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GENERALIZED ALMOST PARA-SASAKIAN MANIFOLDS

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ABSTRACT

In 1976, 1977, I. Sato [3], [4] dicussed on a structure similar to almost contact structure. Also in 1979, K. Matsumoto and I. Sato [1] discussed on p-Sasakian manifolds satisfying certain conditions and in 2011, R. Nivas and A. Bajpai [2] studied on generalized Lorentzian Para-Sasakian manifolds. T. Suguri and S. Nakayama [5] considered D-conformal deformations on almost contact metric structure. In this paper generalized para-Sasakian manifold, Generalized special para-Sasakian manifold, generalised almost para-Sasakian manifold and generalized almost special para-Sasakian manifold have been introduced and some of their properties have been established. A generalized D-conformal transformation in a generalized almost para-contact manifold has also been introduced.

Keywords: Generalized almost p-Sasakian manifold, generalized almost special p-Sasakian manifolds, generalized almost p-co-symplectic manifolds, and generalized D-conformal transformation.

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1. Introduction

Let M_n be an n-dimensional differentiable manifold, on which there are defined a tensor field $F$of type (1, 1), two contravariant vector fields $T_1$and T_2 , two covariant vector fields $A_1$and A_2 and a metric tensor g, satisfying for arbitrary vector fields $X, Y, Z, \ldots$

(1.1) $\overline{\overline{X}} = X - A_1(X)T_1 - A_2(X)T_2,\ \overline{T_1} = 0,\ \overline{T_2} = 0,\ A_1(T_1) = 1,\ A_2(T_2) = 1,$

$\overline{X} \overset{\text{def}}{=} FX,\ \ A_1(\overline{X}) = 0,\ \ A_2(\overline{X}) = 0,$ rank F= n-2

(1.2) g $(\overline{X}, \overline{Y})$ = g $(X, Y) - A_1(X)A_1(Y) - A_2(X)A_2(Y)$, where $A_1(X) = g(X, T_1)$,

$A_2(X) = g(X, T_2),\ \ `F(X, Y) \overset{\text{def}}{=} g(\overline{X},\ Y) = `F(Y, X),$

Then M_n is called a generalized almost Para-Contact manifold (a generalized almost P-Contact manifold) and the structure $(F, T_1, T_2, A_1, A_2, g)$ is called a generalized almost Para-Contact structure.

Let D be a Riemannian connection on M_n, then we have

(1.3) (a) $(D_X`F)(\overline{Y},\ Z) + (D_X`F)(Y, \overline{Z}) + A_1(Y)(D_XA_1)(Z) + A_2(Y)(D_XA_2)(Z) +$

A1ZDXA1Y+A2ZDXA2Y=0

(b) $(D_X`F)(\overline{Y},\ \overline{Z}) + (D_X`F)(Y, Z) + A_1(Y)(D_XA_1)(\overline{Z}) + A_2(Y)(D_XA_2)(\overline{Z}) +$

$A_1(Z)(D_XA_1)(\overline{Y}) + A_2(Z)(D_XA_2)(\overline{Y}) = 0$

(1.4) (a) $(D_X`F)\left(\overline{Y},\ \overline{\overline{Z}}\right) + (D_X`F)\left(\overline{\overline{Y}}, \overline{Z}\right) = 0$

(b) $(D_X`F)\left(\overline{\overline{Y}},\ \overline{\overline{Z}}\right) + (D_X`F)(\ \overline{Y},\ \overline{Z}\ ) = 0$

A generalized almost P-Contact manifold is called a generalized Para-Sasakian manifold (a generalized P-Sasakian manifold) if

(1.5) (a) $2(D_XF)(Y) + \{A_1(Y) + A_2(Y)\}\,\overline{\overline{X}} + g(\overline{X}, \overline{Y})(T_1 + T_2) = 0 \Leftrightarrow$

(b) $2(D_X`F)(Y, Z) + \{A_1(Y) + A_2(Y)\}\ g(\overline{X}, \overline{Z}) + \{A_1(Z) + A_2(Z)\}g(\overline{X}, \overline{Y})\ = 0 \Leftrightarrow$

(c) $D_X T_1 = \overline{X} - T_2, \qquad D_X T_2 = \overline{X} - T_1$

This implies

(1.6) (a) $2(D_X `F)(\overline{Y}, Z) + \{A_1(Z) + A_2(Z)\}`F(X, Y) = 0$

(b) $2(D_X `F)\left(\overline{\overline{Y}}, Z\right) + \{A_1(Z) + A_2(Z)\}g(\overline{X}, \overline{Y}) = 0$

(c) $2(D_X `F)(Y, Z) + A_1(Y)(D_X A_1)(\overline{Z}) + A_2(Y)(D_X A_2)(\overline{Z}) + \{A_1(Z) + A_2(Z)\}g(\overline{X}, \overline{Y}) = 0$

On this manifold, we have

(1.7) (a) $(D_X A_1)(\overline{Y}) = (D_X A_2)(\overline{Y}) = g(\overline{X}, \overline{Y}) \Leftrightarrow$

(b) $(D_X A_1)(Y) + A_2(Y) = (D_X A_2)(Y) + A_1(Y) = `F(X, Y)$

A generalized almost P-Contact manifold is called a generalized Special Para-Sasakian manifold (a generalized SP-Sasakian manifold) if

(1.8) (a) $2(D_X F)(Y) + \{A_1(Y) + A_2(Y)\}\overline{X} + `F(X, Y)(T_1 + T_2) = 0 \Leftrightarrow$

(b) $2(D_X `F)(Y, Z) + \{A_1(Y) + A_2(Y)\}`F(X, Z) + \{A_1(Z) + A_2(Z)\}`F(X, Y) = 0 \Leftrightarrow$

(c) $D_X T_1 = \overline{\overline{X}} - T_2, \qquad D_X T_2 = \overline{\overline{X}} - T_1$

This implies

(1.9) (a) $2(D_X `F)(\overline{Y}, Z) + \{A_1(Z) + A_2(Z)\}g(\overline{X}, \overline{Y}) = 0$

(b) $2(D_X `F)\left(\overline{\overline{Y}}, Z\right) + \{A_1(Z) + A_2(Z)\}`F(X, Y) = 0$

(c) $2(D_X `F)(Y, Z) + A_1(Y)(D_X A_1)(\overline{Z}) + A_2(Y)(D_X A_2)(\overline{Z}) + \{A_1(Z) + A_2(Z)\}`F(X, Y) = 0$

On this manifold, we have

(1.10) (a) $(D_X A_1)(\overline{Y}) = (D_X A_2)(\overline{Y}) = `F(X, Y) \Leftrightarrow$

(b) $(D_X A_1)(Y) + A_2(Y) = (D_X A_2)(Y) + A_1(Y) = g(\overline{X}, \overline{Y})$

Nijenhuis tensor in a generalized almost P-Contact manifold is given by

(1.11) $`N(X,Y,Z) = (D_{\overline{X}}`F)(Y,Z) - (D_{\overline{Y}}`F)(X,Z) - (D_X`F)(Y,\overline{Z}) + (D_Y`F)(X,\overline{Z})$

Where $`N(X,Y,Z) \stackrel{\text{def}}{=} g(N(X,Y),Z)$

2. Generalized Almost Para-Co-symplectic manifold

A generalized almost P-Contact manifold will be called a generalized almost P-Co-symplectic manifold if

(2.1) $2(D_X`F)(Y,Z) + 2(D_Y`F)(Z,X) + 2(D_Z`F)(X,Y) + A_1(X)\{(D_Y A_1)(\overline{Z}) + (D_Z A_1)(\overline{Y})\} +$
$A_2(X)\{(D_Y A_2)(\overline{Z}) + (D_Z A_2)(\overline{Y})\} + A_1(Y)\{(D_X A_1)(\overline{Z}) + (D_Z A_1)(\overline{X})\}$
$+A_2(Y)\{(D_X A_2)(\overline{Z}) + (D_Z A_2)(\overline{X})\} + A_1(Z)\{(D_X A_1)(\overline{Y}) + (D_Y A_1)(\overline{X})\} +$
$A_2(Z)\{(D_X A_2)(\overline{Y}) + (D_Y A_2)(\overline{X})\} = 0$

3. Generalized almost Para-Sasakian manifold

A generalized almost P-Contact manifold will be called a generalized almost Para-Sasakian manifold (a generalized almost P-Sasakian manifold) if

(3.1) $(D_X`F)(Y,Z) + (D_Y`F)(Z,X) + (D_Z`F)(X,Y) + \{A_1(X) + A_2(X)\}g(\overline{Y},\overline{Z}) +$
$\{A_1(Y) + A_2(Y)\}\, g(\overline{X},\overline{Z}) + \{A_1(Z) + A_2(Z)\}\, g(\overline{X},\overline{Y}) = 0$

Therefore, A generalized almost P-Co-symplectic manifold is a generalized almost P-Sasakian manifold if

(3.2) (a) $(D_X A_1)(\overline{Y}) = (D_X A_2)(\overline{Y}) = g(\overline{X}, \overline{Y}) \Leftrightarrow$

(b) $(D_X A_1)(Y) + A_2(Y) = (D_X A_2)(Y) + A_1(Y) = `F(X,Y) \Leftrightarrow$ (c) $D_X T_1 = \overline{X} - T_2$,
$D_X T_2 = \overline{X} - T_1$

Barring X, Y, Z in (1.11) and using equations (3.1), (1.4) (a), we see that a generalized almost P-Sasakian manifold is completely integrable if

(3.3) $(D_{\bar{X}}`F)(\bar{Y},\bar{\bar{Z}}) + (D_{\bar{Y}}`F)(\bar{Z},\bar{\bar{X}}) + (D_{\bar{Z}}`F)(\bar{\bar{X}},\bar{Y}) = 0$

4. Generalized almost Special Para-Sasakian manifold

A generalized almost P-Contact manifold will be called a generalized almost Special Para-Sasakian manifold (a generalized almost SP-Sasakian manifold) if

(4.1) $(D_X`F)(Y,Z) + (D_Y`F)(Z,X) + (D_Z`F)(X,Y)$

$$+\{A_1(X) + A_2(X)\}\,`F(Y,Z) + \{A_1(Y) + A_2(Y)\}\,`F(Z,X) + \{A_1(Z) + A_2(Z)\}\,`F(X,Y) = 0$$

Therefore a generalized P-Co-symplectic manifold is a generalized SP-Sasakian manifold if

(4.2) (a) $(D_X A_1)(\bar{Y}) = (D_X A_2)(\bar{Y}) = `F(X,Y) \Leftrightarrow$

(b) $(D_X A_1)(Y) + A_2(Y) = (D_X A_2)(Y) + A_1(Y) = g(\bar{X}, \bar{Y}) \Leftrightarrow$ (c) $D_X T_1 = \bar{\bar{X}} - T_2$, $D_X T_2 = \bar{\bar{X}} - T_1$

Barring X, Y, Z in (1.8) and using equations (4.1), (1.4) (a), we see that a generalized almost SP-Sasakian manifold is completely integrable if

(4.3) $(D_{\bar{\bar{X}}}`F)(\bar{Y},\bar{Z}) - (D_{\bar{\bar{Y}}}`F)(\bar{Z},\bar{X}) - (D_{\bar{\bar{Z}}}`F)(\bar{X},\bar{Y}) = 2(D_{\bar{X}}`F)(\bar{Y},\bar{\bar{Z}})$

5. Generalized D- Conformal transformation.

Let the corresponding Jacobian map B of the transformation b transforms the structure $(F, T_1, T_2, A_1, A_2, g)$ to the structure $(F, V_1, V_2, \upsilon_1, \upsilon_2, h)$ such that

(5.1) (a) $B\bar{Z} = \overline{BZ}$ (b) $h(BX, BY)ob = e^{\sigma}\, g(\bar{X},\bar{Y}) + e^{2\sigma} A_1(X)A_1(Y) + e^{2\sigma} A_2(X)A_2(Y)$

(c) $V_1 = e^{-\sigma} BT_1,\ V_2 = e^{-\sigma} BT_2$ (d) $\upsilon_1(BX)ob = e^{\sigma} A_1(X),\ \upsilon_2(BX)ob = e^{\sigma} A_2(X)$

Where σ is a differentiable function on M_n, then the transformation is said to be generalized D-conformal transformation. If σ is a constant, the transformation is known as D-homothetic.

Theorem 5.1 The structure $(F, V_1, V_2, \upsilon_1, \upsilon_2, h)$ is generalized almost Para-Contact structure.

Proof. Inconsequence of (1.1), (1.2), (5.1) (b) and (5.1) (d), we have

$$h(B\bar{X}, B\bar{Y})ob = e^{\sigma}\, g(\bar{X},\bar{Y}) = h(BX, BY)ob - e^{2\sigma} A_1(X)A_1(Y) - e^{2\sigma} A_2(X)A_2(Y)$$

$$= h(BX, BY)ob - \{\upsilon_1(BX)ob\}\{\upsilon_1(BY)ob\} - \{\upsilon_2(BX)ob\}\{\upsilon_2(BY)ob\}$$

This implies

(5.2) $$h(B\overline{X}, B\overline{Y}) = h(BX, BY) - v_1(BX)\, v_1(BY) - v_2(BX)\, v_2(BY)$$

Using (1.1), (5.1) (a), (5.1) (c) and (5.1) (d), we obtain

(5.3) $$\overline{\overline{BX}} = B\overline{\overline{X}} = BX - A_1(X)BT_1 - A_2(X)BT_2 = BX - \{v_1(BX)ob\}V_1 - \{v_2(BX)ob\}V_2$$

Also

(5.4) $$\overline{V_1} = e^{-\sigma}\,\overline{BT_1} = 0, \quad \overline{V_2} = e^{-\sigma}\,\overline{BT_2} = 0$$

Equations (5.2), (5.3) and (5.4) prove the statement.

Theorem 5.2 Let E and D be the Riemannian connections with respect to h and g such that

(5.5) (a) $E_{BX}BY = BD_XY + BH(X,Y)$ (b) $`H(X,Y,Z) \overset{\text{def}}{=} g(H(X,Y),Z)$

Then we have

(5.6) $2E_{BX}BY =$

$2BD_XY + B[2e^{\sigma}\{(X\sigma)\, A_1(Y)\, T_1 + (X\sigma)\, A_2(Y)\, T_2 + (Y\sigma)\, A_1(X)\, T_1 + (Y\sigma)\, A_2(X)\, T_2 -$

$({}^{-1}G\nabla\sigma)\, A_1(X)\, A_1(Y) - ({}^{-1}G\nabla\sigma)\, A_2(X)\, A_2(Y)\} + (e^{\sigma} - 1)\{(D_XA_1)(Y) + (D_YA_1)(X) -$

2A1HX,YT1+eσ −1DXA2Y+DYA2X−2A2HX,YT2+eσ

−1A1XDYT1+A2XDYT2+A1YDXT1+A2YDXT2−A1X(−1G∇A1)Y−A2X(−1G∇A2)Y−A1Y

(−1G∇A1)X−A2Y(−1G∇A2)X]

Proof. Using (5.1) (b), we get

$$BX\big(h(BY,BZ)\big)ob = X\{e^{\sigma}\, g\,(\overline{Y},\overline{Z}) + e^{2\sigma} A_1(Y)A_1(Z) + e^{2\sigma} A_2(Y)A_2(Z)\}$$

Consequently

(5.7) $$h(E_{BX}BY, BZ)ob + h(BY, E_{BX}BZ)ob =$$
$$(X\sigma)e^{\sigma}\, g\,(\overline{Y},\overline{Z}) + e^{\sigma}\, g\,(D_X\overline{Y},\overline{Z}) + e^{\sigma}\, g\,(\overline{Y},D_X\overline{Z}) + 2(X\sigma)e^{2\sigma} A_1(Y)A_1(Z)$$
$$+ e^{2\sigma}(D_XA_1)(Y)A_1(Z) + e^{2\sigma}(D_XA_1)(Z)A_1(Y) + e^{2\sigma} A_1(D_XY)A_1(Z)$$
$$+ e^{2\sigma} A_1(D_XZ)A_1(Y) + 2(X\sigma)e^{2\sigma} A_2(Y)A_2(Z) + e^{2\sigma}(D_XA_2)(Y)A_2(Z)$$
$$+ e^{2\sigma}(D_XA_2)(Z)A_2(Y) + e^{2\sigma} A_2(D_XY)A_2(Z) + e^{2\sigma} A_2(D_XZ)A_2(Y)$$

Also

(5.8) $$h(E_{BX}BY, BZ)ob + h(BY, E_{BX}BZ)ob = e^{\sigma}\, g\,(\overline{D_XY},\overline{Z}) + e^{2\sigma} A_1(D_XY)A_1(Z) +$$
$$e^{2\sigma} A_2(D_XY)A_2(Z) + e^{\sigma}\, g\,(\overline{H(X,Y)},\overline{Z}) + e^{2\sigma} A_1\big(H(X,Y)\big)A_1(Z) + e^{2\sigma} A_2\big(H(X,Y)\big)A_2(Z) +$$
$$e^{\sigma}\, g\,(\overline{Y},\overline{H(X,Z)}) + e^{2\sigma} A_1(Y)A_1\big(H(X,Z)\big) + e^{2\sigma} A_2(Y)A_2\big(H(X,Z)\big) + e^{\sigma}\, g\,(\overline{Y},\overline{D_XZ}) +$$
$$e^{2\sigma} A_1(D_XZ)A_1(Y) + e^{2\sigma} A_2(D_XZ)A_2(Y)$$

Inconsequence of (1.3) (a), (5.7) and (5.8), we have

(5.9)

$(X\sigma)g\left(\overline{Y},\overline{Z}\right)+2(X\sigma)e^{\sigma}A_1(Y)A_1(Z)+2(X\sigma)e^{\sigma}A_2(Y)A_2(Z)+(e^{\sigma}-1)\{(D_XA_1)(Y)A_1(Z)+(D_XA_2)(Y)A_2(Z)+(D_XA_1)(Z)A_1(Y)+(D_XA_2)(Z)A_2(Y)\}`H(X,Y,Z)+`H(X,Z,Y)+(e^{\sigma}-1)\{A_1\big(H(X,Y)\big)A_1(Z)+A_2\big(H(X,Y)\big)A_2(Z)+A_1\big(H(X,Z)\big)A_1(Y)+A_2\big(H(X,Z)\big)A_2(Y)\}$

Writing two other equations by cyclic permutation of X,Y,Z and subtracting the third equation from the sum of the first two equations and using symmetry of $`H$ in the first two slots, we get

(5.10) $2`H(X,Y,Z)=2e^{\sigma}\{(X\sigma)A_1(Y)A_1(Z)+(X\sigma)A_2(Y)A_2(Z)+(Y\sigma)A_1(Z)A_1(X)+(Y\sigma)A_2(Z)A_2(X)-(Z\sigma)A_1(X)A_1(Y)-(Z\sigma)A_2(X)A_2(Y)\}+(e^{\sigma}-1)\big[A_1(Z)\big\{(D_XA_1)(Y)+(D_YA_1)(X)-2A_1\big(H(X,Y)\big)\big\}+A_2(Z)\big\{(D_XA_2)(Y)+(D_YA_2)(X)-2A_2\big(H(X,Y)\big)\big\}+A_1(X)\{(D_YA_1)(Z)-(D_ZA_1)(Y)\}+A_2(X)\{(D_YA_2)(Z)-(D_ZA_2)(Y)\}+A_1(Y)\{(D_XA_1)(Z)-(D_ZA_1)(X)\}+A_2(Y)\{(D_XA_2)(Z)-(D_ZA_2)(X)\}\big]$

This implies

(5.11) $2H(X,Y)=2e^{\sigma}\,[(X\sigma)A_1(Y)T_1+(X\sigma)A_2(Y)\,T_2+(Y\sigma)A_1(X)T_1+(Y\sigma)A_2(X)\,T_2-({}^{-1}G\nabla\sigma)A_1(\mathrm{X})A_1(\mathrm{Y})-({}^{-1}G\nabla\sigma)A_2(\mathrm{X})A_2(\mathrm{Y})]+(e^{\sigma}-1)\big[\big\{(D_XA_1)(Y)+(D_YA_1)(X)-2A_1\big(H(X,Y)\big)\big\}T_1+\big\{(D_XA_2)(Y)+(D_YA_2)(X)-2A_2\big(H(X,Y)\big)\big\}\,T_2+A_1(X)(D_YT_1)+A_2(X)(D_Y\,T_2)+A_1(Y)(D_XT_1)+A_2(Y)(D_X\,T_2)-A_1(X)({}^{-1}G\nabla A_1)(\mathrm{Y})-A_2(X)({}^{-1}G\nabla A_2)(\mathrm{Y})-A_1(\mathrm{Y})({}^{-1}G\nabla A_1)(\mathrm{X})-A_2(\mathrm{Y})({}^{-1}G\nabla A_2)(\mathrm{X})]$

Substitution of (5.11) into (5.5) (a) gives (5.6).

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