INVERSE PROPERTY LOOPS AND THE ITS NUCLEUS

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ABSTRACT

In this paper we give a detailed and step by step proof of the coincidence of the nucleus $N$of a loop$L$ that is equiped with the inverse property. We further considered some properties of the inner mappings of N.

KEYWORDS

Loop, Inverse property, Nucleus, Automorphisms, Autotopism, Pseudo-automorphisms, Moufang elements, Moufang Loop.

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INTRODUCTION

A major concern of researchers has been how to clearly present their findings to the general populace. We preseent a detailed and well presented proof of one major theorem of R. Hubert Bruck. There have been several findings in the area of the Loops and some of these researches are those of R. H. Bruck, 1971 as earlier mentioned, H.O. Pflugfelder,1990 and A. Drapal, 2010.

BASIC DEFINITIONS

Definition 1

A nonempty set L with a binary operation is called a Loop.

Definition 2

A groupoid G is said to have the left inverse property if for each $m \in G$ there is atleast one $a \in G$ such that

$$a(mn) = n \text{ for all } n \in G. \text{ i.e } L(m)L(a) = I$$

G is said to have the right inverse property if for each $x \in G$ there is atleast one $b \in G$ such that

$$(nm)b = n \text{ for all } n \in G. \text{ i.e } R(m)R(a) = I$$

If G has both the the left inverse property and the the right inverse propert, then G is said to have the inverse property. It is also called IP-loop.

Definition 3

A Loop satisfying

$$(lm)(nl) = [l(mn)]l$$

is called a Moufang Loop.

Definition 4

A Moufang Loop$L$, is said to have a Nucleus denoted N. N is defined as the set of all elements which associate with every pair of element of L i.e

$a \in N$ iff for all $l, m \in L$

$$(al)m = a(lm)$$
$$(la)\mathrm{m} = \mathrm{l(am)}$$
$$\mathrm{l(ma)} = \mathrm{(lm)a}$$

Definition 5

An Isomorphism from a group $(G,*)$ to itself is called an automorphism of this group. That is a bijection such that if given a mapping $\alpha: G \rightarrow$G, then

$$\alpha(x) * \alpha(y) = \alpha(x * y) \text{ for all } x, y \in G.$$

Definition 6

An element of an inverse property Loop G is called a Moufang element of G provided definition 3 holds in G.

Definition 7

A triple (α, β, γ) of bijecttions is called an isotopism of Loop $(L,\cdot)$ onto a loop $(H,\circ)$ provided

$$x\, \alpha^\circ y\beta = (x \cdot y)\gamma \text{ for all } x, y \in L.$$

Definition 8

An isotopism of $(L,\cdot)$ onto a loop $(L,\cdot)$ is called an autotopism of L and is denoted $A(L)$.

Definition 9

A permutation S of a loop L is called a Pseudo-automorphism of L provided there exists atleast one element $c \in L$, called a Companion of S such that

$$(xS)(yS \cdot c) = xyS \cdot c$$

for all $x, y \in L$

MAIN RESULTS

Theorem 1:

Given an inverse property loop L, with Nucleus N.

Claim1:

The Nuclei N_ρ, N_σ, N_τ of L coincide with N.

Proof Of Claim1:

To proof claim 1, we shall need to show that

i. An element $a \in L$ is in N_ρ
ii. N element $a^{-1} \in L$ and hence a is in N_σ
iii. $N_\rho = N_\sigma = N_\tau$

Proof of Claim1i and ii:

By definition 4:

$a \in N$ iff for all $b, c \in L$

$$(ab)c = a(bc)$$
$$(ba)\mathrm{c} = \mathrm{b(ac)}$$
$$\mathrm{b(ca)} = \mathrm{(bc)a}$$

So $a \in L$ is in N_ρ if and only if

$$(ab)c = a(bc)$$

for all $b, c \in L$

If this then holds it means that

$$[(ab)c]^{-1} = [a(bc)]^{-1}$$

Iff

$$c^{-1}(ab)^{-1} = (bc)^{-1}a^{-1}$$

Iff

$$c^{-1}(b^{-1}a^{-1}) = (c^{-1}b^{-1})a^{-1}$$

Thus a^{-1} and hence a is in N_ρ.

Proof of Claim 1iii:

Using the proof of Claim1i and ii, we see that by symmetry $N_\rho = N_\sigma$, Now again from

$$(ab)c = a(bc)$$

We see that if we multiply both sides by $(ab)^{-1}$, we will have that:

$$(ab)^{-1}(ab)c = (ab)^{-1}[a(bc)]$$
$$c = (ab)^{-1}[a(bc)]$$
$$c=[(ab)^{-1}a](bc)$$

for all $b, c \in L$

Again our goal is to show that $N_\rho = N_\sigma = N_\tau$, so we shall here introduce some new elements so as to link the first two equalities to the third.

So set $p = (ab)^{-1}$ and $q = (bc)$.

So that we now have

$$c = [(ab)^{-1}a](bc)]$$
$$c=(pa)q = p(aq)$$

for all all $p, q \in L$

With the above we see that the converse

$$\mathrm{b(ca) = (bc)a}$$

for all all $\mathrm{b}, c \in L$ also holds since

$$[(ba)c]^{-1} = [b(ac)]^{-1}$$

Iff

$$c^{-1}(ba)^{-1} = (ac)^{-1}b^{-1}$$

Iff

$$c^{-1}(a^{-1}b^{-1}) = (c^{-1}a^{-1})b^{-1}$$

And so by symmetry again we see that $N_\rho = N_\sigma = N_\tau$ nas required.

Theorem 2:

$(\mathrm{L(a), I, L(a)})$ is an autotopism

Claim2:

$(\mathrm{L(a), I, L(a)})$ is an autotopism

Proof of Claim2:

By definitions 9 and 10 we see that

$$xL(a) \cdot yI = ax \cdot y = xyL(a)$$

Therefore $(\mathrm{L(a), I, L(a)})$ is an autotopism.

Theorem 3:

A Pseudo-automorphism S with companion c, induces an automorphism on N.

Proof of Theorem 3:

Assume S is as stated in the theorem, then by definition 4 and definition 9, we have

$$(ax)y = a(xy) \text{ and}$$
$$(xS)(yS\cdot c) = xyS\cdot c$$

for all $x, y, z \in L$

We shall set the equation as

$$w = (ax)y = a(xy)$$

So we can write:

$$wS\cdot c = [(ax)y]S\cdot c$$
$$= [(ax)S][yS\cdot c]$$

And since

$$w = (ax)y = a(xy)$$

We have

$$wS\cdot c = [a(xy)]S\cdot c$$
$$= [aS][(xy)S\cdot c]$$
$$= [aS][xSyS\cdot c] \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

for all $x, y, z \in L$

Now setting $yS\cdot c = y$ we see that

$$wS\cdot c = [aS][([xS\cdot y]$$

So setting $y = 1$ in eqution (1), we will have that :

$$wS\cdot c = [a(xy)]S\cdot c = [a(x1)]S\cdot c = [aS][([xS\cdot 1] = (ax)S = aS\cdot xS \ldots\ldots\ldots\ldots\ldots . . (2)$$

for all $x \in L$.

Thus by equation (1), (2) and the property $(ax)y = a(xy)$ we have that $aS \in N$

Similarly using the same steps as above we are able to see that $a^{-1}S \in N$

And finally by this result $(ax)S = aS\cdot xS$ and definition 5 we conclude that S induces an automorphism of N. Hence the proof.

CONCLUSION

This paper has succeded in simplifying major expressions in Brucks's work and it is with hope that many more of such deep and detailed works will be presented to the populace.

REFERENCES

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