AN ALGORITHM FOR SOLVING A FRACTIONAL CAPACITATED TRANSPORTATION PROBLEM WITH ENHANCED FLOW

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Abstract:

This paper presents an algorithm to solve a fractional capacitated transportation problem with enhanced flow. Sometimes, situations arise where either reserve stocks have to be kept at the supply points say, for emergencies or there may be extra demand in the markets. In such situations, the total flow needs to be controlled or enhanced. In this paper, a special class of transportation problems is studied where in the total transportation flow is enhanced to a known specified level. A related transportation problem is formulated and it is established that special type of feasible solution called corner feasible solution of related transportation problem bear one to one correspondence with the feasible solution of the given enhanced flow problem . The optimal solution to enhanced flow problem may be obtained from the optimal solution to related transportation problem. A real life example of a company is included in support of theory.

Keywords: *Capacitated transportation problem, enhanced flow, related problem, corner feasible solution.*

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1 Introduction :

Capacitated transportation problem finds its application in a variety of real world problems such as tele-communication networks, production- distribution systems, rail and urban road systems where there is scarcity of resources such as vehicles, docks, equipment capacity etc. These are bounded variation transportation problems. Many researchers like Gupta and Arora [3] ,Misra and Das[8], Dahiya and Verma [1] have contributed in this field. Jain and Arya [5] studied the inverse version of capacitated transportation problem in which the unit transportation cost of some cells in the original problem are adjusted as little as possible so that the given feasible solution becomes an optimal one. Xie et al. [11] developed a technique for duration and cost optimization for transportation problem.

A special class of transportation problem is a fractional transportation problem where the objective function to be optimized is a ratio of two linear functions. Optimization of a ratio of criteria often describes some kind of an efficiency measure for a system . Fractional programs finds its application in a variety of real world problems such as stock cutting problem , resource allocation problems , routing problem for ships and planes , cargo – loading problem , inventory problem and many other problems. Gupta and Arora [4] studied paradox in a fractional capacitated transportation problem. Dinkelbach [2] studied non- linear fractional programming in 1967. Swarup [10] gave a transportation technique in linear fractional functional programming in 1966. Jain and Saxena [6] studied time minimizing transportation problem with fractional bottleneck objective function.

Sometimes situations arise when because of extra demand in the market , the total flow needs to be enhanced , compelling some of the factories to increase their production in order to be able to meet the extra demand . The total flow from the factories in the market is now increased by an amount of extra demand .In such cases , flow needs to be enhanced.Khurana and Arora [7] studied enhanced flow in a fixed charge indefinite quadratic transportation problem.Khurana and Arora [8] studied linear plus linear fractional transportation problem with restricted and enhanced flow. This motivated us to study enhanced flow in a fractional capacitated transportation problem. In this paper, we shall be discussing the case when the flow gets enhanced due to extra demand in the market for a fractional capacitated transportation problem.

2.Problem Formulation:

Consider a fractional capacitated transportation problem given by:

$$\text{(P1): } \min\left[\frac{\sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij}}{\sum_{i\in I}\sum_{j\in J} d_{ij}x_{ij}}\right]$$

subject to

$$\sum_{j\in J} x_{ij} \geq a_i;\ \forall i \in I \qquad (1)$$

$$\sum_{i\in I} x_{ij} \geq b_j;\ \forall j \in J \qquad (2)$$

$$\sum_{i\in I}\sum_{j\in J} x_{ij} = P \quad \text{where } P > \max\left(\sum_{i\in I} a_i, \sum_{j\in J} b_j\right) \qquad (3)$$

$$l_{ij} \le x_{ij} \le u_{ij}\,;\forall\ \ i, j\ \in I\times J \qquad (4)$$

$I = \{1, 2, \ldots m\}$ is the index set of m origins.

$J = \{1, 2, \ldots, n\}$ is the index set of n destinations.

x_{ij} = number of units transported from i^{th} origin to the j^{th} destination .

c_{ij} = The actual cost of transporting one unit of a commodity from i^{th} origin to the j^{th} destination.

d_{ij} = the standard cost of transporting per unit of a commodity transported from i^{th} origin to the j^{th} destination.

l_{ij} and u_{ij} are the bounds on number of units to be transported from i^{th} origin to j^{th} destination.

In order to solve the problem (P1), we consider the following related problem (P2) with an additional supply point and an additional destination point.

$$\text{(P2): } \min z = \left[\frac{\sum_{i\in I'}\sum_{j\in J'} c'_{ij} y_{ij}}{\sum_{i\in I'}\sum_{j\in J'} d'_{ij} y_{ij}}\right]$$

subject to

$$\sum_{j\in J'} y_{ij} = a_i';\forall i \in I'$$

$$\sum_{i\in I'} y_{ij} = b_j';\forall j \in J'$$

$$l_{ij} \le y_{ij} \le u_{ij}\,;\forall\ \ i,j\ \in IxJ$$

$$0 \le y_{m+1,\,j} \le \sum_{i\in I} u_{ij} - b_j;\forall j \in J$$

$$0 \le y_{i,\,n+1} \le \sum_{j\in J} u_{ij} - a_i;\forall i \in I$$

$y_{m+1,\,n+1} \ge 0$ and integers.

$$a_i' = \sum_{j\in J} u_{ij};\forall i\in I \quad , \quad a_{m+1}' = \sum_{i\in I}\sum_{j\in J} u_{ij} - P = b_{n+1}', \quad b_j' = \sum_{i\in I} u_{ij};\forall j\in J,$$

$$c'_{ij} = c_{ij} \;, \forall i\in I,\, j\in J,\;\; c'_{m+1,j} = c'_{i,n+1} = 0 \quad \forall i\in I,\;\; \forall j\in J,\;\; c'_{m+1,n+1} = M$$

$$d'_{ij} = d_{ij} \ ,\forall i\in I,\ j\in J,\ \ d'_{m+1,j} = d'_{i,n+1} = 0 \quad \forall i\in I,\ \ \forall j\in J \ \ d'_{m+1,n+1} = M$$

$$I' = I \cup \ m+1 \qquad J' = J \cup \ n+1$$

3.Preliminary Result:

Result 1: A feasible solution $X^0 = \{\overset{\circ}{x}_{ij}\}_{I\times J}$ of problem (P2) with objective function value $\dfrac{N^\circ}{D^\circ}$ will be a local optimum basic feasible solution iff the following conditions holds.

$$\delta^1_{ij} = \frac{\theta_{ij}\left[D^\circ(c_{ij}-z^1_{ij}) - N^\circ(d_{ij}-z^2_{ij})\right]}{D^\circ\left[D^\circ + \theta_{ij}(d_{ij}-z^2_{ij})\right]} \ge 0; \forall (i,j)\in N_1$$

$$\delta^2_{ij} = -\frac{\theta_{ij}\left[D^\circ(c_{ij}-z^1_{ij}) - N^\circ(d_{ij}-z^2_{ij})\right]}{D^\circ\left[D^\circ - \theta_{ij}(d_{ij}-z^2_{ij})\right]} \ge 0; \forall (i,j)\in N_2 \qquad (5)$$

and if X^0 is an optimal solution of (P2),then $\delta^1_{ij} \ge 0; \forall (i,j)\in N_1$ and $\delta^2_{ij} \ge 0; \forall (i,j)\in N_2$ where $N^\circ = \sum_{i\in I}\sum_{j\in J} c_{ij}x^\circ_{ij}$, $D^\circ = \sum_{i\in I}\sum_{j\in J} d_{ij}x^\circ_{ij}$, B denotes the set of cells (i,j) which are basic and N_1 and N_2 denotes the set of non basic cells (i,j) which are at their lower bounds and upper bounds respectively.

$u^1_i, u^2_i, v^1_j, v^2_j; i\in I, j\in J$ are the dual variables such that $u^1_i + v^1_j = c_{ij}$, $\forall (i,j)\in B$; $u^2_i + v^2_j = d_{ij}$, $\forall (i,j)\in B$; $u^1_i + v^1_j = z^1_{ij}$, $\forall (i,j)\notin B$; $u^2_i + v^2_j = z^2_{ij}$, $\forall (i,j)\notin B$

Proof: Let $X^0 = \{\overset{\circ}{x}_{ij}\}_{I\times J}$ be a basic feasible solution of problem (P2) with equality constraints. Let z^0 be the corresponding value of objective function. Then

$$z^\circ = \left[\frac{\sum_{i\in I}\sum_{j\in J} c_{ij}x^\circ_{ij}}{\sum_{i\in I}\sum_{j\in J} d_{ij}x^\circ_{ij}}\right] = \frac{N^\circ}{D^\circ} \ \text{(say)}$$

$$= \left[\frac{\sum_{i\in I}\sum_{j\in J}(c_{ij}-u^1_i-v^1_j)x^\circ_{ij} + \sum_{i\in I}\sum_{j\in J}(u^1_i+v^1_j)x^\circ_{ij}}{\sum_{i\in I}\sum_{j\in J}(d_{ij}-u^1_i-v^1_j)x^\circ_{ij} + \sum_{i\in I}\sum_{j\in J}(u^2_i+v^2_j)x^\circ_{ij}}\right]$$

$$= \left[\frac{\sum\sum_{(i,j)\in N_1}(c_{ij}-u^1_i-v^1_j)l_{ij} + \sum\sum_{(i,j)\in N_2}(c_{ij}-u^1_i-v^1_j)u_{ij} + \sum_{i\in I}\sum_{j\in J}(u^1_i+v^1_j)x^\circ_{ij}}{\sum\sum_{(i,j)\in N_1}(d_{ij}-u^2_i-v^2_j)l_{ij} + \sum\sum_{(i,j)\in N_2}(d_{ij}-u^2_i-v^2_j)u_{ij} + \sum_{i\in I}\sum_{j\in J}(u^2_i+v^2_j)x^\circ_{ij}}\right]$$

$$= \left[\frac{\sum\sum_{(i,j)\in N_1} (c_{ij} - z^1_{ij}) l_{ij} + \sum\sum_{(i,j)\in N_2} (c_{ij} - z^1_{ij}) u_{ij} + \sum_{i\in I} a_i u^1_i + \sum_{j\in J} b_j v^1_j}{\sum\sum_{(i,j)\in N_1} (d_{ij} - z^2_{ij}) l_{ij} + \sum\sum_{(i,j)\in N_2} (d_{ij} - z^2_{ij}) u_{ij} + \sum_{i\in I} a_i u^2_i + \sum_{j\in J} b_j v^2_j} \right]$$

Let some non basic variable $x_{ij} \in N_1$ undergoes change by an amount θ_{rs} where θ_{rs} is given by $\min \begin{Bmatrix} u_{rs} - l_{rs} \\ x^\circ_{ij} - l_{ij} \text{ for all basic cells (i, j) with a } (-\theta) \text{ entry in } \theta - \text{loop} \\ u_{ij} - x^\circ_{ij} \text{ for all basic cells (i, j) with a } (+\theta) \text{ entry in } \theta - \text{loop} \end{Bmatrix}$ Then new value of the objective function $\hat{z}$ will be given by

$$\hat{z} = \frac{N^\circ + \theta_{rs}(c_{rs} - z^1_{rs})}{D^\circ + \theta_{rs}(d_{rs} - z^2_{rs})}$$

$$\hat{z} - z^\circ = \left[\frac{N^\circ + \theta_{rs}(c_{rs} - z^1_{rs})}{D^\circ + \theta_{rs}(d_{rs} - z^2_{rs})} - \frac{N^\circ}{D^\circ} \right]$$

$$= \frac{\theta_{rs}\left[D^\circ (c_{rs} - z^1_{rs}) - N^\circ (d_{rs} - z^2_{rs}) \right]}{D^\circ \left[D^\circ + \theta_{rs}(d_{rs} - z^2_{rs}) \right]} = \delta^1_{rs} \text{(say)}$$

Similarly, when some non basic variable $x_{pq} \in N_2$ undergoes change by an amount θ_{pq} then

$$\hat{z} - z^\circ = -\frac{\theta_{pq}\left[D^\circ (c_{pq} - z^1_{pq}) - N^\circ (d_{pq} - z^2_{pq}) \right]}{D^\circ \left[D^\circ - \theta_{pq}(d_{pq} - z^2_{pq}) \right]} = \delta^2_{pq} \text{(say)}$$

Hence X^0 will be local optimal solution iff $\delta^1_{ij} \geq 0; \forall (i, j) \in N_1$ and $\delta^2_{ij} \geq 0; \forall (i, j) \in N_2$. If X^0 is a global optimal solution of (P2), then it is an optimal solution and hence the result follows.

4.Theoretical Development:

Definition: Corner feasible solution : *A basic feasible solution* $\{y_{ij}\}$ $i \in I'$, $j \in J'$ *to problem (P2) is called a corner feasible solution (cfs) if* $y_{m+1,n+1} = 0$

Theorem 1. *A non corner feasible solution of problem (P2) cannot provide a basic feasible solution to problem(P1).*

Proof: Let $\{y_{ij}\}_{I' \times J'}$ be a non corner feasible solution to problem (P2).Then $y_{m+1,n+1} = \lambda$ (>0)

Thus $\sum_{i\in I'} y_{i,n+1} = \sum_{i\in I} y_{i,n+1} + y_{m+1,n+1}$

$$= \sum_{i\in I} y_{i,n+1} + \lambda$$

$$= \sum_{i\in I}\sum_{j\in J} u_{ij} - P$$

Therefore $\sum_{i\in I} y_{i,n+1} = \sum_{i\in I}\sum_{j\in J} u_{ij} - \ P + \lambda$

Now, for $i \in I$,

$$\sum_{j\in J'} y_{ij} = a_i^{'} = \sum_{j\in J} u_{ij}$$

$$\sum_{i\in I}\sum_{j\in J'} y_{ij} = \sum_{i\in I}\sum_{j\in J} u_{ij}$$

$$\sum_{i\in I}\sum_{j\in J} y_{ij} + \sum_{i\in I} y_{i,n+1} = \sum_{i\in I}\sum_{j\in J} u_{ij}$$

$$\sum_{i\in I}\sum_{j\in J} y_{ij} + \sum_{i\in I}\sum_{j\in J} u_{ij} - \ P + \lambda \ = \sum_{i\in I}\sum_{j\in J} u_{ij}$$

Therefore, $\sum_{i\in I}\sum_{j\in J} y_{ij} = P + \lambda$

This implies that total quantity transported from the sources in I to the destinations in J is P + λ > P, a contradiction to assumption that total flow is P and hence $\{y_{ij}\}_{I' \ xJ'}$ can not provide a feasible solution to problem (P1).

Lemma 1: *There is a one –to-one correspondence between the feasible solution to problem (P1) and the corner feasible solution to problem (P2).*

Proof: Let $\{x_{ij}\}_{IxJ}$ be a feasible solution of problem (P1).

So, By relation (4) we have $x_{ij} \le u_{ij}$ which implies $\sum_{j\in J} x_{ij} \le \sum_{j\in J} u_{ij}$ (6)

By relation (1) and (6), we get

$$a_i \le \sum_{j\in J} x_{ij} \le \sum_{j\in J} u_{ij} = a_i^{'}$$

Similarly, $b_j \le \sum_{i\in I} x_{ij} \le \sum_{i\in I} u_{ij} = b_j^{'}$

Define $\{y_{ij}\}_{I' \ xJ'}$ by the following transformation

$y_{ij} = x_{ij} \ ,i \in I, \ j \in J$ (7)

$y_{i,\ n+1} = \sum_{j\in J} u_{ij} - \sum_{j\in J} x_{ij}; \forall i \in I$ (8)

$y_{m+1,\ j} = \sum_{i\in I} u_{ij} - \sum_{i\in I} x_{ij}; \forall j \in J$ (9)

$y_{m+1,n+1} = 0$ (10)

It can be shown that $\{y_{ij}\}$ so defined is cfs to problem (P2).

Relation (4) and (7) implies that $l_{ij} \le y_{ij} \le u_{ij}$ for all $i \in I, j \in J$

Relation (1) and (8) implies that $0 \le y_{i,n+1} \le \sum_{j \in J} u_{ij} - a_i ; \forall i \in I$

Relation (2) and (9) implies that $0 \le y_{m+1,j} \le \sum_{i \in I} u_{ij} - b_j ; \forall j \in J$

Relation (10) implies that $y_{m+1,n+1} \ge 0$

Also for $i \in I$, relation (7) and (8) implies that,

$$\sum_{j \in J'} y_{ij} = \sum_{j \in J} y_{ij} + y_{i,n+1} = \sum_{j \in J} x_{ij} + \sum_{j \in J} u_{ij} - \sum_{j \in J} x_{ij} = \sum_{j \in J} u_{ij} = a_i$$

For i = m+1

$$\sum_{j \in J'} y_{m+1,j} = \sum_{j \in J} y_{ij} + y_{m+1,n+1} = \sum_{j \in J}\left(\sum_{i \in I} u_{ij} - \sum_{i \in I} x_{ij}\right)$$

$$= \sum_{i \in I}\sum_{j \in J} u_{ij} - \sum_{i \in I}\sum_{j \in J} x_{ij}$$

$$= \sum_{i \in I}\sum_{j \in J} u_{ij} - P = a'_{m+1}$$

Therefore, $\sum_{j \in J'} y_{ij} = a'_i ; i \in I'$

Similarly, it can be shown that $\sum_{i \in I'} y_{ij} = b'_j ; j \in J'$

Therefore $\{y_{ij}\}_{I' \times J'}$ is a cfs to problem (P2).

Conversely, let $\{y_{ij}\}_{I' \times J'}$ be a cfs to problem(P2).Define x_{ij} , $i \in I$, $j \in J$ by the following transformation.

$x_{ij} = y_{ij}$, $i \in I, j \in J$ (11)

It implies that $l_{ij} \le x_{ij} \le u_{ij}$, $i \in I, j \in J$

Now for $i \in I$, the source constraints in problem (P2) implies

$$\sum_{j \in J'} y_{ij} = a'_i = \sum_{j \in J} u_{ij}$$

$$\sum_{j \in J} y_{ij} + y_{i,n+1} = \sum_{j \in J} u_{ij}$$

$$\Rightarrow a_i \le \sum_{j \in J} y_{ij} \le \sum_{j \in J} u_{ij} \quad (\text{since } 0 \le y_{i,n+1} \le \sum_{j \in J} u_{ij} - a_i ; \forall i \in I)$$

Hence, $\sum_{j\in J} y_{ij} \geq a_i$, $i \in I$ and subsequently $\sum_{j\in J} x_{ij} \geq a_i$, $i \in I$

Similarly, for $j \in J$, $\sum_{j\in J} y_{ij} \geq b_j; \forall j \in J$ and subsequently $\sum_{i\in I} x_{ij} \geq b_j$; $\forall\ j \in J$

For i= m+1

$$\sum_{j\in J'} y_{m+1,j} = a'_{m+1} = \sum_{i\in I}\sum_{j\in J} u_{ij} - P$$

$$\Rightarrow \sum_{j\in J} y_{m+1,j} = \sum_{i\in I}\sum_{j\in J} u_{ij} - P \text{ because } y_{m+1,n+1} = 0 \qquad (12)$$

Now, for $j \in J$ the destination constraints in problem (P2) give

$$\sum_{i\in I} y_{ij} + y_{m+1,j} = \sum_{i\in I} u_{ij}$$

Therefore, $\sum_{i\in I}\sum_{j\in J} y_{ij} + \sum_{j\in J} y_{m+1,j} = \sum_{i\in I}\sum_{j\in J} u_{ij}$

By relation (12), we have $\sum_{i\in I}\sum_{j\in J} y_{ij} = \sum_{i\in I}\sum_{j\in J} u_{ij} - \sum_{j\in J} y_{m+1,j} = P$

$$\Rightarrow \sum_{i\in I}\sum_{j\in J} x_{ij} = P$$

Therefore $\{x_{ij}\}_{IxJ}$ is a feasible solution to problem (P1)

Remark 1: *If problem (P2) has a cfs ,then since $c'_{m+1,n+1}$=M and $d'_{m+1,n+1}$= M, it follows that non corner feasible solution can not be an optimal solution of problem (P1) .*

Lemma 2: *The value of the objective function of problem (P1) at a feasible solution $\{x_{ij}\}_{I\times J}$ is equal to the value of the objective function of problem (P2) at its corresponding cfs $\{y_{ij}\}_{I'xJ'}$ and conversely.*

Proof: The value of the objective function of problem (P2) at a feasible solution $\{y_{ij}\}_{I'xJ'}$ is

$$z = \left[\frac{\sum_{i\in I'}\sum_{j\in J'} c'_{ij} y_{ij}}{\sum_{i\in I'}\sum_{j\in J'} d'_{ij} y_{ij}}\right]$$

$$= \left[\frac{\sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij}}{\sum_{i\in I}\sum_{j\in J} d_{ij}x_{ij}}\right] \text{because} \begin{Bmatrix} c'_{ij} = c_{ij}, \forall i \in I, j \in J \\ d'_{ij} = d_{ij}, \forall i \in I, j \in J \\ x_{ij} = y_{ij}, \forall i \in I, j \in J \\ c'_{i,n+1} = c'_{m+1,j} = 0; \forall i \in I, j \in J \\ d'_{i,n+1} = d'_{m+1,j} = 0; \forall i \in I, j \in J \\ y_{m+1,n+1} = 0 \end{Bmatrix}$$

= the value of the objective function of problem (P1) at the corresponding cfs $\{x_{ij}\}_{I \times J}$

The converse can be proved in a similar way.

Lemma 3: *There is a one –to-one correspondence between the optimal solution to problem (P1) and optimal solution among the corner feasible solution to problem (P2).*

Proof: Let $\{\overset{\circ}{x}_{ij}\}_{I \times J}$ be an optimal solution to problem (P1) yielding objective function value z^0 and $\{\overset{\circ}{y}_{ij}\}_{I' \times J'}$ be the corresponding cfs to problem (P2). Then by Lemma 2, the value yielded by $\{\overset{\circ}{y}_{ij}\}_{I' \times J'}$ is z^0. If possible, let $\{\overset{\circ}{y}_{ij}\}_{I' \times J'}$ be not an optimal solution to problem (P2). Therefore, there exists a cfs $\{y'_{ij}\}$ to problem (P2) with the value $z^1 < z^0$. Let $\{x'_{ij}\}$ be the corresponding feasible solution to problem (P1). Then by lemma 2,

$$z^1 = \left[\frac{\sum_{i\in I}\sum_{j\in J} c_{ij}x'_{ij}}{\sum_{i\in I}\sum_{j\in J} d_{ij}x'_{ij}}\right]$$

which is less than z^0, a contradiction to the assumption that $\{\overset{\circ}{x}_{ij}\}_{I \times J}$ is an optimal solution to problem (P1). Hence $\{\overset{\circ}{y}_{ij}\}_{I' \times J'}$ must be an optimal solution to problem (P2). Similarly it can be proved that an optimal corner feasible solution to problem (P2) will give an optimal solution to problem (P1).

Theorem 2: *Optimizing problem (P2) is equivalent to optimizing problem (P1) provided problem (P1) has a feasible solution.*

Proof: As problem (P1) has a feasible solution , by lemma 1,there exists a cfs to problem (P2).Thus by remark 1,an optimal solution to problem (P2) will be a cfs. Hence , by lemma 3,an optimal solution to problem (P1) can be obtained.

5.Algorithm:

Step 1 : Given a fractional capacitated transportation problem (P1) with enhanced flow, form a related transportation problem (P2) by introducing a dummy source and a dummy destination with $a'_i = \sum_{j\in J} u_{ij}; \forall i \in I$, $a'_{m+1} = \sum_{i\in I}\sum_{j\in J} u_{ij} - P = b'_{n+1}$, $b'_j = \sum_{i\in I} u_{ij}; \forall j \in J$,

$c'_{ij} = c_{ij}$, $\forall i\in I, j\in J$, $c'_{m+1,j} = c'_{i,n+1} = 0$ $\forall i\in I$, $\forall j\in J$, $c'_{m+1,n+1} = M$

$d'_{ij} = d_{ij}$, $\forall i\in I, j\in J$, $d'_{m+1,j} = d'_{i,n+1} = 0$ $\forall i\in I$, $\forall j\in J$ $d'_{m+1,n+1} = M$

Step 2: Find an initial basic feasible solution to (P2) with respect to variable cost only. Let B be its corresponding basis.

Step 3(a) : Calculate θ_{ij} ,$(c_{ij}-z^1_{ij})$, $(d_{ij} - z^2_{ij})$, N^0 , D^0 for all non basic cells such that

$u^1_i + v^1_j = c_{ij}$ $\forall (i, j) \in B$

$u^2_i + v^2_j = d_{ij}$ $\forall (i, j) \in B$

$u^1_i + v^1_j = z^1_{ij}$ $\forall (i, j) \in N_1$ and N_2

$u^2_i + v^2_j = z^2_{ij}$ $\forall (i, j) \in N_1$ and N_2

N^0 = value of $\sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij}$ at the current basic feasible solution corresponding to the basis B

D^0 = value of $\sum_{i\in I}\sum_{j\in J} d_{ij}x_{ij}$ at the current basic feasible solution corresponding to the basis B.

θ_{ij} = level at which a non basic cell (i,j) enters the basis replacing some basic cell of B.

N_1 and N_2 denotes the set of non basic cells (i,j) which are at their lower bounds and upper bounds respectively.

Note: $u^1_i, v^1_j, u^2_i, v^2_j$ are the dual variables which are determined by using the above equations and taking one of the u_i's or v_j's. as zero.

Step3(b): Calculate N°, D° where $N^{\circ} = \sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij}, D^{\circ} = \sum_{i\in I}\sum_{j\in J} d_{ij}x_{ij}$

Step3(c): Calculate A_{ij}^{1} and A_{ij}^{2} where $A_{ij}^{1} = \theta_{ij}(c_{ij} - z_{ij}^{1}); \forall (i,j) \notin B$ and $A_{ij}^{2} = \theta_{ij}(d_{ij} - z_{ij}^{2}); \forall (i,j) \notin B$.

Step 4(a): Find $\Delta_{ij} = D^{0}(c_{ij} - z_{ij}^{1}) - N^{0}(d_{ij} - z_{ij}^{2}); \forall (i,j) \notin B$

Step 4(b): Find $\delta_{ij}^{1}\ ; \forall (i,j) \in N_1$ and $\delta_{ij}^{2}\ ; \forall (i,j) \in N_2$ where

$$\delta_{ij}^{1} = \left[\frac{\theta_{ij}\Delta_{ij}}{D^{\circ}\left[D^{\circ} + A_{ij}^{2}\right]}\right]; \forall (i,j) \in N_1 \text{ and}$$

$$\delta_{ij}^{2} = \left[-\frac{\theta_{ij}\Delta_{ij}}{D^{\circ}\left[D^{\circ} - A_{ij}^{2}\right]}\right]; \forall (i,j) \in N_2$$

where N_1 and N_2 denotes the set of non basic cells (i,j) which are at their lower bounds and upper bounds respectively.

If $\delta_{ij}^{1} \geq 0; \forall (i,j) \in N_1$ and $\delta_{ij}^{2} \geq 0; \forall (i,j) \in N_2$ then the current solution so obtained is the optimal solution to (P2´) and subsequently to (P2).Then go to step (5). Otherwise some $(i,j) \in N_1$for which $\delta_{ij}^{1} < 0$ or some $(i,j) \in N_2$for which $\delta_{ij}^{2} < 0$ will enter the basis. Go to step 2.

Step 5: Find the optimal cost $Z = \frac{N^{0}}{D^{0}}$

6. Problem of the manager of ABC company

ABCcompany deals in selling cement bags from its two major distribution centres (i) at Mumbai and Chennai. The company has retail outlets (j) at Delhi, U.P and M.P. The actual cost of transporting one cement bag from Mumbai to Delhi, U.P and M.P is □ 2, 3 and 1 respectively. Similarly, the cost figures for Chennai are □ 1, 2 and 2 respectively. The standard cost of transporting one cement bag from Mumbai to Delhi , U.P and M.P are □ 1, 2 and 2 respectively while the cost figures for Chennai are □ 4 , 4 and 6 respectively. Mumbai and Chennai distribution centres can supply atleast 40 and 30 bags respectively. In emergency situations when there is an extra demand in the market , the company can increase its supply to 80 units. The manager of the company wishes to determine the number of bags to be supplied from each distribution centre to each retail outlet so that the ratio of total actual cost of transportation and total standard transportation cost is minimized and the total number of cement bags transported is 80 to meet the demand in the market. The minimum and maximum number of cement bags that can be supplied from each distribution centre to each retail outlet is also given below.

	Delhi	U.P	M.P
Mumbai	1 to 20	2 to 10	0 to 20

Chennai	0 to 10	2 to 20	1 to 30

Solution: In order to solve the problem of the manager of the company, we can formulate the above data as a fractional capacitated transportation problem (P1) with enhanced flow. The given data is summarized in the form of the table .Table 1 gives the values of $c_{ij,}d_{ij}$, a_i,b_j for i=1,2 and j=1,2,3

Table 1:Problem (P1)

	D_1	D_2	D_3	a_i
O_1	2 3	3 4	1 5	40
O_2	1 4	2 4	2 6	30
b_j	20	10	30	

Note: O_1 and O_2 are distribution centres(i) at Mumbai and Chennai. D_1 , D_2 and D_3 are the retail outlets (j) at Delhi, U.P and M.P respectively. Values in the upper left corners are c_{ij}'s which shows the actual cost of transporting one cement bag from the i^{th} distribution centre to j^{th} retail outlet.Values in lower left corners are d_{ij}'s that shows the standard cost of transporting one cement bag from the i^{th} distribution centre to j^{th} retail outlet where i=1,2.and j=1,2,3.

Let x_{ij} be the number of cement bags transported from the i^{th} distribution centre to the j^{th} retail outlet.

$1\leq x_{11} \leq 20$, $2 \leq x_{12} \leq 10$, $0 \leq x_{13} \leq 20$,$0\leq x_{21} \leq 10$, $2 \leq x_{22} \leq 20$, $1 \leq x_{23} \leq 30$

The enhanced flow is P = 80 where P =80> $\max\left(\sum_{i=1}^{2} a_i = 70, \sum_{j=1}^{3} b_j = 60\right)$. Introduce a dummy origin and a dummy destination in Table 1 with $c_{i4} = 0 = d_{i4}$ for all i = 1,2 and $c_{3j} = 0 = d_{3j}$ for all j = 1,2,3 . $c_{34} = d_{34} = M$ where M is a large positive number. Also we have $0\leq x_{14} \leq 10$, $0\leq x_{24} \leq 30$, $0 \leq x_{31} \leq 10$, $0 \leq x_{32} \leq 20$, $0 \leq x_{33} \leq 20$ .In this way , we form the problem (P2). Now we find an initial basic feasible solution of problem (P2) which is given in table 2 below.

Table 2: Initial basic feasible solution of problem (P2)

	D_1	D_2	D_3	D_4	a_i'	u_i^1	u_i^2
O_1	2 $\overline{20}$ 3	3 $\underline{2}$ 4	1 $\overline{20}$ 5	0 **8** 0	50	0	0
O_2	1 **10** 4	2 **8** 4	2 **20** 6	0 **22** 0	60	0	0
O_3	0 0	0 $\overline{20}$ 0	0 **10** 0	M M	30	-2	-6
b_j'	30	30	50	30			
v_j^1	1	2	2	0			
v_j^2	4	4	6	0			

Note: Entries of the form $\underline{a}$ and $\overline{b}$ represent non basic cells which are at their lower and upper bounds respectively. Entries in bold are basic cells.

$N^0 = 132$,$D^0 = 360$

Table 3: Computation of δ_{ij}^1 and δ_{ij}^2

NB	O_1D_1	O_1D_2	O_1D_3	O_3D_1	O_3D_2
θ_{ij}	0	6	2	10	10
$c_{ij} - z_{ij}^1$	1	1	-1	1	0
$d_{ij} - z_{ij}^2$	-1	0	-1	2	2
δ_{ij}^1		0.0166		0.007	
δ_{ij}^2	0		0.00349		0.0215

Since $\delta_{ij}^1 \geq 0; \forall$ (i,j) $\in N_1$ and $\delta_{ij}^2 \geq 0; \forall$(i,j) $\in N_2$, therefore the solution in table 2 is an optimal solution of problem (P2) and hence yields an optimal solution of (P1). Therefore, the distribution centre at Mumbai should sent 20 , 2 and 20 cement bags to retail outlets at Delhi, U.P and M.P respectively while the distribution centre at Chennai should transport 10, 8 and 20 cement bags to retail outlets at Delhi, U.P and M.P respectively. The minimum ratio of actual cost of transportation to the standard cost of transportation is $z = 132/360 = 0.367$

Conclusion: In order to solve a fractional capacitated transportation problem , a related transportation problem is formed and it is shown that the optimal solution to enhanced flow problem may be obtained from the optimal solution to the related transportation problem.

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