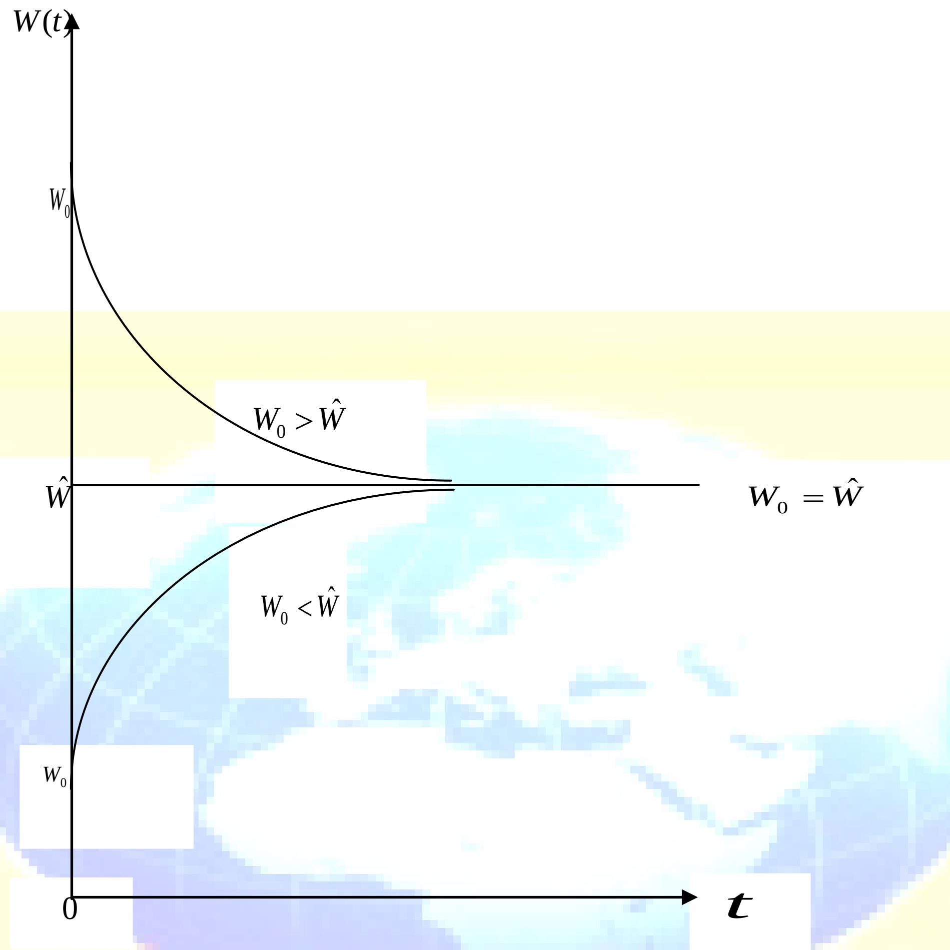

Figure 3.1: Stability Analysis of the Wage Function

Figure 3.1 shows that when $W_0 = \hat{W}$, *then* $W(t) = \hat{W}$, which is a constant function. If $W_0 > \hat{W}$, *then* $W(t)$ decreases asymptotically towards $\hat{W}$, *and if* $W_0 < \hat{W}$ *then* $W(t)$ increases asymptotically towards the equilibrium wage rate $\hat{W}$. The results therefore shows that as $t \to \infty$, the functions (2.16) and (2.18) approach the equilibrium wage rate and becomes stable so long as $\psi < 0$ *and* $\tau < 0$.

We now turn to investigating the solution when $\left(\frac{\tau}{\psi}\right)^2 + 4\left(\frac{\sigma + \lambda}{\psi}\right) < 0$. In this case, the general wage function (2.19) has been developed. If the initial conditions are used, a particular function

(2.20) is obtained. The function (2.20) is now investigated for stability, i.e. since $\alpha = -\frac{\tau}{2\psi}$ is the real part of the complex root if we let $\psi < 0$ *and* $\tau < 0$, then $\alpha < 0$. The wage function (2.20) therefore becomes dynamically stable. The time path in this case is one with periodic fluctuation of period $\frac{2\pi}{\beta}$; that is, there is a complete cycle every time t increases by $\frac{2\pi}{\beta}$, where β is as defined in (2.9). In view of the multiplicative factor $\exp \alpha t$ the fluctuation is damped. The time path, which starts from the initial wage, $W(t)\big|_{t=0} = W_0$ converges to an inter-temporal equilibrium wage $W(t) = \hat{W}$ in a cyclical fashion. This is illustrated in figure 3.2.

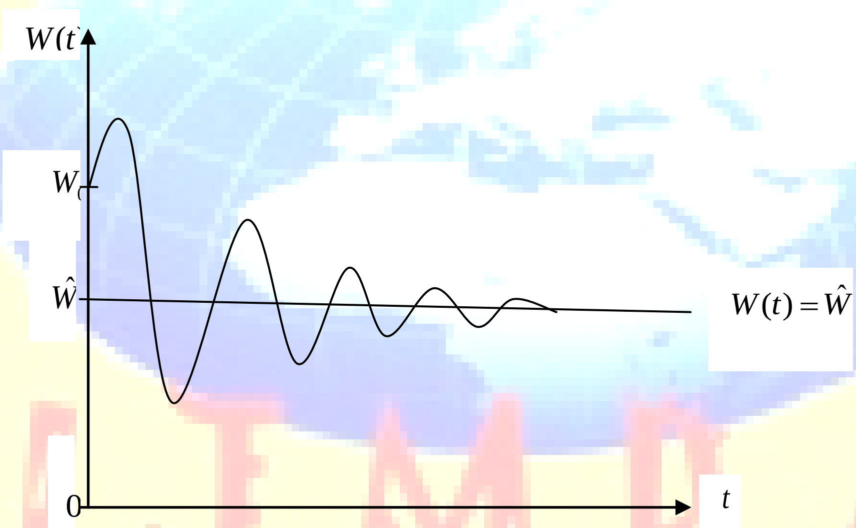

Figure 3.2: Periodic Stability Analysis of the Wage

Figure 3.2 show that since $\alpha < 0$, the exponential factor $\exp \alpha t$ continually decreases as $t \to \infty$ and each successive cycle gives smaller amplitude than the preceding one and the ripples naturally dies slowly.

4. Conclusion

In this paper, the method of undetermined coefficients has been used to solve wage equation. The subsequent wage function has been analyzed and interpreted for stability. The equation incorporates speculative parameters operating in free range. The variations of these parameters have caused stability and instability of the wage function in certain circumstances. Where the wage function takes an exponential form with particular assumptions, as time approaches infinity, it asymptotically approaches the equilibrium wage rate. Where as in a case of an exponential and a periodic factor, the time path shows a periodic function whose successive cycles elicit smaller amplitudes, which eventually dies naturally at equilibrium wage rate as time approaches infinity. It is also observed that though an algorithm of undetermined coefficients is just as effective as differential operator shown in [6] and variation of parameters demonstrated in [7]; it is fast with limited algebra than the method of variation of parameters but relatively has more algebra than the operator method.

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