HANKEL TYPE TRANSFORMABLE DISTRIBUTION SPACES AND ITS CHARACTERIZATION

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ABSTRACT

In this paper, we have established new properties of distribution spaces of slow growth and of exponential growth that are Hankel transformable. We have also obtained representations of those generalized functions as initial values of solutions of the Kepinski type equation. Further we have analyzed Hankel positive definite functions and generalized functions. Finally characterizations of Hankel transformable distributions having bounded above or bounded below support on $(0,\infty)$ are obtained.

Keywords: Hankel type transformation, Kepinski type equation, distributions, positive definite, Hankel approximate identity.

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1. Introduction: The Hankel type transformation is defined

$$h_{\alpha,\beta}(\phi)(y)=\int_0^\infty (xy)^{\alpha+\beta}J_{\alpha-\beta}(xy)\phi(x)dx,\quad y\in(0,\infty),$$

where $J_{\alpha-\beta}$ represents the Bessel type function of the first kind and order $\alpha-\beta$. We will consider throughout this paper that $(\alpha-\beta)>-\frac{1}{2}$. Just like in Zemanian[23], we introduce the $H_{\alpha,\beta}$ that consists of all those complex valued and smooth functions ϕ on $(0,\infty)$ such that

$$\rho_{m,k}^{\alpha,\beta}(\phi)=\sup_{x\in(0,\infty)}(1+x^2)^m\left|\left(\frac{1}{x}D\right)^k\left(x^{2\beta-1}\phi(x)\right)\right|<\infty,$$

for every $m,k\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$. On $H_{\alpha,\beta}$ we consider the topology generated by the family $\{\rho_{m,k}^{\alpha,\beta}\}_{m,k\in\mathbb{N}_0}$ of seminorms. Thus $H_{\alpha,\beta}$ is a Frechet space. The Hankel type transformation is an automorphism of $H_{\alpha,\beta}$ (Refer to [23, Lemma 8]). We denote by $H'_{\alpha,\beta}$ the dual of $H_{\alpha,\beta}$ and elements of $H'_{\alpha,\beta}$ are distributions of slow growth. If $T\in H'_{\alpha,\beta}$ then the Hankel type transform $h'_{\alpha,\beta}T$ of T defined by

$$\langle h'_{\alpha,\beta}T,\phi\rangle=\langle T,h_{\alpha,\beta}\phi\rangle,\quad \phi\in H_{\alpha,\beta}.$$

Let $a>0$. Following Zemanian[24], we define the space $B_{\alpha,\beta,a}$ that consists of all those functions $\phi\in H_{\alpha,\beta}$ such that $\phi(x)=0, x>a$. $B_{\alpha,\beta,a}$ is closed subspace of $H_{\alpha,\beta}$. We can characterize Hankel type transform $h_{\alpha,\beta}(B_{\alpha,\beta,a})$ of $B_{\alpha,\beta,a}$ (Refer to [24, Theorem 1]). It is clear that $B_{\alpha,\beta,a}$ is continuously contained in $B_{\alpha,\beta,b}$ provided that $0<a<b$. The space $B_{\alpha,\beta}=\bigcup_{a>0}B_{\alpha,\beta,a}$ is endowed with the inductive topology. We can establish the topological properties of the spaces $H_{\alpha,\beta}$, $B_{\alpha,\beta,a}$ and their duals as in [1] and [2]. Waphare[21] studied the Hankel type transform of distributions of exponential growth. We introduce the space $M_{\alpha,\beta}$ that consists of all those complex valued and smooth functions ϕ defined on $(0,\infty)$ satisfying

$$\eta_{m,k}^{\alpha,\beta}(\phi)=\sup_{x\in(0,\infty)}e^{mx}\left|\left(\frac{1}{x}D\right)^k\left(x^{2\beta-1}\phi(x)\right)\right|<\infty,$$

for every $m,k \in \mathbb{N}_0$. $M_{\alpha,\beta}$ is a Frechet space when we consider in $M_{\alpha,\beta}$ the topology generated by $\{\eta_{m,k}^{\alpha,\beta}\}_{m,k\in \mathbb{N}_0}$. $M'_{\alpha,\beta}$ represents the dual space of $M_{\alpha,\beta}$ and the elements of $M'_{\alpha,\beta}$ are distributions of exponential growth.

By $Q_{\alpha,\beta}$ we denote the space of all those functions Φ verifying the following two conditions:

(i) $z^{2\beta-1}\Phi(z)$ is an even and entire function, and

(ii) for every $m,k \in \mathbb{N}_0$,

$$\omega_{m,k}^{\alpha,\beta}(\Phi) = \sup_{|\operatorname{Im} z|\le k} (1+|z|^2)^m |z^{2\beta-1}\Phi(z)| < \infty.$$

The topology of $Q_{\alpha,\beta}$ is the one generated by $\{\omega_{m,k}^{\alpha,\beta}\}_{m,k\in \mathbb{N}_0}$. In Waphare[21, Theorem 2.4], it is proved that the Hankel type transformation $h_{\alpha,\beta}$ is an isomorphism from $M_{\alpha,\beta}$ onto $Q_{\alpha,\beta}$. The $h_{\alpha,\beta}$-transformation is defined on the dual spaces $M'_{\alpha,\beta}$ and $Q'_{\alpha,\beta}$ as the transpose of $h_{\alpha,\beta}$ on $Q_{\alpha,\beta}$ and $M_{\alpha,\beta}$ respectively. Cholewinski[8], Haimo[15], Hirschman[16] have investigated a convolution operation for a version of the Hankel transformation closely connected to h_μ. After doing a change of variable by taking into account the results in [16], we can define a convolution of the Hankel type transformation $h_{\alpha,\beta}$. In particular if f and g are $L_1(x^{2\alpha}dx,(0,\infty))$ the Hankel type convolution $f\#g$ of f and g is defined as

$$(f\#g)(x) = \int_0^\infty f(y)(\tau_x g)(y)dy, \; x\in(0,\infty),$$

where the Hankel type translation $\tau_x, x\in(0,\infty)$ is given by

$$(\tau_x g)(y) = \int_0^\infty D_{\alpha,\beta}(x,y,z)g(z)dz, \; x,y\in(0,\infty),$$

and

$$D_{\alpha,\beta}(x,y,z) = \int_0^\infty t^{2\beta-1}(xt)^{\alpha+\beta}J_{\alpha-\beta}(xt)(yt)^{\alpha+\beta}J_{\alpha-\beta}(yt)(zt)^{\alpha+\beta}J_{\alpha-\beta}(zt)dt, \; x,y,z\in(0,\infty).$$

J.de Sousa-Pinto[19] started the investigation about the Hankel convolution in generalized functions. He defined the Hankel convolution of order $\mu=0$ on distributions of compact support on $(0,\infty)$. Betancor and Marrero([3],[4],[5],[6]), Marrero and Betancor[17], Betancor and Gonzalez[2], Waphare[21] and Betancor and Rodriguez-mesa[7] have studied the Hankel convolution on distribution spaces of slow growth and of exponential growth. In Marrero and Betancor [17, Proposition 2.1(i)], it is established that the Hankel translation τ_x, $x \in (0,\infty)$, defines a continuous mapping from H_μ into itself. The Hankel type convolution $T\#\phi$ of $T \in H'_{\alpha,\beta}$ and $\phi \in H_{\alpha,\beta}$ is defined as

$$(T\#\phi)(x) = \langle T, \tau_x \phi \rangle,\ x \in (0,\infty) \tag{1.1}$$

The Hankel type convolution is studied on $M'_{\alpha,\beta}$ in Waphare[21]. If $T \in M'_{\alpha,\beta}$ and $\phi \in M_{\alpha,\beta}$ the $\#$-convolution $T\#\phi$ of T and ϕ is also defined by (1.1).

In the subsequent section we represent the generalized functions in $H'_{\alpha,\beta}$ and $M'_{\alpha,\beta}$ as initial values of solutions of the Kepinski type equations[22, p.99]

$$\Delta_{\alpha,\beta,x} U = \frac{\partial}{\partial t} U,$$

where $\Delta_{\alpha,\beta,x} = x^{2\beta-1} D x^{4\alpha} D x^{2\beta-1}$.

Throughout this paper C will always represent a positive constant not necessarily the same in each occurrence.

2. Hankel type transformable generalized functions as initial values of solutions of Kepinski type equations :

In this section we obtain representations of the elements of $H'_{\alpha,\beta}$ and $M'_{\alpha,\beta}$ as the initial values of solutions of the Kepinski type equation [22,p.99].

We will denote by E the function defined by

$$E(x,t) = x^{2\alpha} (2t)^{-(3\alpha+\beta)} e^{-\frac{x^2}{4t}},\ x,t \in (0,\infty).$$

By an application of [12,(10),p.29] the following formula holds:

$$h_{\alpha,\beta}[E(x,t)](y) = e^{-ty^2},\ y,t \in (0,\infty) \tag{2.1}$$

Following result will be useful in the sequel.

Lemma 2.1: (i) If $\phi \in H_{\alpha,\beta}$ then

$$E(x,t)\#\phi \to \phi,\ as\ t \to 0^+,\ \text{in } H_{\alpha,\beta}. \tag{2.2}$$

(ii) If $\phi \in M_{\alpha,\beta}$ then

$$E(x,t)\#\phi \to \phi,\ as\ t \to 0^+,\ \text{in } M_{\alpha,\beta}. \tag{2.3}$$

Proof: (i) Note that $E(\cdot,t)\in H_{\alpha,\beta}$ for every $t\in(0,\infty)$. Then according to [17, Proposition 2.2(i)], $E(\cdot,t)\#\phi\in H_{\alpha,\beta}$ for each $t\in(0,\infty)$. Let $\phi\in H_{\alpha,\beta}$. By involving the interchange formula [17, (1.3)] and [23,Lemma 8, (2.2)] is equivalent to

$$x^{2\beta-1}h_{\alpha,\beta}(E(\cdot,t))h_{\alpha,\beta}(\phi)\to h_{\alpha,\beta}(\phi),\ as\ t\to 0^+ \text{ in } H_{\alpha,\beta}. \tag{2.4}$$

Write $\psi=h_{\alpha,\beta}(\phi)$. By (2.1) to see (2.4), we have to show that

$$e^{-ty^2}\psi(y)\to\psi(y),\ as\ t\to 0^+,\text{ in } H_{\alpha,\beta}.$$

Let $m,k\in\mathbb{N}$ and $\varepsilon>0$. By Leibniz rule, we have

$$(1+y^2)^m\left(\frac{1}{y}D\right)^k\left(y^{2\beta-1}\psi(y)(e^{-ty^2}-1)\right)=(1+y^2)^m\left(\frac{1}{y}D\right)^k\left(y^{2\beta-1}\psi(y)\right)(e^{-ty^2}-1)$$
$$+\sum_{j=0}^{k-1}\binom{k}{j}(1+y^2)^m\left(\frac{1}{y}D\right)^j\left(y^{2\beta-1}\psi(y)\right)(-2t)^{k-j}e^{-ty^2},\quad t,y\in(0,\infty). \tag{2.5}$$

We can infer that

$$\frac{1}{1+y^2}\left|e^{-ty^2}-1\right|\le\frac{2}{1+y^2},\quad t,y\in(0,\infty),$$

Thus there exists $y_0\in(0,\infty)$ such that for every $y>y_0$ and $t\in(0,\infty)$,

$$\frac{1}{1+y^2}\left|e^{-ty^2}-1\right|\le\varepsilon \tag{2.6}$$

Moreover, we can find $\delta>0$ for which

$$\frac{1}{1+y^2}\left|e^{-ty^2}-1\right|\le\varepsilon,\quad y\le y_0 \text{ and } 0<t<\delta \tag{2.7}$$

Now by combining (2.5), (2.6) and (2.7) we can obtain

$$\rho_{m,k}^{\alpha,\beta}(\psi)(e^{-ty^2}-1)\to 0,\ as\ t\to 0^+.$$

Thus proof of (i) is completed.

(ii) Let $\phi\in M_{\alpha,\beta}$. As $E(\cdot,t)\in M_{\alpha,\beta}$, for each $t\in(0,\infty)$ according to Waphare[21, Lemma 3.2], $E(\cdot,t)\#\phi\in M_{\alpha,\beta}$, for every $t\in(0,\infty)$.

To prove (2.3), by Waphare [21, Theorem 2.4], it is sufficient to prove that

$$e^{-ty^2}\psi(y)\to\psi(y),\ as\ t\to 0^+ \text{ in } Q_{\alpha,\beta} \tag{2.8}$$

where $\psi=h_{\alpha,\beta}(\phi)$

Let $m,k \in \mathbb{N}_0$ and $\varepsilon > 0$. We can write

$$\left|e^{-ty^2}-1\right|\left(1+|y|^2\right)^m\left|y^{2\beta-1}\psi(y)\right| \le C\left(1+|y|^2\right)^m\left|y^{2\beta-1}\psi(y)\right|\left(e^{-t(\operatorname{Re} y)^2}+1\right),\ 0<t<1,\ |\operatorname{Im} y|\le k.$$

Thus there exists $a>0$ such that, for every $y \in \mathbb{C}$ being $|y|\ge a$ and $|\operatorname{Im} y|\le k$,

$$\left|e^{-ty^2}-1\right|\left(1+|y|^2\right)^m\left|y^{2\beta-1}\psi(y)\right| \le \varepsilon,\ 0<t<1.$$

Moreover, we can find $t_0 \in (0,1)$ such that

$$\left|e^{-ty^2}-1\right|\left(1+|y|^2\right)^m\left|y^{2\beta-1}\psi(y)\right| \le \varepsilon,\ |\operatorname{Im} y|\le k, |y|\le a,\ 0<t<t_0.$$

Thus, if $0<t<t_0$, then $\omega_{m,k}^{\alpha,\beta}\left(\psi(e^{-ty^2}-1)\right)\le\varepsilon$.

Thus (2.8) is established. Thus proof is completed.

Theorem 2.1 (Characterization): Let $u \in H'_{\alpha,\beta}$. Define the function U by

$$U(x,t) = \left(u \# E(\cdot,t)\right)(x),\ x,t \in (0,\infty).$$

Then

(i) U is an infinitely differentiable function on $(0,\infty)\times(0,\infty)$ and

$$\Delta_{\alpha,\beta,x} U(x,t) = \frac{\partial}{\partial t} U(x,t),\ x,t\in(0,\infty) \tag{2.9}$$

(ii) For every $T\in(0,\infty)$ there exists $C>0$ and $r \in \mathbb{N}_0$ such that

$$|U(x,t)| \le C x^{2\alpha} t^{-(3\alpha+\beta+2r)}(1+x^2)^r,\ x\in(0,\infty) \text{ and } 0<t<T.$$

(iii) $U(x,t)\to u,\ as\ t\to 0^+$, in the weak* topology of $H'_{\alpha,\beta}$, that is

$$\langle u,\phi\rangle = \lim_{t\to 0^+}\int_0^\infty U(x,t)\phi(x)dx,\ \ \phi\in H_{\alpha,\beta}.$$

Conversely, if U is an infinitely differentiable function on $(0,\infty)\times(0,\infty)$ such that (i) and (ii) hold, then there exists a unique $u \in H'_{\alpha,\beta}$ for which

$$U(x,t) = \left(u \# E(\cdot,t)\right)(x),\ x,t \in (0,\infty).$$

Proof: Let $u \in H'_{\alpha,\beta}$. Since $E(\cdot,t)\in H_{\alpha,\beta}$, $t\in(0,\infty)$, by [17, Proposition 3.5] the function U defined by $U(x,t) = \left(u \# E(\cdot,t)\right)(x)$, $t,x\in(0,\infty)$, is infinitely differentiable and $x^{2\beta-1}U$ is a multiplier of $H_{\alpha,\beta}$ [2, Theorem 2.3]. Moreover by [17, (3.18)], we have

$$\int_0^\infty U(x,t)\phi(x)dx = \langle u, E(\cdot,t)\#\phi\rangle,\ \ \phi\in H_{\alpha,\beta} \text{ and } t\in(0,\infty).$$

Thus according to Lemma 2.1(i), (iii) holds.

To show that U satisfies (2.9), we take into account that by [17, Proposition 4.7(ii)],

$$\left(\frac{\partial}{\partial t}-\Delta_{\alpha,\beta,x}\right)U(x,t)=\left\langle u(y),\frac{\partial}{\partial t}\left(\tau_x E(\cdot,t)\right)(y)-\tau_x\left(\Delta_{\alpha,\beta,x}E(\cdot,t)\right)(y)\right\rangle$$

$$=\left(u\#\left(\frac{\partial}{\partial t}E(\cdot,t)-\Delta_{\alpha,\beta,x}E(\cdot,t)\right)\right)(x),\quad x,t\in(0,\infty).$$

As $\left(\frac{\partial}{\partial t}-\Delta_{\alpha,\beta,x}\right)E(\cdot,t)=0,\quad x,t\in(0,\infty)$, we conclude (i).

To prove (ii). As $u\in H'_{\alpha,\beta}$, there exists $C>0$ and $r\in\mathbb{N}_0$ such that

$$\left|\langle u,\phi\rangle\right|\le C\max_{0\le m,k\le r}\rho_{m,k}^{\alpha,\beta}(\phi),\quad \phi\in H_{\alpha,\beta}. \tag{2.10}$$

Let $k\in\mathbb{N}_0$ and $T\in(0,\infty)$. We can write

$$x^{2\beta-1}\Delta_{\alpha,\beta,x}^{k}=\sum_{i=0}^{k}a_{i,k}x^{2i}\left(\frac{1}{x}D\right)^{k+i}x^{2\beta-1},$$

where $a_{i,k},\ i=0,\cdots,k,$ are suitable real numbers. Then

$$x^{2\beta-1}\Delta_{\alpha,\beta,x}^{k}E(x,t)=\sum_{i=0}^{k}a_{i,k}(-1)^{k+i}x^{2i}(2t)^{-(3\alpha+\beta+k+i)}e^{-\frac{x^2}{4t}},\quad x,t\in(0,\infty),$$

and if $T\in(0,\infty)$,

$$x^{2\beta-1}\left|\Delta_{\alpha,\beta,x}^{k}E(x,t)\right|\le Ct^{-(3\alpha+\beta+2k)}(1+x^2)^{k}e^{-\frac{x^2}{4t}},\quad x\in(0,\infty)\ and\ 0<t<T.$$

Thus according to [17, Proposition 2.1(ii)] and [16, (2), p.310], we can infer that

$$\left|\Delta_{\alpha,\beta,y}^{k}\tau_x\left(E(\cdot,t)\right)(y)\right|\le\int_{|x-y|}^{x+y}D_{\alpha,\beta}(x,y,z)\left|\Delta_{\alpha,\beta,z}^{k}E(z,t)\right|dz$$

$$\le Ct^{-(3\alpha+\beta+2k)}\int_{|x-y|}^{x+y}D_{\alpha,\beta}(x,y,z)z^{2\alpha}(1+z^2)^{k}e^{-\frac{z^2}{4t}}dz$$

$$\le Ct^{-(3\alpha+\beta+2k)}(xy)^{2\alpha}e^{-\frac{(x-y)^2}{8t}}, \tag{2.11}$$

$x,y\in(0,\infty)$ and $0<t<T$.

From (2.10) and (2.11), we conclude that

$$|U(x,t)| \le C t^{-(3\alpha+\beta+2k)} x^{2\alpha} \sup_{0<y<\infty} (1+y^2)^r e^{-\frac{(x-y)^2}{8t}}$$
$$\le C t^{-(3\alpha+\beta+2k)} x^{2\alpha} (1+x^2)^r, \quad x\in(0,\infty) \ and \ 0<t<T.$$

Now to prove the converse, we proceed as in the proof of [11, Theorem 2.4].

Let $m \in \mathbb{N}$ and $T \in (0,\infty)$. Define the function f_m by

$$f_m(t)=0,\ t<0 \text{ and } f_m(t)=\frac{t^{m-1}}{\Gamma(m)},\ t\ge 0.$$

As it is well-known, we can write

$$\left(\frac{d}{dt}\right)^m v(t)=\delta(t)+w(t), \tag{2.12}$$

where v is an infinitely differentiable function on $\mathbb{R}$ such that $v(t)=f_m(t),\ t\le\frac{T}{4}$; and $v(t)=0,\ t\ge\frac{T}{2}$, and w is an infinitely differentiable function on $\mathbb{R}$ having its support contained in $\left[\frac{T}{4},\frac{T}{2}\right]$. Here as usual, δ denotes the Dirac functional.

Now we define

$$\tilde{U}(x,t)=\int_0^\infty U(x,t+s)v(s)ds,\quad 0<t<\frac{T}{2} \text{ and } x\in(0,\infty).$$

Thus $\tilde{U}$ is an infinitely differentiable function on $(0,\infty)\times\left(0,\frac{T}{2}\right)$. Moreover as U satisfies (ii) there exist $C>0$ and $r\in\mathbb{N}_0$ such that

$$|U(x,t)| \le C x^{2\alpha} t^{-(3\alpha+\beta+2r)} (1+x^2)^r, \quad x\in(0,\infty) \text{ and } 0<t<T.$$

Thus, if $m>(\alpha-\beta)+2r-1$ it follows

$$\left|\tilde{U}(x,t)\right| \le C x^{2\alpha}(1+x^2)^r \int_0^{\frac{T}{2}} (t+s)^{-(3\alpha+\beta+2r)} |v(s)| ds$$
$$\le C x^{2\alpha}(1+x^2)^r \int_0^{\frac{T}{2}} s^{-(\alpha-\beta-m+2r)} ds$$
$$\le C x^{2\alpha}(1+x^2)^r,\quad 0<t<\frac{T}{2} \ and \ x\in(0,\infty).$$

Note that it is also deduced that $\tilde{U}$ can be continuously extended to $(0,\infty)\times\left[0,\frac{T}{2}\right)$.

As $\left(\frac{\partial}{\partial t}-\Delta_{\alpha,\beta,x}\right)\tilde{U}(x,t)=0,\quad 0<t<\frac{T}{2}$ and $x\in(0,\infty)$ and by (2.12), we have

$$\left(-\Delta_{\alpha,\beta,x}\right)^m\tilde{U}(x,t)=\left(-\frac{\partial}{\partial t}\right)^m\tilde{U}(x,t)=U(x,t)+\int_0^\infty U(x,t+s)\,w(s)ds, \qquad (2.13)$$

for every $0<t<\frac{T}{2}$ and $x\in(0,\infty)$.

Now we introduce the function H defined by

$$H(x,t)=-\int_0^\infty U(x,t+s)\,w(s)ds,\quad 0<t<\frac{T}{2} \text{ and } x\in(0,\infty).$$

By proceeding as above we can see that

$$\left(\frac{\partial}{\partial t}-\Delta_{\alpha,\beta,x}\right)H(x,t)=0,$$

and

$$|H(x,t)|\le C\,x^{2\alpha}(1+x^2)^r,\qquad 0<t<\frac{T}{2},\ x\in(0,\infty).$$

Also H can be continuously extended to $(0,\infty)\times\left[0,\frac{T}{2}\right)$.

If we define $g(x)=\tilde{U}(x,0),\ x\in(0,\infty)$, and $h(x)=U(x,0),\ x\in(0,\infty)$ the uniqueness of solution implies that

$$\tilde{U}(x,t)=\left(g\#E(\cdot,t)\right)(x),\quad 0<t<\frac{T}{2} \text{ and } x\in(0,\infty),$$

and

$$H(x,t)=\left(h\#E(\cdot,t)\right)(x),\quad 0<t<\frac{T}{2} \text{ and } x\in(0,\infty).$$

Define

$$u=\left(-\Delta_{\alpha,\beta}\right)^m g+h.$$

It is clear that $u\in H'_{\alpha,\beta}$. Moreover by taking into account [17, Proposition 4.7(iii)] and (2.13), it infers

$$\left(u\#E(\cdot,t)\right)(x)=\left(\Delta_{\alpha,\beta}\right)^{m}\left(g\#E(\cdot,t)\right)(x)+\left(h\#E(\cdot,t)\right)(x)$$
$$=\left(\Delta_{\alpha,\beta}\right)^{m}\tilde{U}(x,t)+H(x,t)$$
$$=U(x,t),\quad 0<t<\frac{T}{2}\quad and\quad x\in(0,\infty).$$

By Lemma 2.1(i) we have

$$U(\cdot,t)=u\#E(\cdot,t)\to u,\ as\ t\to 0^{+},$$

in the weak* topology of $H'_{\alpha,\beta}$.

Hence u is the unique element of $H'_{\alpha,\beta}$ fulfilling

$$U(x,t)=\left(u\#E(\cdot,t)\right)(x),\quad x,t\in(0,\infty).$$

This completes the proof.

As a consequence of Theorem 2.1, we can obtain the following.

Corollary 2.1: If $u\in H'_{\alpha,\beta}$ then there exists $C>0,\ r,m\in\mathbb{N}_0$ and two continuous functions g and h such that

$$|h(x)|\le C\,x^{2\alpha}(1+x^2)^r,\quad x\in(0,\infty),$$

$$|g(x)|\le C\,x^{2\alpha}(1+x^2)^r,\quad x\in(0,\infty),$$

and for which $u=\Delta^m_{\alpha,\beta}g+h.$

Proof: Define the function by

$$U(x,t)=\left(u\#E(\cdot,t)\right)(x),\quad x,t\in(0,\infty).$$

Then by using proposition 2.1 of [17] we get desired representation for u. This completes the proof.

Theorem 2.2: Let $u\in M'_{\alpha,\beta}$. Define the function

$$U(x,t)=\left(u\#E(\cdot,t)\right)(x),\quad x,t\in(0,\infty).$$

Then

(i) U is an infinitely differentiable function on $(0,\infty)\times(0,\infty)$ and (2.9) holds.

(ii) For $T>0$ there exists $C>0$ and $r\in\mathbb{N}_0$ such that

$$|U(x,t)|\le C\,x^{2\alpha}t^{-(3\alpha+\beta+2r)}e^{-rx},\quad 0<t<T\ \text{and}\ x\in(0,\infty).$$

(iii) $U(\cdot,t)\to u,\ as\ t\to 0^{+}$, in the weak* topology of $M'_{\alpha,\beta}$.

Conversely, if U is an infinitely differentiable function on $(0,\infty)\times(0,\infty)$ such that (i) and (ii) hold, then there exists a unique $u \in M'_{\alpha,\beta}$ for which

$$U(x,t) = (u\#E(\cdot,t))(x), \;\; x,t \in (0,\infty).$$

Moreover, If $u \in M'_{\alpha,\beta}$ then there exists $C>0$, $r,m \in \mathbb{N}_0$ and two continuous functions g and h such that

$$|h(x)| \le C\, x^{2\alpha} e^{rx}, \;\; x \in (0,\infty),$$

$$|g(x)| \le C\, x^{2\alpha} e^{rx}, \;\; x \in (0,\infty),$$

and $u = \Delta^m_{\alpha,\beta} g + h.$

Proof: Proof follows by proceeding as in the proof of Theorem 2.1 and Corollary 2.1 and by using Lemma 2.1(ii) instead of Lemma 2.1(i) we can establish the corresponding result for the space $M'_{\alpha,\beta}$. Thus proof is completed.

3. Positive definite Hankel type transformable generalized functions:

The Bochner theorem for the Hankel transformation has investigated by Cholewinski, Haimo and Nussbaum[9] and Nussbaum[18]. Following [9] and [18] we say that a function $f \in x^{2\alpha} L_\infty(0,\infty)$ is positive definite provided that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i \bar{a}_j (\tau_{x_i} f)(x_j) \ge 0, \qquad (3.1)$$

for every $n \in \mathbb{N}_0$, $a_i \in \mathbb{C}$, $x_i \in (0,\infty)$, $i = 1,2,\ldots,n.$

With suitable change of variables, from the results in [9] it follows that if f is a positive definite function then there exists a positive measure λ on $[0,\infty)$ such that $\int_0^\infty x^{2\alpha} d\lambda(x) < \infty$ and

$$f(x) = \int_0^\infty (xy)^{\alpha+\beta} J_{\alpha-\beta}(xy)\, d\lambda(y) \quad a.\ e.,\ x \in (0,\infty).$$

Further if $u \in H'_{\alpha,\beta}$ (respectively, $M'_{\alpha,\beta}$) we say that u is a positive generalized function in $H'_{\alpha,\beta}$. (respectively in $M'_{\alpha,\beta}$) when

$$\langle u, \phi\#\bar{\phi}\rangle \ge 0, \;\; \phi \in H_{\alpha,\beta} \text{(respectively, } M_{\alpha,\beta}).$$

We notice that if u is positive definite generalized function in $H'_{\alpha,\beta}$ then u is also a positive generalized function in $M'_{\alpha,\beta}$.

The following result is a Hankel version of [10, Lemma 2.5](see [14, pp.153-155]).

Theorem 3.1: A positive definite continuous function is a positive definite generalized function in $H'_{\alpha,\beta}$ (and hence in $M'_{\alpha,\beta}$). Conversely if a continuous function in $x^{2\alpha}L_\infty(0,\infty)$ is a positive definite generalized function in $M'_{\alpha,\beta}$ then it is a positive definite function. Hence if a continuous function in $x^{2\alpha}L_\infty(0,\infty)$ is a positive generalized function in $H'_{\alpha,\beta}$ then it is a positive definite function.

Proof: Let f be a positive definite continuous function. Since $f\in x^{2\alpha}L_\infty(0,\infty)$, $f\in H'_{\alpha,\beta}$ and according to [17, Proposition 3.5], we can write

$$\left\langle f,\phi\#\bar{\phi}\right\rangle=\left\langle f\#\phi,\bar{\phi}\right\rangle=\int_0^\infty\int_0^\infty(\tau_x f)(y)\phi(x)\bar{\phi}(y)dxdy,\quad f\in H_{\alpha,\beta}.$$

Now, by writing each integral as a limit of sums, from (3.1) we deduce that $\left\langle f,\phi\#\bar{\phi}\right\rangle\geq 0$, for each $\phi\in H_{\alpha,\beta}$.

Now, let f be a continuous function that is a positive definite generalized function in $M'_{\alpha,\beta}$. We will prove that if λ is a finite measure which is concentrated on a bounded set of $(0,\infty)$ then

$$\int_0^\infty\int_0^\infty(\tau_x f)(y)d\lambda(x)d\bar{\lambda}(y)\geq 0.$$

This shows that f is a positive definite function.

Let $(\psi_n)_{n\in\mathbb{N}_0}$ be a Hankel approximate identity in the sense of [3]. That is there exists a sequence $(a_n)_{n\in\mathbb{N}_0}\subset(0,\infty)$ such that $a_n\downarrow 0$ as $n\to\infty$, and the following properties

(i) $\psi_n\in B_{\alpha,\beta,\mathbf{a_n}}$,

(ii) $\psi_n(x)\geq 0,\ x\in(0,\infty)$, and

(iii) $\int_0^\infty\phi_n(x)x^{2\alpha}dx=2^{\alpha-\beta}\Gamma(3\alpha+\beta)$,

hold for every $n\in\mathbb{N}_0$. One can easily note that if $(\psi_n)_{n\in\mathbb{N}_0}$ and $(\Psi_n)_{n\in\mathbb{N}_0}$ are Hankel approximate identities then $(\psi_n\#\Psi_n)_{n\in\mathbb{N}_0}$ also is a Hankel approximate identity.

Moreover if $(\psi_n)_{n\in\mathbb{N}_0}$ is a Hankel approximate identity and f is a continuous function on $(0,\infty)$ such that $f\in x^{2\alpha}L_\infty(0,\infty)$ then for every $x\in(0,\infty)$,

$$\int_0^\infty \left({}_x\psi_n\right)(y) f(y)dy \to f(x), \text{ as } n \to \infty. \tag{3.2}$$

According to [16, (2), p.310] we can write

$$\begin{aligned}\int_0^\infty \left({}_x\psi_n\right)(y) y^{2\alpha} dy &= \int_0^\infty y^{2\alpha} \int_0^\infty D_{\alpha,\beta}(x,y,z)\psi_n(z)dzdy \\ &= \int_0^\infty \psi_n(z) \int_0^\infty y^{2\alpha} D_{\alpha,\beta}(x,y,z)dydz \\ &= \frac{1}{2^{\alpha-\beta}\Gamma(3\alpha+\beta)} x^{2\alpha} \int_0^\infty \psi_n(z) z^{2\alpha} dz \\ &= x^{2\alpha},\end{aligned}$$

for every $n \in \mathbb{N}_0$.

Hence, for every $n \in \mathbb{N}_0$, one has

$$\int_0^\infty \left({}_x\psi_n\right)(y) f(y)dy - f(x) = \int_0^\infty \left({}_x\psi_n\right)(y) y^{2\alpha} \left(y^{2\beta-1} f(y) - x^{2\beta-1} f(x)\right) dy.$$

Let $\varepsilon > 0$. There exists $0 < \delta < x$ such that

$$\left|y^{2\beta-1} f(y) - x^{2\beta-1} f(x)\right| < \varepsilon$$

provided that $|x-y| < \delta$. Hence, since $f \in x^{2\alpha} L_\infty(0,\infty)$, we can find $n_0 \in \mathbb{N}$ such that

$$\begin{aligned}\left|\int_0^\infty \left({}_x\psi_n\right)(y) f(y)dy - f(x)\right| &\le C\left(\int_0^{x-\delta} + \int_{x+\delta}^\infty\right) \left({}_x\psi_n\right)(y) y^{2\alpha} dy + \varepsilon \\ &= \varepsilon, \quad n \in n_0.\end{aligned}$$

In the last equality we have taken into account that

$$\left({}_x\psi_n\right)(y) = \int_{|x-y|}^{x+y} D_{\alpha,\beta}(x,y,z)\psi_n(z)dz \le \int_\delta^\infty D_{\alpha,\beta}(x,y,z)\psi_n(z)dz,$$

$|x-y| > \delta$, and that $\psi_n \in B_{\alpha,\beta,\mathbf{a_n}}$, $n \in \mathbb{N}_0$, for some $\{a_n\}_{n\in\mathbb{N}_0} \subset (0,\infty)$ being $a_n \downarrow 0$ as $n \to \infty$.

Assume that λ is a finite measure and that it is concentrated on $(0,a)$. For every $n \in \mathbb{N}_0$, we define

$$\Psi_n(x) = \int_0^\infty \left({}_x\psi_n\right)(y) d\lambda y, \quad x \in (0,\infty).$$

Note that $\Psi_n(x) = 0$, $x > a + a_n$ and $n \in \mathbb{N}_0$. Indeed, let $n \in \mathbb{N}_0$. According to [16, (2), p.308] we have that

$$\Psi_n(x) = \int_0^a \int_{|x-y|}^{x+y} D_{\alpha,\beta}(x,y,z)\psi_n(z)dzd\lambda(y) = 0, \quad x > a + a_n.$$

Moreover, by [4,(1.2)] and [25, (7)] one has for every $x \in (0,\infty)$ and $n,k \in \mathbb{N}_0$,

$$\left(\frac{1}{x}D\right)^k \left(x^{2\beta-1}\Psi_n(x)\right) = \int_0^\infty h_{\alpha,\beta}\left((xt)^{-(\alpha-\beta)-k} J_{\alpha-\beta+k}(xt)h_{\alpha,\beta}\left(\tau^k_{\alpha,\beta}\psi_n\right)(t)\right)(y)d\lambda(y).$$

Thus for each $n,k \in \mathbb{N}_0$, since the function $z^{-(\alpha-\beta)}J_{\alpha-\beta}(z)$ is bounded on $(0,\infty)$, and by taking into account [25, Lemma 5.4-1 and Theorem 5.4-1], we can obtain

$$\sup_{x\in(0,\infty)}\left|\left(\frac{1}{x}D\right)^k \left(x^{2\beta-1}\Psi_n(x)\right)\right| < \infty.$$

Thus we conclude that for every $n \in \mathbb{N}_0$, $\Psi_n \in B_{\alpha,\beta}$, and then $\Psi_n \in M_{\alpha,\beta}$ and

$$\left\langle f, \Psi_n \# \overline{\Psi}_n \right\rangle \geq 0 \tag{3.3}$$

on the other side, $\lambda \in Q'_{\alpha,\beta,\#}$. Indeed for every $\phi \in H_{\alpha,\beta}$, we have

$$\begin{aligned}\left\langle h'_{\alpha,\beta}(\lambda),\phi\right\rangle &= \left\langle \lambda, h_{\alpha,\beta}(\phi)\right\rangle \\ &= \int_0^\infty h_{\alpha,\beta}(\phi)(x)d\lambda(x) \\ &= \int_0^\infty \phi(y)\int_0^\infty (xy)^{\alpha+\beta} J_{\alpha-\beta}(xy)d\lambda(x)dy.\end{aligned}$$

Hence

$$h'_{\alpha,\beta}(\lambda)(y) = \int_0^\infty (xy)^{\alpha+\beta} J_{\alpha-\beta}(xy)d\lambda(x), \quad y \in (0,\infty).$$

For $k \in \mathbb{N}_0$, we have

$$\left(\frac{1}{y}D\right)^k \left(y^{2\beta-1}h'_{\alpha,\beta}(\lambda)(y)\right) = \int_0^\infty x^{2k+2\alpha}(xy)^{-(\alpha-\beta)-k} J_{\alpha-\beta+k}(xy)d\lambda(x), \quad y \in (0,\infty).$$

As $z^{-(\alpha-\beta)}J_{\alpha-\beta}(z)$ is bounded on $(0,\infty)$ and λ is supported on a bounded set on $(0,\infty)$, it follows that

$$\sup_{y\in(0,\infty)}\left|\left(\frac{1}{y}D\right)^k \left(y^{2\beta-1}h'_{\alpha,\beta}(\lambda)(y)\right)\right| < \infty.$$

Therefore to prove that $y^{2\beta-1}h'_{\alpha,\beta}(\lambda)$ is a multiplier of $H_{\alpha,\beta}$ ([2, Theorem 2.3]). Thus from [17, Proposition 4.2] we deduce that $\lambda \in Q'_{\alpha,\beta,\#}$.

Now, it follows from [17, Proposition 4.7] that

$$\langle f, \Psi_n \# \overline{\Psi}_n \rangle = \langle f, (\lambda \# \psi_n) \# (\overline{\lambda} \# \psi_n) \rangle$$
$$= \langle f, (\lambda \# \overline{\lambda}) \# (\psi_n \# \psi_n) \rangle$$
$$= \langle f \# \lambda, \overline{\lambda} \# (\psi_n \# \psi_n) \rangle$$
$$= \int_0^\infty (f \# \lambda)(x) \int_0^\infty \tau_x (\psi_n \# \psi_n)(y) d\overline{\lambda}(y) dx$$
$$= \int_0^\infty \int_0^\infty \tau_x (\psi_n \# \psi_n)(x) (f \# \lambda)(x) dx d\overline{\lambda}(y), \quad n \in \mathbb{N}_0 .$$

But $f \# \lambda$ is continuous function on $(0, \infty)$, therefore we can write

$$\left| x^{2\beta-1} (f \# \lambda)(x) \right| \leq \int_0^\infty x^{2\beta-1} \left| (\tau_x f)(y) \right| d\left|\lambda\right|(y)$$
$$\leq \int_0^\infty y^{2\alpha} d\left|\lambda\right|(y), \qquad x \in (0, \infty).$$

Thus $f \# \lambda \in x^{2\alpha} L_\infty (0, \infty)$. Since $(\psi_n \# \psi_n)_{n \in \mathbb{N}_0}$ is a Hankel approximate identity, therefore by (3.2) and by using dominated convergence theorem we conclude that

$$\int_0^\infty \int_0^\infty \tau_y (\psi_n \# \psi_n)(x) (f \# \lambda)(x) dx d\overline{\lambda}(y) \rightarrow \int_0^\infty \int_0^\infty (\tau_x f)(y) d\lambda(x) d\overline{\lambda}(y), \text{ as } n \rightarrow \infty .$$

Then from (3.3), we can infer that

$$\int_0^\infty \int_0^\infty (\tau_x f)(y) d\lambda(x) d\overline{\lambda}(y) \geq 0.$$

This completes the proof.

Now we give a characterization of the positive definite generalized functions that involve the heat kernel function E.

Theorem 3.2(Characterization): Let $u \in H'_{\alpha,\beta}$. Then the following are equivalent.

(i) u is a positive definite generalized function in $H'_{\alpha,\beta}$.

(ii) The function $U(x,t) = (u \# E(\cdot, t))(x)$, $x \in (0, \infty)$, is a positive definite function, for every $t \in (0, \infty)$.

Proof : (i)$\Rightarrow$(ii): In view of Theorem 3.1, it is enough to prove that

$$\langle U(\cdot, t), \phi \# \overline{\phi} \rangle \geq 0, \quad \phi \in H_{\alpha,\beta} \text{ and } t \in (0, \infty).$$

By using [17, (1.3)] and [12, (10), p.29], one can easily obtain that

$$E(\cdot, t) = E\left(\cdot, \frac{t}{2}\right) \# E\left(\cdot, \frac{t}{2}\right), \quad t \in (0, \infty).$$

Then, as $E(\cdot,t)\in H_{\alpha,\beta}$, $t\in(0,\infty)$, [17, Proposition 3.5] leads to

$$\left\langle U(\cdot,t),\phi\#\overline{\phi}\right\rangle=\left\langle u\#E(\cdot,t),\phi\#\overline{\phi}\right\rangle$$

$$=\left\langle u,\left(\phi\#E\left(\cdot,\frac{t}{2}\right)\right)\#\overline{\left(\phi\#E\left(\cdot,\frac{t}{2}\right)\right)}\right\rangle\geq 0,\quad t\in(0,\infty)\ and\ \ \phi\in H_{\alpha,\beta},$$

because u is positive definite generalized function in $H'_{\alpha,\beta}$.

(ii)$\Rightarrow$(i): Let $\phi\in H_{\alpha,\beta}$. According to Theorem 3.1 we have

$$\left\langle U(\cdot,t),\phi\#\overline{\phi}\right\rangle\geq 0,\quad t\in(0,\infty).$$

Thus, [17, Proposition 2.2(i)] and Theorem 2.1(iii) imply that $\left\langle u,\phi\#\overline{\phi}\right\rangle\geq 0$. Thus (i) holds. This completes the proof.

In a similar way we can establish the corresponding property for $M'_{\alpha,\beta}$.

Theorem 3.3: Let $u\in M'_{\alpha,\beta}$. Then the following properties are equivalent.

(i) u is a positive definite generalized function in $M'_{\alpha,\beta}$.

(ii) The function $U(x,t)=(u\#E(\cdot,t))(x)$, $x\in(0,\infty)$, is a positive definite function for every $t\in(0,\infty)$.

An immediate consequence of Theorem 3.2 and 3.3 is the following corollary.

Corollary 3.1: Let $u\in H'_{\alpha,\beta}$. Then u is a positive definite generalized function in $H'_{\alpha,\beta}$ if and only if u is a positive definite generalized function in $M'_{\alpha,\beta}$.

4. Distributions in $H'_{\alpha,\beta}$ having Hankel type Transforms zero outside the interval $(0,a)$ or outside the interval (a,∞):

A Hankel version of the Paley-Wiener theorem was established in [3]. As in [3], we introduce the space $E_{\alpha,\beta}$ that consists of all those complex valued and smooth functions ϕ on $(0,\infty)$ such that, for every $k\in\mathbb{N}$

$$\lim_{x\to 0^+}\left(\frac{1}{x}D\right)^k\left(x^{2\beta-1}\phi(x)\right)\text{ exists.}$$

$E_{\alpha,\beta}$ is endowed with the topology generated by the family $\{\eta_{m,k}^{\alpha,\beta}\}_{m\in\mathbb{N}_0-0,\,k\in\mathbb{N}}$ of seminorms, where

$$\eta_{m,k}^{\alpha,\beta}(\phi)=\sup_{x\in(0,m)}\left|\left(\frac{1}{x}D\right)^k \left(x^{2\beta-1}\phi(x)\right)\right|,\quad \phi\in E_{\alpha,\beta},$$

for every $m\in\mathbb{N}_0-0$ and $k\in\mathbb{N}_0$.

We say that a functional $T\in H'_{\alpha,\beta}$ is in $E'_{\alpha,\beta}$, the dual space of $E_{\alpha,\beta}$ if and only if there exists $a=a(T)>0$ such that $\langle T,\phi\rangle=0$ for every $\phi\in E_{\alpha,\beta}$ being $\phi(t)=0,\ \ t<b,$ for some $b>a$ [3, Proposition 4.4]. Moreover the elements of $E'_{\alpha,\beta}$ were characterized as follows:

A functional $T\in H'_{\alpha,\beta}$ is in $E'_{\alpha,\beta}$ if and only if the Hankel transform $F=h'_{\alpha,\beta}T$ of T satisfies the following:

(i) $z^{2\beta-1}F(z)$ is an even and entire function, and

(ii) there exists $C,\ A>0$ and $r\in\mathbb{N}_0$ such that

$$\left|z^{2\beta-1}F(z)\right|\le C\left(1+|z|\right)^r e^{A|\mathrm{Im}\,z|},\ z\in\mathbb{C}.$$

In this section, we obtain a new characterization of the elements $T\in H'_{\alpha,\beta}$ that are also in $E'_{\alpha,\beta}$. Also we characterize the functionals in $H'_{\alpha,\beta}$ that are, for some $a>0$, zero inside the interval (a,∞), that is those $T\in H'_{\alpha,\beta}$ such that $\langle T,\phi\rangle=0,\ \ \phi\in B_{\alpha,\beta,a}$.

Theorem 4.1: Let $T\in H'_{\alpha,\beta}$ and $a>0$. Then following are equivalent:

(i) $\langle h'_{\alpha,\beta}T,\phi\rangle=0$ for every $\phi\in H_{\alpha,\beta}$ such that $supp\,\phi\subset(a,\infty)$,

(ii) $\lim_{k\to\infty}R^{-2k}\Delta_{\alpha,\beta}^k T=0$, in the weak* topology of $H'_{\alpha,\beta}$, for every $R>a$.

Proof: (i)$\Rightarrow$(ii): By [25, Lemma 5.4-1,(6) and Theorem 5.4-1], we note that for every $R>0$, the sequence $\{R^{-2k}\Delta_{\alpha,\beta}^k T\}_{k\in\mathbb{N}_0}$ converges to zero in the weak* topology of $H'_{\alpha,\beta}$ if and only if, the sequence $\{R^{-2k}x^{2k}h'_{\alpha,\beta}T\}_{k\in\mathbb{N}_0}$ converges to zero in the weak* topology in $H'_{\alpha,\beta}$.

Let $a<\varepsilon<\eta<R.$ Define $\psi\in C^\infty(0,\infty)$ such that $\psi(y)=1,\ \ y\in(0,\varepsilon),$ and $\psi(y)=0,\ \ y\in(\eta,\infty).$ Then $h'_{\alpha,\beta}T=\psi\, h'_{\alpha,\beta}T.$

Let now $m \in \mathbb{N}_0$ and $\phi \in H_{\alpha,\beta}$. Lebniz rule leads to

$$\left(\frac{1}{y}D\right)^m \left(y^{2\beta-1}x^{2k}R^{-2k}\phi(y)\psi(y)\right) = R^{-2k}\sum_{j=0}^{m}\binom{m}{j}2k(2k-2)\cdots(2k-2j+2)y^{2(k-j)}\left(\frac{1}{y}D\right)^{m-j}$$

$$\times\left(y^{2\beta-1}\phi(y)\psi(y)\right),\ k\in\mathbb{N}_0 \text{ and } y\in(0,\infty).$$

Hence, since $\psi(y)=0,\ y>\eta,$ where $\eta<R,$ we conclude that

$$\rho_{n,m}^{\alpha,\beta}\left(y^{2k}R^{-2k}\phi\psi\right)\to 0, \text{as } k\to\infty,$$

for every $\phi\in H_{\alpha,\beta}$, and then

$$\left\langle R^{-2k}y^{2k}h'_{\alpha,\beta}(T),\phi\right\rangle = \left\langle h'_{\alpha,\beta}(T),R^{-2k}y^{2k}\phi\psi\right\rangle\to 0, \text{ as } k\to\infty,$$

for every $\phi\in H_{\alpha,\beta}$. Thus we have proved (ii).

(ii)$\Rightarrow$(i): Let $R>a$ and $\phi\in H_{\alpha,\beta}$ such that $\phi(x)=0,\ x\le a+\varepsilon,$ for some $\varepsilon>0.$ Since $R^{-2k}\Delta_{\alpha,\beta}^{k}T\to 0,$ as $k\to\infty$, in the weak* topology of $H'_{\alpha,\beta}$, [25, Theorem 5.4-1] implies that the sequence $\left\{R^{-2k}y^{2k}h'_{\alpha,\beta}T\right\}_{k\in\mathbb{N}_0}$ is weakly* (or, equivalently, strongly) bounded in $H'_{\alpha,\beta}$.

Moreover, for every $k,l,m\in\mathbb{N}_0$ we can write that

$$\left(1+x^2\right)^l\left(\frac{1}{x}D\right)^m\left(x^{2\beta-1}x^{-2k}R^{2k}\phi(x)\right) = R^{2k}\sum_{j=0}^{m}c_j(k)\left(1+x^2\right)^l\left(\frac{1}{x}D\right)^j\left(x^{2\beta-1}\phi(x)\right)x^{-2(k+m-j)},\ x\in(0,\infty),$$

where $c_j(k)$ is a polynomial in k, for every $j=0,\cdots,m.$

Hence it follows

$$\left|\left(1+x^2\right)^l\left(\frac{1}{x}D\right)^m\left(x^{2\beta-1}x^{-2k}R^{2k}\phi(x)\right)\right| \le C\sum_{j=0}^{m}\left(\frac{R}{x}\right)^{2k}\left|c_j(k)\right|\left|\left(1+x^2\right)^l\left(\frac{1}{x}D\right)^j\left(x^{2\beta-1}\phi(x)\right)\right|$$

$$\le C\sum_{j=0}^{m}\left(\frac{R}{a+\varepsilon}\right)^{2k}\left|c_j(k)\right|\rho_{l,j}^{\alpha,\beta}(\phi),\ x\in(0,\infty)$$

$x\in(0,\infty)$ and $k,l,m\in\mathbb{N}_0$

Thus we conclude that $R^{2k}x^{-2k}\phi\to 0,$ as $k\to\infty$ in $H_{\alpha,\beta}$, provided that $R<a+\varepsilon.$

Hence, for each $a<R<a+\varepsilon,$

$$\lim_{k\to\infty}\left\langle R^{-2k}x^{2k}h'_{\alpha,\beta}T,R^{2k}x^{-2k}\phi\right\rangle=0.$$

Hence we prove that $\left\langle h'_{\alpha,\beta}T,\phi\right\rangle=0.$ This completes the proof.

Now, we prove the following result which can be seen as a dual version of Theorem 4.1.

Theorem 4.2: Let $T \in H'_{\alpha,\beta}$ and $a > 0$. The following are equivalent:

(i) $\langle h'_{\alpha,\beta} T, \phi \rangle = 0$, for every $\phi \in B_{\alpha,\beta,b}$ with $b < a$.

(ii) There exists a unique sequence $(L_k)_{k\in\mathbb{N}_0}$ in $H'_{\alpha,\beta}$ such that $L_0 = T$, $\Delta_{\alpha,\beta} L_{k+1} = L_k$, $k \in \mathbb{N}_0$ and $\lim_{k\to\infty} R^{-2k} h'_{\alpha,\beta} L_k = 0$ in the weak* topology of $H'_{\alpha,\beta}$ for every $R > \frac{1}{a}$.

Proof: (i) $\Rightarrow$(ii): For every $k \in \mathbb{N}_0$, define

$$L_k = h'_{\alpha,\beta}\left((-x^2)^{-k} h'_{\alpha,\beta} T\right) \tag{4.1}$$

Note that, since $h'_{\alpha,\beta} T$ vanishes in $(0,a)$, $L_k \in H'_{\alpha,\beta}$ for each $k \in \mathbb{N}_0$. Let $R > \frac{1}{a}$ and $\frac{1}{R} < \delta < a$. Now choose a function $\psi \in C^\infty(0,\infty)$ such that $\psi(x) = 0,\ 0 < x < \delta$, and $\phi(x) = 1,\ x > \frac{a+\delta}{2}$. From (i) it follows that

$$h'_{\alpha,\beta} T = \psi\, h'_{\alpha,\beta} T.$$

Moreover, for every $\phi \in H_{\alpha,\beta}$,

$$\langle R^{-2k} h'_{\alpha,\beta} L_k, \phi \rangle = \langle (-1)^k (Rx)^{-2k} h'_{\alpha,\beta} T, \phi(x) \rangle$$
$$= \langle h'_{\alpha,\beta} T, (-1)^k (Rx)^{-2k} \phi(x)\psi(x) \rangle.$$

From [2, Theorem 2.3], as ψ is a multiplier of $H_{\alpha,\beta}$ therefore $\phi\psi \in H_{\alpha,\beta}$. By taking into account that $\psi(x) = 0,\ 0 < x < \delta$, being $R\delta > 1$, by proceeding as in the proof of part (i) $\Rightarrow$(ii) in Theorem 4.1, we can obtain

$$\langle R^{-2k} h'_{\alpha,\beta} L_k, \phi \rangle \to 0, \text{ as } k \to \infty.$$

We now prove that $(L_k)_{k\in\mathbb{N}_0}$, where L_k is defined by (4.1) for every $k \in \mathbb{N}_0$, is the unique sequence satisfying the conditions in (ii).

Assume that, for every $k \in \mathbb{N}_0$, $\mathrm{L}_k \in H'_{\alpha,\beta}$, being $\mathrm{L}_0 = 0$, $\Delta_{\alpha,\beta} \mathrm{L}_{k+1} = \mathrm{L}_k$, $k \in \mathbb{N}_0$ and $\lim_{k\to\infty} R^{-2k} \mathrm{L}_k = 0$, in the weak* topology of $H'_{\alpha,\beta}$, for every $R > \frac{1}{a}$.

Now define the $H'_{\alpha,\beta}$-valued function F by

$$F(\lambda) = \sum_{k=0}^{\infty} \lambda^{2k} \mathrm{L}_{k+1} \tag{4.2}$$

that is holomorphic in $|\lambda| < a.$ It is easy to see that

$$\Delta_{\alpha,\beta,x} F(\lambda) = \sum_{k=0}^{\infty} \lambda^{2k} \mathrm{L}_k = \lambda^2 F(\lambda), \quad |\lambda| < a.$$

Then according to [25, Theorem 5.5-2, (8)] it follows

$$-x^2 h'_{\alpha,\beta} F(\lambda) = \lambda^2 h'_{\alpha,\beta} F(\lambda), \quad |\lambda| < a. \tag{4.3}$$

Since, for every $\lambda \neq 0$, the function $f(x) = x^2 + \lambda^2$, $x \in (0,\infty)$, is a multiplier in $H_{\alpha,\beta}$ (see [25, Lemma 5.3-1]). From (4.3) we can infer that

$$\langle h'_{\alpha,\beta} F(\lambda), \phi \rangle = \left\langle (x^2 + \lambda^2) h'_{\alpha,\beta} F(\lambda), \frac{\phi}{x^2 + \lambda^2} \right\rangle = 0,$$

$\phi \in H_{\alpha,\beta}$ and $0 < \lambda < a.$

Hence $F(\lambda) = 0$, $0 < |\lambda| < a.$ From the representation (4.2) we conclude that $\mathrm{L}_k = 0$, $k \in \mathbb{N}_0$.

Hence the uniqueness of the sequence $(\mathrm{L}_k)_{k \in \mathbb{N}_0}$ satisfying the properties in (ii) is established.

(ii) $\Rightarrow$(i): Let $(L_k)_{k \in \mathbb{N}_0}$ be a sequence in $H'_{\alpha,\beta}$ satisfying the properties in (ii). Let $\phi \in B_{\alpha,\beta,b}$, where $0 < b < a$, and let $R \in \left(\frac{1}{a}, \frac{1}{b} \right)$. we have

$$\begin{aligned} \langle h'_{\alpha,\beta} T, \phi \rangle &= \langle h'_{\alpha,\beta} (\Delta^k_{\alpha,\beta} L_k), \phi \rangle \\ &= \langle (-x^2)^k h'_{\alpha,\beta} L_k, \phi \rangle \\ &= \langle R^{-2k} h'_{\alpha,\beta} L_k, (-1)^k (xR)^{2k} \phi \rangle, \quad k \in \mathbb{N}_0 \end{aligned}$$

Since $\phi(x) = 0$, $x \geq b$, being $Rb < 1$, $(xR)^{2k} \phi \to 0$, as $k \to \infty$, in $H_{\alpha,\beta}$. Hence by taking into account that $R^{-2k} h'_{\alpha,\beta} L_k \to 0$, as $k \to \infty$, in the weak* topology of $H'_{\alpha,\beta}$, it concludes that $\langle h'_{\alpha,\beta} T, \phi \rangle = 0.$ Thus proof is completed.

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