# SOME NEW FAMILIES OF STRONGLY PRIME GRAPHS

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**ABSTRACT:**

A graph $G = (V, E)$ with $n$ vertices is said to admit prime labeling if its vertices can be labeled with distinct positive integers not exceeding $n$ such that the label of each pair of adjacent vertices are relatively prime. A graph $G$ which admits prime labeling is called a prime graph and a graph $G$ is said to be a strongly prime graph if for any vertex $v$ of $G$ there exists a prime labeling $f$ satisfying $f(v) = 1$. In the present work we investigate some classes of graphs and subdivision of some classes of graphs which admit strongly prime labeling.

**Keywords:** Graph labeling, prime labeling, prime graph, strongly prime graph.

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**1 Introduction:**

In this paper, We consider only simple, finite, undirected and non trivial graph $G=(V(G),E(G))$ with vertex set $V(G)$ and edge set $E(G)$. The set of vertices adjacent to a vertex $u$ of $G$ is denoted by $N(u)$. For notations and terminology we refer to Bondy and Murthy [1].Two integers a and b are said to be relatively prime if their greatest common divisor is 1. Relatively prime numbers play an important role in both analytic and algebraic number theory. The notion of a prime labeling was introduced by Roger Etringer and was discussed in a paper by Tout.A [8].

Many researchers have studied prime graph. For example Fu.H [3] have proved that path $P_n$ on $n$ vertices is a prime graph. Deresky.T [2] have proved that the cycle $C_n$ with $n$ vertices is a prime graph. Lee.S [5] have proved that wheel $W_n$ is a prime graph iff $n$ is even. Around 1980 Roger Etringer conjectured that all trees having prime labeling which is not settled till today. In [6] S.Meena and K.Vaithiligam have investigated the Prime labeling for some helm related graph.

In [9] S.K.Vaidya and Udayan M.Prajapati have introduced Strongly prime graph and has proved $C_n$, $P_n$ and $K_{1,n}$ are strongly prime graphs and $W_n$ is a strongly prime graph for every even integer $n \geq 4$, in Some new results on prime graph. In [7] Sharon Philomena. V and K. Thirusangu have investigated Square and cube difference labeling of cycle cactus, special tree and a new key graphs . Graph labeling can also be applied in areas such as communication network, mobile telecommunications and medical field. Latest Dynamic survey on graph labeling we refer to Gallian [4]. Vast amount of literature is available on different types of graph labeling. More than 1000 research papers have been published so far in last four decades. We give a brief summary of definitions which are useful for this paper.

**Definition 1.1:** If the vertices of the graph are assigned values subject to certain conditions then it is known as graph labeling.

**Definition 1.2.** Let $G=(V(G),E(G))$ be a graph with $p$ vertices. A bijection $f : V(G) \to \{1,2,\ldots\ldots p\}$ is called a prime labeling if for each edge $e=uv$, $\gcd\{f(u), f(v)\}=1$. A graph which admits prime labeling is called a prime graph.

**Definition 1.3.** A graph G is said to be a strongly prime graph if for any vertex $v$ of G there exists a prime labeling $f$ satisfying $f(v)=1$.

**Definition 1.4.** A key graph is a graph obtained from $K_2$ by appending one vertex of $C_5$ to one end point and Hoffman tree $P_n \odot K_1$ to the other end point of $K_2$.

**Definition 1.5.**Let $e=uv$ be an edge of a graph $G$ and $w$ is not a vertex of $G$. Then edge $e$ is said to be subdivided when it is replaced by edges $e'=uw$ and $e''=wv$.

**Definition 1.6.** If every edge of graph G is subdivided, then the resulting graph is called barycentric subdivision of graph G .

**Definition 1.7.** The Crown graph $C_n^*$ is obtained from a cycle $C_n$ by attaching a pendent edge at each vertex of the $n$-cycle.

**Definition 1.8.** The graph $P_n \odot K_1$ is called a comb $C_{bn}$.

## 2. Strongly Prime Labeling Of Some Graphs

### Theorem 2.1:

The key graph $C_5 \odot P_n$ is a strongly prime graph for all integer $n \geq 1$.

### Proof:

Let $G$ be the key graph $C_5 \odot P_n$ with vertex set $V(G)=\{w_1,w_2,..w_5,v_1,v_2,...v_n,u_1,u_2,...u_n\}$ and the edge set

$E(G)=\{w_iw_{i+1} / 1\leq i\leq 4\}\cup\{v_iv_{i+1} / 1\leq i\leq n-1\}\cup\{v_iu_i / 1\leq i\leq n\}\cup\{w_1w_5\}\cup\{w_1v_1\}$ .Here $|V(G)|=2n+5$.

Let $a$ be the vertex for which we assign label 1 in our labeling method. Then we have the following cases:

**Case (i):** When $a$ is any arbitrary vertex of $C_5$.

Let $a=w_j$ for some $j\in\{1,2,...,5\}$ then the function $f:V(G)\rightarrow\{1,2,...,2n+5\}$ defined by

$$f(w_i)=\begin{cases}5+i-j+1 & if\ i=1,2,...,j-1;\\ i-j+1 & if\ i=j,j+1,...,5;\end{cases}$$

$$f(v_i)=5+2i \qquad if\ i=1,2,...,n;$$

$f(u_i) = 5 + 2i - 1 \qquad if\ i = 1, 2, ..., n;$

is a prime labeling for $C_5 \odot P_n$ with $f(a) = f(w_j) = 1$.

Thus $f$ is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of $a = w_j$ in $C_5 \odot P_n$.

**Case (ii):** When $a$ is any arbitrary vertex of Hoffman graph.

**Subcase (i):**

Let $a = v_j$ for some $j \in \{1, 2, ..., n\}$ then the function $f : V(G) \rightarrow \{1, 2, ..., 2n+5\}$ defined by

$f(w_i) = i + 2 \qquad if\ i = 3, 4, 5;$

$f(w_1) = 4,\ f(w_2) = 3;$

$$f(v_i) = \begin{cases} 2(n+i-j)+7 & if\ i = 1, 2, ..., j-1; \\ 2(i-j)+7 & if\ i = j+1, j+2, ..., n; \end{cases}$$

$f(v_j) = 1;$

$$f(u_i) = \begin{cases} 2(n+i-j)+6 & if\ i = 1, 2, ..., j-1; \\ 2(i-j)+6 & if\ i = j+1, j+2, ..., n; \end{cases}$$

$f(u_j) = 2;$

is a prime labeling for $C_5 \odot P_n$ with $f(a) = f(v_j) = 1$. Thus $f$ is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of $a = v_j$ in $C_5 \odot P_n$.

**Subcase (ii):**

Let $a = u_j$ for some $j \in \{1, 2, ..., n\}$ then define a labeling $f_2$ using the labeling $f$ defined in subcase (i) as follows: $f_2(u_j) = f(v_j),\ f_2(v_j) = f(u_j)$ for $j \in \{1, 2, ..., n\}$ and $f_2(v) = f(v)$ for all the remaining vertices. Then the resulting labeling $f_2$ is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex $a = u_j$ in $C_5 \odot P_n$ graph. Thus from all the cases described above $G$ is a strongly prime graph.

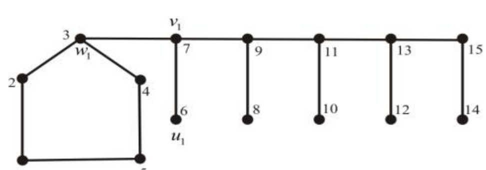


**Figure 1. A prime labeling of $C_5 \odot P_n$ having $w_4$ as label 1**

**Theorem 2.2:**

The graph $G$ obtained by attaching $K_{1,3}$ at each vertex of a cycle $C_n$ is a strongly prime graph for all integers $n \geq 3$.

**Proof:**

Let $C_n$ be the cycle $u_1, u_2, ..., u_n, u_1$. Let $v_i x_i y_i z_i$ be the vertices of i[th] copy of $K_{1,3}$ in which $v_i$ is the central vertex. Identify $z_i$ with $u_i, 1 \leq i \leq n$. Let the resultant graph be $G$. Now the vertex set $V(G) = \{u_1, u_2, ... u_n, v_1, v_2, ... v_n, x_1, x_2, ... x_n, y_1, y_2, ... y_n\}$ and the edge set $E(G) = \{u_i u_{i+1} / 1 \leq i \leq n-1\} \cup \{u_n u_1\} \cup \{u_i v_i, x_i v_i, y_i v_i / 1 \leq i \leq n\}$, here $|V(G)| = 4n$.

Let $a$ be the vertex for which we assign label 1 in our labeling method. Then we have the following cases:

**Case (i):**

Let $a = u_j$ for some $j \in \{1, 2, ..., n\}$ then the function $f : V(G) \to \{1, 2, ..., 4n\}$ defined by

$$f(u_i) = \begin{cases} 4n + 4i - 4j + 1 & if \ i = 1, 2, ... j-1; \\ 4i - 4j + 1 & if \ i = j, j+1, ... n; \end{cases}$$

$$f(v_i) = \begin{cases} 4n + 4i - 4j + 3 & if \ i = 1, 2, ... j-1; \\ 4i - 4j + 3 & if \ i = j, j+1, ... n; \end{cases}$$

$$f(x_i) = \begin{cases} 4n + 4i - 4j + 2 & if \ i = 1, 2, ... j-1; \\ 4i - 4j + 2 & if \ i = j, j+1, ... n; \end{cases}$$

$$f(y_i) = \begin{cases} 4n + 4i - 4j + 4 & if \ i = 1, 2, ... j-1; \\ 4i - 4j + 4 & if \ i = j, j+1, ... n; \end{cases}$$

is a prime labeling for $G$ with $f(a) = f(u_j) = 1$. Thus $f$ is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of $a = u_j$ in $G$.

**Case (ii):**

Let $a = x_j$ for some $j \in \{1, 2, ..., n\}$ then define a labeling $f_2$ using the labeling $f$ defined in case (i) as follows: $f_2(u_j) = f(x_j), f_2(x_j) = f(u_j)$ for $j = 1, 2, ..., n$ and $f_2(v) = f(v)$ for all the remaining vertices. Then the resulting labeling $f_2$ is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex $a = x_j$ in $G$.

**Case (iii):**

Let $a = y_j$ for some $j \in \{1, 2, ..., n\}$ then define a labeling $f_3$ using the labeling $f$ defined in case (i) as follows: $f_3(u_j) = f(y_j), f_3(y_j) = f(u_j)$ for $j = 1, 2, ..., n$ and $f_3(v) = f(v)$ for all the remaining vertices. Then the resulting labeling $f_3$ is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex $a = y_j$ in $G$.

**Case (iv):**

Let $a = v_j$ for some $j \in \{1, 2, ..., n\}$ then define a labeling $f_4$ using the labeling $f$ defined in case (ii) as follows: $f_4(v_j) = f_2(x_j), f_4(x_j) = f_2(v_j)$ for $j = 1, 2, ..., n$ and $f_4(v) = f_2(v)$ for all the remaining vertices. Then the resulting labeling $f_4$ is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex $a = v_j$ in $G$. Thus from all the cases described above $G$ is a strongly prime graph.

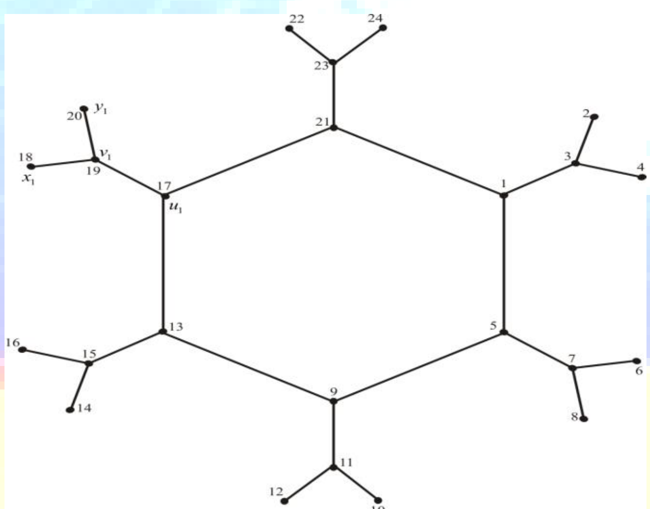


**Figure 2. A prime labeling of a graph $G$ obtained by attaching $K_{1,3}$ at each vertex of a cycle $C_n$ having $u_3$ as label 1**

## 3. Strongly Prime Labeling Of Some Subdivision Of Graphs:

### Theorem 3.1:

The graph $G$ obtained from subdividing the pendant edges of the crown graph $C_n^*$ is a strongly prime graph for all integers $n \geq 2$.

### Proof:

Let $C_n^*$ be the crown graph with vertices $v_1, v_2, \ldots v_n, u_1, u_2, \ldots u_n$. Let $w_1, w_2, \ldots w_n$ be the corresponding new vertices which is subdividing the pendant edges of $C_n^*$. Then the resulting graph be $G$. Now the vertex set $V(G) = \{u_i, v_i, w_i / 1 \le i \le n\}$ and the edge set $E(G) = \{u_i w_i, w_i v_i / 1 \le i \le n\} \cup \{u_i u_{i+1} / 1 \le i \le n-1\} \cup \{u_n u_1\}$. Here $|V(G)| = 3n$.

Let $a$ be the vertex for which we assign label 1 in our labeling method. Then we have the following cases:

**Case (i):**

Let $a = u_j$ for some $j \in \{1, 2, \ldots, n\}$ then define the function $f : V(G) \to \{1, 2, \ldots, 3n\}$ defined by

$$f(u_i) = \begin{cases} 3n + 3i - 3j + 1 & \text{if } i = 1, 2, \ldots j-1; \\ 3i - 3j + 1 & \text{if } i = j, j+1, \ldots n; \end{cases}$$

$$f(v_i) = \begin{cases} 3n + 3i - 3j + 3 & \text{if } i = 1, 2, \ldots j-1; \\ 3i - 3j + 3 & \text{if } i = j, j+1, \ldots n; \end{cases}$$

$$f(w_i) = \begin{cases} 3n + 3i - 3j + 2 & \text{if } i = 1, 2, \ldots j-1; \\ 3i - 3j + 2 & \text{if } i = j, j+1, \ldots n; \end{cases}$$

is a prime labeling for $G$ with $f(a) = f(u_j) = 1$. Thus $f$ is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of $a = u_j$ in $G$.

**Case (ii):**

Let $a = v_j$ for some $j \in \{1, 2, \ldots, n\}$ then define a labeling $f_2$ using the labeling $f$ defined in case (i) as follows: $f_2(u_j) = f(v_j), f_2(v_j) = f(u_j)$ for $j = 1, 2, \ldots, n$ and $f_2(v) = f(v)$ for all the remaining vertices. Then the resulting labeling $f_2$ is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex $a = v_j$ in $G$.

**Case (iii):**

Let $a = w_j$ for some $j \in \{1, 2, \ldots, n\}$ then define a labeling $f_3$ using the labeling $f_2$ defined in case (ii) as follows: $f_3(w_j) = f_2(v_j), f_3(v_j) = f_2(w_j)$ for $j = 1, 2, \ldots, n$ and $f_3(v) = f_2(v)$ for all the remaining vertices. Then the resulting labeling $f_3$ is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex $a = w_j$ in $G$. Thus from all the cases described above $G$ is a strongly prime graph.

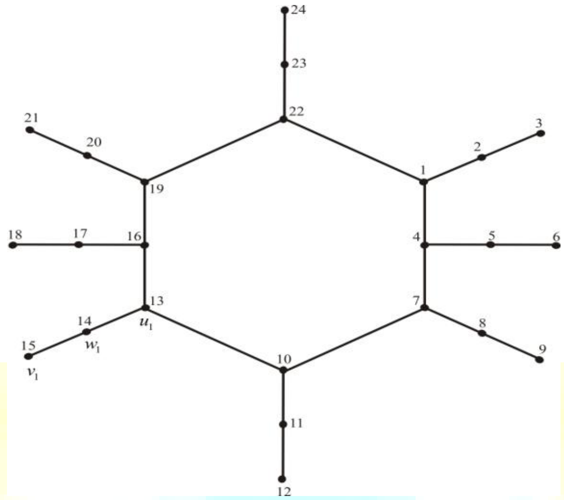


**Figure 3. A prime labeling of a graph $G$ obtained from subdividing the pendant edges of the crown graph $C_n^*$ having $u_5$ as label 1**

## Theorem 3.2:

The graph $G$ obtained from subdividing the edges of the path $P_n$ of the comb graph $C_{bn}$ is a strongly prime graph for all integers $n \geq 2$.

## Proof:

Let $C_{bn}$ be the comb graph with vertices $v_1, v_2, \ldots v_n, u_1, u_2, \ldots u_n$. Let $w_1, w_2, \ldots w_{n-1}$ be the corresponding new vertices which is subdividing the edges of the path $P_n$ in $C_{bn}$. Then the resulting graph be $G$. Now the vertex set $V(G) = \{u_i, v_i, w_k \,/\, 1 \leq i \leq n, 1 \leq k \leq n-1\}$ and the edge set of $G$ is $E(G) = \{u_i v_i, u_k w_k, w_k u_{k+1} \,/\, 1 \leq i \leq n, 1 \leq k \leq n-1\}$. Here $|V(G)| = 3n\text{-}1$.

Let $a$ be the vertex for which we assign label 1 in our labeling method. Then we have the following cases:

**Case (i):** When $n$ is odd.

**Subcase (i):**

Let $a = u_j$ for some $j \in \{1, 2, \ldots, n\}$ then the function $f : V(G) \to \{1, 2, \ldots, 3n\text{-}1\}$ defined by

$$f(w_i) = \begin{cases} 3n + 3i - 3j + 2 & \text{if } i = 1, 2, \ldots j-1; \\ 3i - 3j + 3 & \text{if } i = j, j+1, \ldots n-1; \end{cases}$$

$$\left.\begin{aligned} f(u_i) &= 3i - 3j + 2 \\ f(v_i) &= 3i - 3j + 1 \end{aligned}\right\} \quad \text{if } i = \begin{cases} j+1, j+3, \ldots n & \text{for } j \text{ is even}; \\ j+1, j+3, \ldots n-1 & \text{for } j \text{ is odd}; \end{cases}$$

$$\left.\begin{array}{l} f(u_i)=3i-3j+1 \\ f(v_i)=3i-3j+2 \end{array}\right\} \quad if \ \ i=\begin{cases} j, j+2,\ldots n-1 & for \ \ j \ is \ even; \\ j, j+2,\ldots n & for \ \ j \ is \ odd; \end{cases}$$

$$\left.\begin{array}{l} f(u_i)=3n+3i-3j+1 \\ f(v_i)=3n+3i-3j \end{array}\right\} \quad if \ \ i=\begin{cases} 1,3,5,\ldots j-1 & for \ \ j \ is \ even; \\ 1,3,5,\ldots j-2 & for \ \ j \ is \ odd; \end{cases}$$

$$\left.\begin{array}{l} f(u_i)=3n+3i-3j \\ f(v_i)=3n+3i-3j+1 \end{array}\right\} \quad if \ \ i=\begin{cases} 2,4,6,\ldots j-2 & for \ \ j \ is \ even; \\ 2,4,6,\ldots j-1 & for \ \ j \ is \ odd; \end{cases}$$

is a prime labeling for $G$ with $f(a)=f(u_j)=1$. Thus $f$ is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of $a=u_j$ in $G$.

**Subcase (ii):**

Let $a=w_j$ for some $j\in\{1,2,\ldots,n-1\}$ then define a labeling $f_2$ using the labeling $f$ defined in subcase (i) of case (i) as follows: $f_2(u_j)=f(w_j), f_2(w_j)=f(u_j)$ for $j=1,2,\ldots,n\text{-}1$ and $f_2(v)=f(v)$ for all the remaining vertices. Then the resulting labeling $f_2$ is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex $a=w_j$ in $G$.

**Subcase (iii):**

Let $a=v_j$ for some $j\in\{1,2,\ldots,n\}$ then define a labeling $f_3$ using the labeling $f$ defined in subcase (ii) of case (i) as follows: $f_3(v_j)=f_2(w_j), f_3(w_j)=f_2(v_j)$ for $j=1,2,\ldots,n$ and $f_3(v)=f_2(v)$ for all the remaining vertices. Then the resulting labeling $f_3$ is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex $a=v_j$ in $G$.

**Case (ii):** When $n$ is even.

**Subcase (i):**

Let $a=u_j$ for some $j\in\{1,2,\ldots,n\}$ then the function $f:V(G)\to\{1,2,\ldots,3n\text{-}1\}$ defined by

$$f(w_i)=\begin{cases} 3n+3i-3j+2 & if \ \ i=1,2,\ldots j-1; \\ 3i-3j+3 & if \ \ i=j, j+1,\ldots n-1; \end{cases}$$

$$\left.\begin{array}{l} f(u_i)=3i-3j+2 \\ f(v_i)=3i-3j+1 \end{array}\right\} \quad if \ \ i=\begin{cases} j+1, j+3,\ldots n-1 & for \ \ j \ is \ even; \\ j+1, j+3,\ldots n & for \ \ j \ is \ odd; \end{cases}$$

$$\left.\begin{array}{l} f(u_i)=3i-3j+1 \\ f(v_i)=3i-3j+2 \end{array}\right\} \quad if \ \ i=\begin{cases} j, j+2,\ldots n & for \ \ j \ is \ even; \\ j, j+2,\ldots n-1 & for \ \ j \ is \ odd; \end{cases}$$

$$\left.\begin{aligned} f(u_i) &= 3n+3i-3j \\ f(v_i) &= 3n+3i-3j+1 \end{aligned}\right\} \quad if \;\; i = \begin{cases} 1,3,5,\ldots j-1 & for \;\; j \; is \; even; \\ 1,3,5,\ldots j-2 & for \;\; j \; is \; odd; \end{cases}$$

$$\left.\begin{aligned} f(u_i) &= 3n+3i-3j+1 \\ f(v_i) &= 3n+3i-3j \end{aligned}\right\} \quad if \;\; i = \begin{cases} 2,4,\ldots j-2 & for \;\; j \; is \; even; \\ 2,4,\ldots j-1 & for \;\; j \; is \; odd; \end{cases}$$

is a prime labeling for $G$ with $f(a) = f(u_j) = 1$. Thus $f$ is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of $a = u_j$ in $G$.

**Subcase (ii):**

Let $a = w_j$ for some $j \in \{1,2,\ldots,n-1\}$ then define a labeling $f_4$ using the labeling $f$ defined in sub case (i) of case (ii) as follows: $f_4(u_j) = f(w_j)$, $f_4(w_j) = f(v_j)$ for $j = 1,2,\ldots,n\text{-}1$ and $f_4(v) = f(v)$ for all the remaining vertices. Then the resulting labeling $f_4$ is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex $a = w_j$ in $G$.

**Subcase (ii):**

Let $a = v_j$ for some $j \in \{1,2,\ldots,n\}$ then define a labeling $f_5$ using the labeling $f_4$ defined in subcase (ii) of case (ii) as follows: $f_5(v_j) = f_4(w_j)$, $f_5(w_j) = f_4(v_j)$ for $j = 1,2,\ldots,n$ and $f_5(v) = f_4(v)$ for all the remaining vertices. Then the resulting labeling $f_5$ is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex $a = v_j$ in $G$. Thus from all the cases described above $G$ is a strongly prime graph.

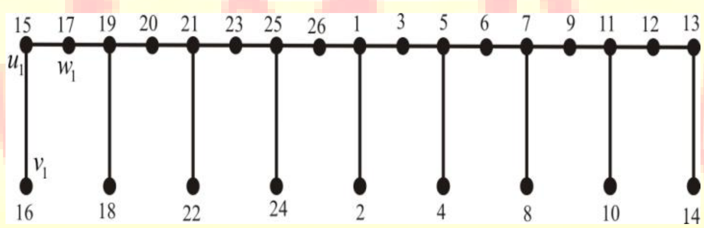


**Figure 4. A prime labeling of a graph $G$ obtained from subdividing the edges of the path $P_n$ of the comb graph $C_{bn}$ having $u_5$ as label 1 ( n, j is odd )**

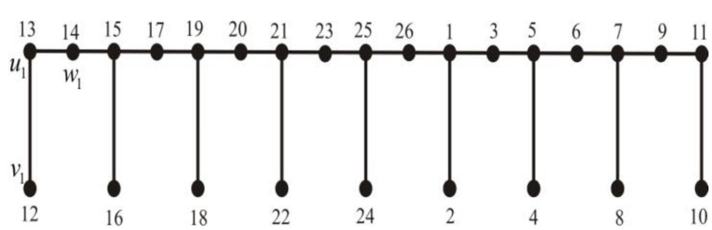

**Figure 5. A prime labeling of a graph $G$ obtained from subdividing the edges of the path $P_n$ of the comb graph $C_{bn}$ having $u_6$ as label 1 ( n is odd, j is even )**

## Theorem 3.3:

The graph $G$ obtained from subdividing the pendant edges of the comb graph $C_{bn}$ is a strongly prime graph for all integers $n \geq 2$.

## Proof:

Let $C_{bn}$ be the comb graph with vertices $v_1, v_2, ... v_n, u_1, u_2, ... u_n$. Let $w_1, w_2, ... w_n$ be the corresponding new vertices which is subdividing the pendant edges of $C_{bn}$. Then the resulting graph be $G$. Now the vertex set $V(G) = \{u_i, v_i, w_i / 1 \leq i \leq n\}$ and the edge set $E(G) = \{u_i w_i, w_i v_i / 1 \leq i \leq n\} \cup \{u_i u_{i+1} / 1 \leq i \leq n-1\}$. Here $|V(G)| = 3n$.

Let $a$ be the vertex for which we assign label 1 in our labeling method. Then we have the following cases:

**Case (i):**

Let $a = u_j$ for some $j \in \{1, 2, ..., n\}$ then the function $f : V(G) \rightarrow \{1, 2, ..., 3n\}$ defined by

$$f(u_i) = \begin{cases} 3n + 3i - 3j + 1 & if \ \ i = 1, 2, ... j-1; \\ 3i - 3j + 1 & if \ \ i = j, j+1, ... n; \end{cases}$$

$$f(v_i) = \begin{cases} 3n + 3i - 3j + 3 & if \ \ i = 1, 2, ... j-1; \\ 3i - 3j + 3 & if \ \ i = j, j+1, ... n; \end{cases}$$

$$f(w_i) = \begin{cases} 3n + 3i - 3j + 2 & if \ \ i = 1, 2, ... j-1; \\ 3i - 3j + 2 & if \ \ i = j, j+1, ... n; \end{cases}$$

is a prime labeling for $G$ with $f(a) = f(u_j) = 1$. Thus $f$ is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of $a = u_j$ in $G$.

**Case (ii):**

Let $a = v_j$ for some $j \in \{1, 2, ..., n\}$ then define a labeling $f_2$ using the labeling $f$ defined in case (i) as follows: $f_2(u_j) = f(v_j), f_2(v_j) = f(u_j)$ for $j = 1, 2, ..., n$ and $f_2(v) = f(v)$ for all the remaining vertices. Then the resulting labeling $f_2$ is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex $a = v_j$ in $G$.

**Case (iii):**

Let $a = w_j$ for some $j \in \{1,2,...,n\}$ then define a labeling $f_3$ using the labeling $f_2$ defined in case (ii) as follows: $f_3(w_j) = f_2(v_j), f_3(v_j) = f_2(w_j)$ for $j = 1,2,...,n$ and $f_3(v) = f_2(v)$ for all the remaining vertices. Then the resulting labeling $f_3$ is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex $a = w_j$ in $G$. Thus from all the cases described above $G$ is a strongly prime graph.

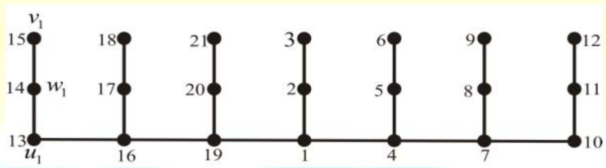


**Figure 6. A prime labeling of a graph $G$ obtained from subdividing the pendant edges of the comb graph $C_{bn}$ having $u_4$ as label 1**

## Theorem 3.4:

The graph $G$ obtained from subdividing the edges of the cycle $C_5$ in the key graph $C_5 \square P_n$ is a strongly prime graph for all integers $n \geq 2$.

## Proof:

Let $C_5 \square P_n$ be the key graph with vertices $w_1, w_2,...w_5, u_1, u_2,...u_n, v_1, v_2,...v_n$. Let $w_1', w_2',...w_5'$ be the corresponding new vertices which is subdividing the edges of the cycle $C_5$ in key graph.

Then the resulting graph be $G$. Now the vertex set $V(G) = \{v_i, u_i / 1 \leq i \leq n, w_i, w_i' / 1 \leq i \leq 5\}$ and the edge set $E(G) = \{v_i v_{i+1} / 1 \leq i \leq n-1\} \cup \{v_i u_i / 1 \leq i \leq n\} \cup \{w_1 v_1\}$

$$\cup \{w_i w_i' / 1 \leq i \leq n\} \cup \{w_i' w_{i+1} / 1 \leq i \leq n-1\} \cup \{w_5' w_1\}.$$ Here $|V(G)| = 2n+10$.

Let $a$ be the vertex for which we assign label 1 in our labeling method. Then we have the following cases:

**Case (i):** When $a$ is any arbitrary vertex of a cycle.

**Subcase (i):**

Let $a = w_j$ for some $j \in \{1,2,...,5\}$ then define the function $f : V(G) \to \{1,2,...,2n+10\}$ as

$$f(w_i) = \begin{cases} 10 + 2(i-j) + 1 & \text{if } i = 1,2,..., j-1; \\ 2(i-j) + 1 & \text{if } i = j, j+1,...,5; \end{cases}$$

$$f(w_i^{'}) = \begin{cases} 10+2(i-j)+2 & \text{if } i=1,2,...,j-1; \\ 2(i-j)+2 & \text{if } i=j,j+1,...,5; \end{cases}$$

$$f(v_i) = 10+2i-1 \qquad \text{if } i=1,2,...,n;$$

$$f(u_i) = 10+2i \qquad \text{if } i=1,2,...,n;$$

is a prime labeling for $G$ with $f(a) = f(w_j) = 1$. Thus $f$ is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of $a = w_j$ in $G$.

**Subcase (ii):**

Let $a = w_j^{'}$ for some $j \in \{1,2,...,5\}$ then the function $f_1$ using the labeling $f$ defined in subcase (i) as follows:

$$f_1(w_i) = \begin{cases} 10+2(i-j) & \text{if } i=1,2,...j; \\ 2(i-j) & \text{if } i=j+1,j+2,...5; \end{cases}$$

$$f_1(w_i^{'}) = \begin{cases} 10+2(i-j)+1 & \text{if } i=1,2,...j-1; \\ 2(i-j)+1 & \text{if } i=j,j+1,...5; \end{cases}$$

and $f_1(v) = f(v)$ for all the remaining vertices. Then the resulting labeling $f_1$ is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex $a = w_j^{'}$ in $G$.

**Case (ii):** When $a$ is any arbitrary vertex of path $P_n$.

Let $a = v_j$ for some $j \in \{1,2,...,n\}$ then define the function $f : V(G) \to \{1,2,...,2n+10\}$ as

$$f(w_i^{'}) = 2i+2 \qquad \text{if } i=1,2,...5;$$

$$f(w_i) = 2i+1 \qquad \text{if } i=2,3,...5;$$

$$f(w_1) = 13;$$

$$f(v_i) = \begin{cases} 2(n+i-j+5)+1 & \text{if } i=1,2,...j-1; \\ 2(i-j+5)+1 & \text{if } i=j+3,j+4,...n; \end{cases}$$

$$f(v_j) = 1;$$

$$f(v_{j+1}) = 3;$$

$$f(v_{j+2}) = f(w_1)+3;$$

$$f(u_i) = \begin{cases} 2(n+i-j+5)+2 & \text{if } i=1,2,...j-1; \\ 2(i-j+5)+2 & \text{if } i=j+3,j+4,...n; \end{cases}$$

$f(u_j) = 2;$

$f(u_{j+1}) = f(w_1) + 1;$

$f(u_{j+2}) = f(w_1) + 2;$

is a prime labeling for $G$ with $f(a) = f(v_j) = 1$. Thus $f$ is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of $a = v_j$ in $G$.

[In this case, if $f(v_1)$ is a multiple of 13 then keep the above labeling $f$ defined in case (ii) as same and change the labels ( $f(w_i)$, $f(w_i^{'})$ ) as $f(w_i) = 2i + 2$ $if$ $i = 1, 2, ...5;$ $f(w_i^{'}) = 2i + 3$ $if$ $i = 1, 2, ...5;$ ]

**Case (iii):**

Let $a = u_j$ for some $j \in \{1, 2, ..., n\}$ then define a labeling $f_2$ using the labeling $f$ defined in case (ii) as follows: $f_2(u_j) = f(v_j), f_2(v_j) = f(u_j)$ for $j \in \{1, 2, ..., n\}$ and $f_2(v) = f(v)$ for all the remaining vertices. Then the resulting labeling $f_2$ is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex $a = u_j$ in $G$. Thus from all the cases described above $G$ is a strongly prime graph.

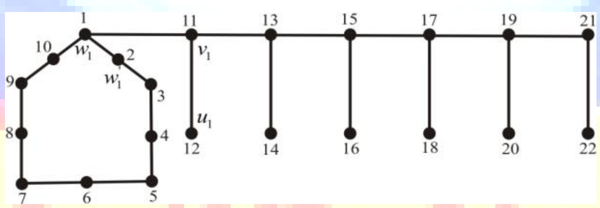


**Figure 7. A prime labeling of a graph $G$ obtained from subdividing the edges of the cycle $C_5$ in $C_5 \square P_n$ having $w_1$ as label 1**

## Theorem 3.5:

The graph $G$ obtained from subdividing the edges of the cycle $C_5$ and pendant edges in the key graph $C_5 \square P_n$ is a strongly prime graph for all integers $n \geq 2$.

## Proof:

Let $C_5 \square P_n$ be the key graph with vertices $w_1, w_2, ...w_5, u_1, u_2, ...u_n, v_1, v_2, ...v_n$. Let $w_1^{'}, w_2^{'}, ...w_5^{'}$ and $u_1^{'}, u_2^{'}, ...u_n^{'}$ be the corresponding new vertices which is subdividing the edges of the cycle $C_5$ and pendant edges in $C_5 \square P_n$. Then the resulting graph be $G$. Now the vertex set

$V(G)=\{u_i,v_i,u_i^{'}/1\le i\le n,\ w_i,w_i^{'}/1\le i\le 5\}$ and the edge set $E(G)=\{v_iv_{i+1}/1\le i\le n-1\}\cup\{w_1v_1\}\cup\{w_iw_i^{'}/1\le i\le n\}\cup\{w_i^{'}w_{i+1}/1\le i\le n-1\}$

$\cup\{w_5^{'}w_1\}\cup\{u_iu_i^{'},v_iu_i^{'}/1\le i\le n\}$.Now $|V(G)|=3n+10$.

Let $a$ be the vertex for which we assign label 1 in our labeling method. Then we have the following cases:

**Case (i):** When $a$ is any arbitrary vertex of $C_5$.

**Subcase (i):**

Let $a=w_j$ for some $j\in\{1,2,...,5\}$ then defined the function $f:V(G)\to\{1,2,...,3n+10\}$ as

$$f(w_i)=\begin{cases}2(i-j+5)+1 & if\ i=1,2,...,j-1;\\ 2(i-j)+1 & if\ i=j,j+1,...,5;\end{cases}$$

$$f(w_i^{'})=\begin{cases}2(i-j+5)+2 & if\ i=1,2,...,j-1;\\ 2(i-j)+2 & if\ i=j,j+1,...,5;\end{cases}$$

$$f(v_i)=10+3i \qquad if\ i=1,2,...,n;$$

$$f(u_i)=10+3i-2 \qquad if\ i=1,2,...,n;$$

$$f(u_i^{'})=10+3i-1 \qquad if\ i=1,2,...,n;$$

is a prime labeling for $G$ with $f(a)=f(w_j)=1$.Thus $f$ is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of $a=w_j$ in $G$.

**Subcase (ii):**

Let $a=w_j^{'}$ for some $j\in\{1,2,...,5\}$ then define a labeling $f_2$ using the labeling $f$ defined in subcase (i) of case (i) as follows:

$$f_2(w_i)=\begin{cases}2(i-j+5) & if\ i=1,2,...j;\\ 2(i-j) & if\ i=j+1,j+2,...5;\end{cases}$$

$$f_2(w_i^{'})=\begin{cases}2(i-j+5)+1 & if\ i=1,2,...j-1;\\ 2(i-j)+1 & if\ i=j,j+1,...5;\end{cases}$$

and $f_2(v)=f(v)$ for all the remaining vertices. Then the resulting labeling $f_2$ is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex $a=w_j^{'}$ in $G$.

**Case (ii):** When $a$ is any arbitrary vertex of subdivision of pendant edges in a Hoffman graph.

**Subcase (i):**

Let $a = v_j$ for some $j \in \{2,3,...n\}$ then the function $f : V(G) \to \{1,2,...,3n+10\}$ defined by

$$f(w_i) = \begin{cases} 2(i+3)+1 & \text{if } i = 1,2,3; \\ 2i-3 & \text{if } i = 4,5; \end{cases}$$

$$f(w_i^{'}) = \begin{cases} 2(i+3)+2 & \text{if } i = 1,2; \\ 2i-2 & \text{if } i = 3,4,5; \end{cases}$$

$$f(v_i) = \begin{cases} 3(n+i-j+5)-2 & \text{if } i = 1,2,...j-1; \\ 3(i-j+5)-2 & \text{if } i = j+1, j+2,...n; \end{cases}$$

$$f(u_i) = \begin{cases} 3(n+i-j+5)-4 & \text{if } i = 1,2,..., j-1; \\ 3(i-j+5)-4 & \text{if } i = j+1, j+2,...n; \end{cases}$$

$$f(u_i^{'}) = \begin{cases} 3(n+i-j+5)-3 & \text{if } i = 1,2,..., j-1; \\ 3(i-j+5)-3 & \text{if } i = j+1, j+2,...n; \end{cases}$$

$f(v_j) = 1;$

$f(u_j) = 2;$

$f(u_j^{'}) = 3;$

is a prime labeling for $G$ with $f(a) = f(v_j) = 1$.Thus $f$ is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of $a = v_j$ in $G$.

**Subcase (ii):**

Let $a = u_j^{'}$ for some $j \in \{2,3,...,n\}$ then define a labeling $f_3$ using the labeling $f$ defined in subcase (i) of case (ii) as follows: $f_3(v_j) = f(u_j^{'})$, $f_3(u_j^{'}) = f(v_j)$ for $j \in \{1,2,...,n\}$ and $f_3(v) = f(v)$ for all the remaining vertices. Then the resulting labeling $f_3$ is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex $a = u_j^{'}$ in $G$.

**Subcase (iii):**

Let $a = u_j$ for some $j \in \{2,3,...n\}$ then define a labeling $f_4$ using the labeling $f_3$ defined in subcase (ii) of case (ii) as follows: $f_4(u_j) = f_3(u_j^{'})$, $f_4(u_j^{'}) = f_3(u_j)$ for $j \in \{1,2,...,n\}$ and

$f_4(v) = f_3(v)$ for all the remaining vertices. Then the resulting labeling $f_4$ is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex $a = u_j$ in $G$.

**Subcase (iv):**

Let $a = v_1$ then define a labeling $f_5$ using the labeling $f$ defined in subcase (i) of case (ii) as follows: $f_5(w_i) = 2i+1$ for $i = 2,3,4,5$, $f_5(w_i^{'}) = 2i+2$ for $i = 1,2,...5$, $f_5(w_1) = 13$ and $f_5(v) = f(v)$ for all the remaining vertices. Then the resulting labeling $f_5$ is a prime labeling and also it is possible to assign label 1 to $a = v_1$ in $G$.

**Subcase (iv)a:**

Let $a = u_1^{'}$ then in the above labeling $f_5$ defined in subcase (iv) interchange the labels of $v_1$ and $u_1^{'}$. Then the resulting labeling $f_6$ is a prime labeling and also it is possible to assign label 1 to $a = u_1^{'}$ in $G$.

**Subcase (iv)b:**

Let $a = u_1$ then in the above labeling $f_6$ defined in subcase ((iv)a) interchange the labels of $u_1$ and $u_1^{'}$. Then the resulting labeling $f_7$ is a prime labeling and also it is possible to assign label 1 to $a = u_1$ in $G$.Thus from all the cases described above $G$ is a strongly prime graph.

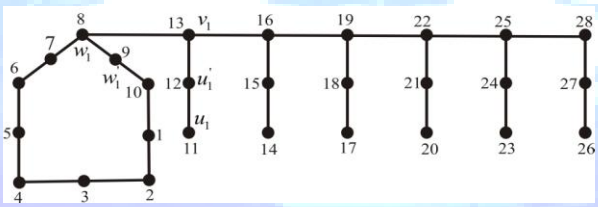


**Figure 8. A prime labeling of a graph $G$ obtained from subdividing the edges of the cycle $C_5$ and pendant edges in the key graph $C_5 \square P_n$ having $w_2^{'}$ as label 1**

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**References**

[1] J.A. Bondy and U.S.R.Murthy 1976, *Graph Theory and Applications* (North-Holland), New York.

[2] T. Deretsky, S.M. Lee and J. Mitchem 1991, *On Vertex Prime Labeling of Graphs in Graph Theory*, Combinatorics and Applications, Vol. 1,Wiley, New York 359-369.

[3] H.C. Fu and K.C.Huang 1994, *On Prime labeling Discrete Math*, 127, 181-186.

[4] Gallian J. A 2009, *A dynamic survey of graph labeling*, The Electronic Journal of Combinations 16 # DS6.

[5] S.M.Lee, L.Wui and J.Yen 1988, *On the amalgamation of Prime graphs Bull*, Malaysian Math.Soc.(Second Series) 11, 59- 67.

[6] S. Meena and K. Vaithilingam 2013, *Prime labeling for some helm related graphs*, International Journal of Innovative Research in Science, Engineering and Technology Vol.2, Issue 4, 1075-1085

[7] V. Sharon Philomena and K. Thirusangu 2014, Square and cube difference labeling of cycle cactus, special tree and a new key graphs, *Annals of Pure and Applied Mathematics Vol. 8, No. 2, 115-121.*

[8] A.Tout A.N.Dabboucy and K.Howalla 1982, *Prime labeling of graphs*, Nat. Acad. Sci letters 11, 365-368.

[9] S.K.Vaidya and Udayan M.Prajapati 2012, *Some new results on prime graph*, Open Journal of Discrete Mathematics, 2 , 99-104.