WAVELET GALERKIN SCHEME FOR PARTIAL DIFFERENTIAL EQUATIONS ARISING IN FLUID FLOW THROUGH POROUS MEDIA

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ABSTRACT:

The wavelet based numerical solution has recently developed theory and application for the solution of ordinary differential equations (ODE) and partial differential equations (PDE). Wavelet analysis is a numerical concept which allows to represents a function in terms of basis functions, called wavelets, which are localized both in location and scale. Nowadays Wavelet Galerkin method is most frequently used scheme. In this paper we have developed a wavelet Galerkin scheme for non-linear partial differential equations. The present scheme is efficient, accurate and has got several advantages over other numerical methods.

Keywords: Wavelet Galerkin method, Daubechies wavelet, Connection coefficients

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Introduction:

Wavelet theory[2,3] has generated a tremendous interest in many area of research in mathematics, physics, computer science and engineering.Wavelet theory provides various basis functions and multi-resolution methods. In fact, in the wavelet expansion of a function many coefficients are negligible and by discarding these coefficients we can obtain a sparse but accurate approximate representation. Moreover, using wavelet decomposition it is possible to detect singularities, irregular structure and transient phenomena exhibit by the analyzed function. In the last years, the good features of wavelets have generated a huge interest in different areas of applied mathematics, physics and engineering. Wavelet basis functions have many properties that make them desirable as a basis for a Galerkin approach for solving partial differential equations. However, these properties rely on the assumption that the partial differential equation is periodic in the computational domain and do not carry over when the domain of the partial differential equation is bounded. Orthogonality, for example, is lost when the basis functions are truncated at a boundary because the domain of integration is a finite interval.

Daubechies Wavelets:

The family of compactly supported wavelets constructed by Daubechies is widely used in mathematics. The Daubechies set of wavelets provide an orthogonal basis with which to approximate functions. The basic recuriosn, or dyadic, or multiresolution equation takes the form[2,3,5,10]

$$\phi(x) = \sum_{j=0}^{L-1} p_j \phi(2x - j) \tag{1}$$

and the equation

$$\psi(x) = \sum_{j=2-L}^{1} (-1)^j p_{1-j} \phi(2x - j) \tag{2}$$

where $\phi(x)$ and $\psi(x)$ are called scaling function and mother wavelet, respectively.

The coefficients p_k in the two-scale relation (1) are called the wavelet filter coefficients. Daubechies established these wavelet filter coefficients to satisfy the following conditions:

$$\sum_{j=0}^{L-1} p_j = 2 \tag{3}$$

$$\sum_{j=0}^{L-1} p_j p_{j-m} = \delta_{0,m} \tag{4}$$

$$\sum_{j=2-L}^{1} (-1)^j p_{1-j} p_{j-2m} = 0 \quad \text{for integer } m \tag{5}$$

$$\sum_{j=0}^{L-1} (-1)^j j^m p_j = 0, \quad m = 0,1,\ldots\ldots, L/2 - 1 \tag{6}$$

where $\delta_{0,m}$ is the Kronecker delta function.

Correspondingly, the constructed scaling function $\phi(x)$ and wavelet $\psi(x)$ have the following properties:

$$\int_{-\infty}^{\infty}\phi(x)\,dx=1 \tag{7}$$

$$\int_{-\infty}^{\infty}\phi(x-j)\,\phi(x-m)\,dx=\delta_{j,m} \tag{8}$$

$$\int_{-\infty}^{\infty}\phi(x)\,\psi(x-m)\,dx=0 \quad \text{for integer } m \tag{9}$$

$$\int_{-\infty}^{\infty}x^{m}\psi(x)\,dx=0, \quad k=0,1,.....,L/2-1 \tag{10}$$

Wavelet Galerkin Method:

The Galerkin method belongs to the family of weighted residuals methods, where the solution to a partial differential equation, $u(x,t)$ is approximated by a finite series of functions $\phi_k(x)$ as follows:

$$u(x,t)=\sum_{k=1}^{p}a_k(t)\,\phi_k(x) \tag{11}$$

where $\phi_k(x)$ are the basis or trial function, $a_k(t)$ are coefficients to be determined (possibly time-dependent) that satisfy the partial differential equation, and p are the number of functions. In general the approximate solution does not satisfy the partial differential equation exactly, and substituting its value results in a residual, which in turn is minimized in some sense.

Consider the following boundary value problem (BVP)[8]

$$Lu(x,t)-f(x,t)=0 \qquad ,x\in\Omega,t>0 \tag{12}$$

Substituting the trial solution (1), we get

$$L\sum_{k=1}^{p}a_k(t)\,\phi_k(x)-f(x,t)=R(x,t) \tag{13}$$

The method of weighted residuals minimizes $R(x,t)$ by forcing it to be zero in the domain Ω using weight functions $w_j(x)$ such that, for every weight function,

$$\int_{\Omega}R(x,t)\,w_j(x)\,dx=0 \qquad \text{for } j=1,2,.........,q$$

where q is the number of weight functions to be determined, depending on the boundary conditions and number of scaling functions. This discretization leads to a system of linear or non-linear ordinary differential equations where the values for $a_k(t)$ can be determined. In the

Galerkin method, the weight functions are chosen to be the basis functions of Daubecheis wavelet.

In wavelet Galerkin method we choose wavelet basis as weight functions. Having multiresoltuion analysis, $V_j, j \in Z$ with scaling function $\phi(x)$, one can use $\phi_{j,k}(x)$ as the basis functions for the Galerkin method. We know that the set $\{\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k) : k \in Z\}$ forms an orthonormal basis of V_j. Thus at approximation level j, one can take the orthogonal projection of u onto V_j in the following manner

$$u(x,t) \approx \Gamma_j u(x,t) = \sum_{k=1}^{p} a_{j,k}(t)\,\phi_{j,k}(x), \quad a_{j,k}(t) = \langle u(x,t), \phi_{j,k}(x)\rangle \tag{14}$$

and this will provide an accurate approximation to u. Furthermore, for some j, V_j will capture all details of the original function.

Definition of the connection coefficients on an interval:

Let ϕ, ψ be the scaling function and wavelet function[10] of an orthonormal resolution analysis with compact support $[N_1, N_2]$ $(N_1 < N_2)$. We have

$$\phi(x) = \sum_k a_k \phi(2x - k) \tag{15}$$

$$\int x^l \psi(x)\,dx = 0 \quad l = 0,1,......N-1 \tag{16}$$

where $a_k = 0$ for $k < N_1$ and $k > N_2$, $N \geq 1$.

Equation (15) is called dilation equation or two scale equation, and equation(16) is the vanishing moments condition. Let

$$\phi_l(x) = \phi(x - l), l \in Z \tag{17}$$

$$\phi_l^n(x) = \frac{d^n \phi_l(x)}{d x_n}, n \in Z^+ \tag{18}$$

For $l, m \in \Lambda^j$, $d_1, d_2 \geq 0$ we define the 2-term connection coefficients on an interval (at level j) by

$$\Omega_{l\,m}^{i,d_1 d_2} = \int_0^{2^j} \phi_l^{d_1}(x)\,\phi_m^{d_2}(x)\,d x \tag{19}$$

Here we assume ϕ has necessary derivatives. For $l, m, k \in \Lambda^j, d_1, d_2, d_3 \geq 0$ we define 3-term connection coefficients on an interval (at level j)

$$\Omega_{lmk}^{i,d_1 d_2 d_3} = \int_0^{2^j} \phi_l^{d_1}(x)\, \phi_m^{d_2}(x)\, \phi_k^{d_3}(x)\, dx \tag{20}$$

Similarly we can define any n-term $(n>3)$ connection coefficients on an interval.

It is easy to check that

$$\Omega_{lm}^{j+1,d_1 d_2} = \Omega_{l-2^j\; m-2^j}^{j,d_1 d_2} \tag{21}$$

$$\Omega_{lmk}^{j+1,d_1 d_2 d_3} = \Omega_{l-2^j\; m-2^j\; k-2^j}^{j,d_1 d_2 d_3} \tag{22}$$

Hence the calculation of the connection coefficients at level j $(j \geq 1)$ can be reduced to the calculation of the connection coefficients on $[0,1]$ (at level $j=0$):

$$\Omega_{lm}^{0,d_1 d_2} = \int_0^1 \phi_l^{d_1}(x)\, \phi_m^{d_2}(x)\, dx$$

$$\Omega_{lmk}^{0,d_1 d_2 d_3} = \int_0^1 \phi_l^{d_1}(x)\, \phi_m^{d_2}(x)\, \phi_m^{d_3}(x)\, dx$$

In the following we denote

$$\Omega_{lm}^{d_1 d_2} = \Omega_{lm}^{0,d_1 d_2}, \quad \Omega_{lmk}^{d_1 d_2 d_3} = \Omega_{lmk}^{0,d_1 d_2 d_3} \tag{23}$$

2-term connection coefficients:

Let $\Omega_{lm}^{d_1 d_2}$ be a column vector which $2N-3$ components are the connection coefficients[10] defined as

$$\Omega_{lm}^{d_1 d_2} = \int_{-\infty}^{\infty} \phi_l^{d_1}\, \phi_m^{d_2}\, dx$$

Taking derivatives of the scaling function $\phi(x)$, assuming it is d times differentiable gives

$$\phi^d(x) = 2^d \sum_{k=0}^{N-1} a_k \phi^d(2x)$$

After simplification and considering it for all $\Omega_{lm}^{d_1 d_2}$, gives a system of linear equations $\Omega^{d_1 d_2}$ with as unknown vector:

$$T\Omega^{d_1 d_2} = \frac{1}{2^{d-1}} \Omega^{d_1 d_2}$$

where $d = d_1 + d_2$

and

$$T_{l,q} = \sum_p a_p a_{q-2l+p}$$

These are the so called scaling equations. But this system is homogenous; and thus does not have a unique nonzero solution. In order to make the system inhomogeneous, one equation is added which is derived from the moment equations of the scaling function ϕ. This is the normalization equation

$$d! = (-1)^d \sum_l M_l^d \Omega_l^{0d}$$

The moments M_i^k of ϕ_i are defined as

$$M_i^k = \int_{-\infty}^{\infty} x^k \phi_i(x)\, dx$$

The scaling function $\phi(x)$ is normalized by definition so that $M_0^0 = 1$. By induction on k, Latto et al. derive an explicit formula to compute the moments

$$M_i^j = \frac{1}{2(2^j - 1)} \sum_{k=0}^{j} \binom{j}{k} i^{j-k} \sum_{l=0}^{k-1} \binom{k}{l} \left(\sum_{i=0}^{N-1} a_i i^{k-1} \right) \quad (24)$$

where the a_i are the Daubechies wavelet coefficients.

Finally, the system will be

$$\begin{pmatrix} T - \frac{1}{2^{d-1}} I \\ M^d \end{pmatrix} \Omega^{d_1 d_2} = \begin{pmatrix} 0 \\ d! \end{pmatrix} \quad (25)$$

where M^d is a row vector with all the M_l^d. This is an over-determined $(2N-3)+1 * (2N-3)$ system which has as its unique solution the exact solution of the connection coefficients $\Omega_l^{d_1 d_2}$. Similarly 3-term and 4-term connection coefficients are evaluated.

Wavelet Galerkin Scheme:

Problem 1: Consider the governing equation for the two phase flow through a porous medium[4] is a quasi-linear first-order partial differential equation is given by

$$\frac{\partial \sigma}{\partial t} + \frac{W}{m} f'(\sigma) \frac{\partial \sigma}{\partial x} = 0 \quad (26)$$

or

$$\frac{\partial \sigma}{\partial t} + \frac{W}{m} \frac{\partial f(\sigma)}{\partial x} = 0$$

where

$$f'(\sigma) = \frac{d f(\sigma)}{d\sigma}\frac{\partial f(\sigma)}{\partial x} = \frac{d f(\sigma)}{d\sigma}\frac{\partial \sigma}{\partial x}$$

We introduce dimensionless variables of length and time:

$$\xi = \frac{x}{L} \quad , \quad \tau = \frac{Wt}{mL} \qquad \tau \geq 0, 0 \leq \xi \leq 1 \tag{27}$$

where L is the length of the reservoir, mL is the pore volume for a reservoir of unit cross-sectional area.

The dimensionless time τ can be interpreted as the ratio of the volume of liquid injected into the reservoir.

From equation (27)

$$x = L\xi \quad , \quad t = \frac{mL}{W}\tau \tag{28}$$

$$\partial x = L\partial\xi \quad , \quad \partial t = \frac{mL}{W}\partial\tau$$

Substituting these in equation (26) we get

$$\frac{W}{mL}\frac{\partial\sigma}{\partial\tau} + \frac{W}{mL}\frac{\partial f(\sigma)}{\partial\xi} = 0 \tag{29}$$

Or

$$\frac{\partial\sigma}{\partial\tau} + \frac{\partial f(\sigma)}{\partial\xi} = 0$$

Considering $f(\sigma) = \mu\sigma$ equation (29) reduces to

$$\frac{\partial\sigma}{\partial\tau} + \mu\frac{\partial\sigma}{\partial\xi} = 0 \tag{30}$$

Now for applying Wavelet Galerkin Method[8] to equation (30) we multiply it by Daubechies Wavelet basis function $\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k)$ and integrating it over (0 , 1) we get,

$$\int_0^1 \frac{\partial\sigma}{\partial\tau}\phi_{jk}(\xi)\,d\xi + \mu\int_0^1 \frac{\partial\sigma}{\partial\xi}\phi_{jk}(\xi)\,d\xi = 0$$

$$\int_0^1 \frac{\partial\sigma}{\partial\tau}\phi_{jk}(\xi)\,d\xi + \mu\left[\sigma\phi_{jk}(\xi)\right]_0^1 - \mu\int_0^1 \sigma\phi'_{jk}(\xi)\,d\xi = 0 \tag{31}$$

Now we set,

$$\sigma(\tau,\xi)=\sum_{l=1}^{n}a_l(\tau)\,\phi_{jl}(\xi) \tag{32}$$

Substituting equation (32) into equation (31) we get,

$$\int_0^1\sum_{l=1}^{n}a_l^{'}(\tau)\,\phi_{jl}(\xi)\,\phi_{jk}(\xi)\,d\xi-\mu\int_0^1\sum_{l=1}^{n}a_l(\tau)\,\phi_{jl}(\xi)\,\phi^{'}_{jk}(\xi)\,d\xi=0 \tag{33}$$

Equation (33) reduces to

$$\sum_{l=1}^{n}a_l^{'}(t)\int_0^1\phi_{jl}(\xi)\,\phi_{jk}(\xi)\,d\xi-\mu\sum_{l=1}^{n}a_l(\tau)\int_0^1\phi_{jl}(\xi)\,\phi^{'}_{jk}(\xi)\,d\xi=0 \tag{34}$$

Equation (34) reduces to

$$M\dot{A}+NA=0 \tag{35}$$

where

$$M=\int_0^1\phi_{jl}(\xi)\,\phi_{jk}(\xi)\,d\xi$$

$$N=\int_0^1\phi_{jl}(\xi)\,\phi^{'}_{jk}(\xi)\,d\xi$$

Problem 2: Consider the governing Richard's Equation[1] for unsteady horizontal flow as follow:

$$F\frac{\partial h}{\partial t}=\frac{\partial}{\partial x}\left(K_sK_r\frac{\partial h}{\partial x}\right) \tag{36}$$

where h is the pressure head, k_s is the saturated hydraulic conductivity, k_r is the relative hydraulic conductivity, x is the horizontal dimension, t is time and F is the water capacity

$$F=\frac{\partial\theta}{\partial h} \tag{37}$$

where θ is the moisture content. Let us consider first linear moisture content vs. pressure head and relative hydraulic conductivity vs. pressure head curves as follows:

$$\theta=\theta_r+\frac{\theta_s-\theta_r}{h_s-h_r}(h-h_r) \tag{38}$$

where θ_r is a small residual moisture content occurring to h_r, θ_s is the maximum moisture content, and h_s (usually, $h_s=0$) is the pressure head corresponding to θ_s

$$k_r=\frac{h-h_r}{h_s-h_r} \tag{39}$$

Applying Wavelet Galerkin Method[8] on both sides of equation (36) by Daubechies Wavelet basis function $\phi_{j,k}(x)=2^{j/2}\phi(2^jx-k)$ and integrating with respect to x over [0,L] we get

$$\int_0^L F\frac{\partial h}{\partial t}\phi_{jk}(x)\,dx=\int_0^L\frac{\partial}{\partial x}\left[K_sK_r\frac{\partial h}{\partial x}\right]\phi_{jk}(x)\,dx$$

$$\int_0^L F\frac{\partial h}{\partial t}\phi_{jk}(x)\,dx=\left[K_sK_r\frac{\partial h}{\partial x}\phi_{jk}(x)\right]_0^L-\int_0^L K_sK_r\frac{\partial h}{\partial x}\phi'_{jk}(x)\,dx \tag{40}$$

Now we set,

$$h(x,t)=\sum_{l=1}^{n}h_l(t)\,\phi_{jl}(x) \tag{41}$$

Substituting equation (41) into equation (40) we get,

$$\int_0^L\sum_{l=1}^{n}h'_l(t)\,\phi_{jl}(x)\,\phi_{jk}(x)\,dx=-K_sK_r\int_0^L\sum_{l=1}^{n}h_l(t)\,\phi'_{jl}(x)\,\phi'_{jk}(x)\,dx \tag{42}$$

Equation (42) reduces to

$$\sum_{l=1}^{n}h'_l(t)\int_0^L\phi_{jl}(x)\,\phi_{jk}(x)\,dx=-K_sK_r\sum_{l=1}^{n}h_l(t)\int_0^L\phi'_{jl}(x)\,\phi'_{jk}(x)\,dx \tag{43}$$

Equation (43) reduces to

$$M\dot{h}+Nh=0 \tag{44}$$

where

$$M=\int_0^L\phi_{jl}(x)\,\phi_{jk}(x)\,dx$$

$$N=\int_0^L\phi'_{jl}(x)\,\phi'_{jk}(x)\,dx$$

Problem 3: Consider the desired advective diffusion equation[9]

$$\frac{\partial C}{\partial t}+\frac{\partial(uC)}{\partial x}=D\frac{\partial^2 C}{\partial x^2} \tag{45}$$

First we develop a weak formulation, from which we will derive the discretization. Multiplying both sides of advection- diffusion equation by a test function, $v(\cdot)\in V$ and integrating, we get

$$\int_0^1\frac{\partial C}{\partial t}v\,dx+u\int_0^1\frac{\partial C}{\partial x}v\,dx=D\int_0^1\frac{\partial^2 C}{\partial x^2}v\,dx \tag{46}$$

Let the trial solution of partial differential equation (46) be

$$C(x,t)=\sum_{k=-N+1}^{n}a_k(t)\,\phi_{j,k}(x) \tag{47}$$

Where $n=2^j$ and $\phi_{j,k}(x)=2^{j/2}\phi(2^j x-k)$ and $\phi(x)$ is a scaling function of Daubechies compactly supported wavelet with vanishing moment $N/2$. Now taking test function as $\phi_{j,p}(x)=2^{j/2}\phi(2^j x-p)$, we get a system of simultaneous differential equation

$$\int_0^1 \sum_{k=-N+1}^{n} \dot{a}_k(t)\,\phi_{j,k}(x)\,\phi_{j,p}(x)\,dx =$$

$$-u\int_0^1 \sum_{k=-N+1}^{n} a_k(t)\,\phi'_{j,k}(x)\,\phi_{j,p}(x)\,dx - D\int_0^1 \sum_{k=-N+1}^{n} a_k(t)\,\phi'_{j,k}(x)\,\phi'_{j,p}(x)\,dx$$

$$\dot{a}_p(t) = -u\sum_{k=-N+1}^{n} a_k(t)\,\Lambda_{k,p}^{1,0} - D\sum_{k=-N+1}^{n} a_k(t)\,\Omega_{k,p}^{1,1} \qquad (48)$$

Where $\Lambda_{k,p}^{1,0} = \int_0^1 \phi'_{j,k}(x)\,\phi_{j,p}(x)\,dx$

$$\Omega_{k,p}^{1,1} = \int_0^1 \phi'_{j,k}(x)\,\phi'_{j,p}(x)\,dx$$

which is the required Wavelet Galerkin Model.

Conclusion:

In the present paper a new wavelet Galerkin scheme is developed for solving non-linear PDEs. The approach used in obtaining this model is a weighted residual approach which leads differential equation into a system of simultaneous differential equation. The system of simultaneous differential equation can be solved by various techniques to obtain the required results. In the present chapter our objective is to present a Wavelet Galerkin model which is an effective model in comparison with other models like FEM model, FDM model etc. The Wavelet Galerkin model is derived in a simplified manner and may reduce the computational cost in comparison with other numerical models.

References:

1. F.T. Tracy(1995),1-D,2-D and 3-D analytical solutions of unsaturated flow in groundwater, Jouranal of Hydrology, vol.170, pg. 199-214.
2. I.Daubechies,(1988), Orthonormal bases of compactly supported wavelets, Communications on Pure andApplied Mathematics, vol. 41, no. 7, pp. 909–996.
3. J. Besora(2004),Galerkin Wavelet Method for global waves in 1D,KTH Numerical Analysis and Computer Science, Sweden,pp.1-43.
4. M. Al-Mannai, N.S. Khabeev(2012),Mathematical modeling of Oil Recovery, International Journal of Pure and Applied Mathematics,vol.77, no. 2,pg. 191-198.
5. M.Q. Chen, C. Hwang and Y.P. Shih, The computation of Wavelet-Galerkin approximation on a bounded interval",International Journal for Numerical Methods in Engineering,1996, vol. 39, pp. 2921-2944.
6. O.V. Vasilyev et. Al.(1997), Solving PDEs using Wavelets, Computer in Physics,vol.11,pp.429.
7. P. Suji, "Bulking of bar by Wavelet galerkin method",2012,pp. 1-85.
8. R.C. Mittal,S. Kumar(2010),A numerical study of stationary solution of viscous Burgers' equation using wavelet, International journal of Computer Mathematics, volume 87, issue 6,pp. 1326-1337.
9. S.A. Socolofsky and G.H. Jrika(2004), Advection Diffusion Equation,pp. 1-4
10. V. Mishra and S. Jindal(2011),Wavelet Galerkin Solutions of ordinary differential equations, International journal of Math. Analysis. Volume 5, issue 9,pp.407-424.