AFFINE CONNECTION IN AN L-CONTACT MANIFOLD

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ABSTRACT

In 1988, K. Matsumoto and I. Mihai [1] discussed on a certain transformation in a Lorentzian Para-Sasakian manifold. T. Suguri and S. Nakayama [3] considered D-conformal deformations on almost contact metric structure. In 1972, R.S. Mishra [2] discussed on affine connexion in an almost Grayan manifold. The purpose of this paper is to study D-conformal transformation in an L-Contact manifold. Affine connection in an L-Contact manifold has also been discussed.

Keywords: L-Contact structure, D-conformal transformation, affine connection.

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1. Introduction

An n-dimensional differentiable manifold M_n, on which there are defined a tensor field F of type (1, 1), a vector field T, a 1-form A and a Lorentzian metric g, satisfying for arbitrary vector fields $X, Y, Z, \ldots$

(1.1) $\bar{\bar{X}} = -X - A(X)T, \bar{T} = 0, A(T) = -1, \ \bar{X} \stackrel{\text{def}}{=} FX, A(\bar{X}) = 0$, rank $F = n - 1$

(1.2) $g\,(\bar{X}, \bar{Y}) = g\,(X, Y) + A(X)A(Y)$, where $A(X) \stackrel{\text{def}}{=} g(X, T)$,

$`F(X, Y) \stackrel{\text{def}}{=} g\left(\bar{X},\ Y\right) = -g\left(\bar{Y},\ X\right) = -`F(Y, X)$,

Then M_n is called a Lorentzian contact manifold (an L-Contact manifold) and the structure (F, T, A, g) is known as Lorentzian contact structure (an L-Contact structure).

Let D be a Riemannian connection on M_n, then we have

(1.3) (a) $(D_X`F)\left(\bar{Y},\ Z\right) - (D_X`F)\left(Y, \bar{Z}\right) + A(Y)(D_XA)(Z) + A(Z)(D_XA)(Y) = 0$

(b) $(D_X`F)\left(\bar{Y},\ \bar{\bar{Z}}\right) = (D_X`F)\left(\bar{\bar{Y}}, \bar{Z}\right)$

(1.4) (a) $(D_X`F)\left(\bar{Y},\ \bar{Z}\right) + (D_X`F)(Y, Z) + A(Y)(D_XA)\left(\bar{Z}\right) - A(Z)(D_XA)\left(\bar{Y}\right) = 0$

(b) $(D_X`F)\left(\bar{\bar{Y}}, \bar{\bar{Z}}\right) + (D_X`F)\left(\ \bar{Y},\ \bar{Z}\ \right) = 0$

An L –Contact manifold is called an L-Cosymplectic manifold if

(1.5) $D_XF = 0$

2. D-conformal transformation

Let the corresponding Jacobian map J of the transformation b transforms the structure $(F, T, A, g\ )$ to the structure $(F, V, v, h\ )$ such that

(2.1) (a) $J\bar{Z} = \overline{JZ}$ (b) $h(\,JX, JY)ob = e^{\sigma}\,\text{g}\left(\bar{X}, \bar{Y}\right) - e^{2\sigma}\,A(X)A(Y)$

(c) $V = e^{-\sigma}\,JT$, (d) $v(\,JX\,)ob = e^{\sigma}\,A(X)$, Where σ is a differentiable function on M_n, then the transformation is called D-conformal transformation.

If σ is a constant, the transformation is known as D-homothetic.

Theorem 2.1 The structure (F, V, v, h) is Lorentzian contact.

Proof. Inconsequence of (1.1), (1.2), (2.1) (b) and (2.1) (d), we have

$$h(J\overline{X}, J\overline{Y})ob = e^{\sigma} g(\overline{X}, \overline{Y}) = h(JX, JY)ob + e^{2\sigma} A(X)A(Y)$$
$$= h(JX, JY)ob + \{v(JX)ob\}\{v(JY)ob\}$$

This implies

(2.2) $\quad h(J\overline{X}, J\overline{Y}) = h(JX, JY) + v(JX)\, v(JY)$

Making the use of (1.1), (2.1) (a), (2.1) (c) and (2.1) (d), we get

(2.3) $\quad \overline{\overline{JX}} = J\overline{\overline{X}} = -JX - A(X)JT = -JX - \{v(JX)ob\}V$

Also

(2.4) $\quad \overline{V} = e^{-\sigma}\overline{JT} = 0,$

Equations (2.2), (2.3) and (2.4) prove the statement.

Theorem 2.2 Let E and D be the Riemannian connections with respect to h and g such that

(2.5) (a) $\quad E_{JX}JY = JD_XY + JH(X,Y)$ $\qquad$ (b) $\quad `H(X,Y,Z) \overset{\text{def}}{=} g(H(X,Y),Z)$

Then

(2.6) $\quad 2E_{JX}JY =$

$2JD_XY - J[2e^{\sigma}\{(X\sigma)A(Y)\,T + (Y\sigma)A(X)\,T - ({}^{-1}G\nabla\sigma)A(X)A(Y)\} + (e^{\sigma} -$

1DXAY+DYAX− 2AHX,YT+eσ −1AXDYT+AYDXT−AX(−1G∇A)Y−AY(−1G∇A)X]

Proof. Inconsequence of (2.1) (b), we have

$$JX(h(JY, JZ))ob = X\{e^{\sigma} g(\overline{Y}, \overline{Z}) - e^{2\sigma} A(Y)A(Z)\}$$

From (2.1) (b) and (2.5), we have

(2.7) $\quad h(E_{JX}JY, JZ)ob + h(JY, E_{JX}JZ)ob = e^{\sigma} g(\overline{D_XY}, \overline{Z}) - e^{2\sigma} A(D_XY)A(Z) +$

$e^{\sigma} g(\overline{H(X,Y)}, \overline{Z}) - e^{2\sigma} A(H(X,Y))A(Z) + e^{\sigma} g(\overline{Y}, \overline{H(X,Z)}) - e^{2\sigma} A(Y)A(H(X,Z)) +$

$e^{\sigma} g(\overline{Y}, \overline{D_XZ}) - e^{2\sigma} A(D_XZ)A(Y)$

Also

(2.8)

$$h(E_{JX}JY, JZ)ob + h(JY, E_{JX}JZ)ob = (X\sigma)e^{\sigma}\, g(\overline{Y}, \overline{Z}) + e^{\sigma}\, g(D_X\overline{Y}, \overline{Z}) + e^{\sigma}\, g(\overline{Y}, D_X\overline{Z}) - 2(X\sigma)e^{2\sigma} A(Y)A(Z) - e^{2\sigma}(D_XA)(Y)A(Z) - e^{2\sigma}(D_XA)(Z)A(Y) - e^{2\sigma} A(D_XY)A(Z) - e^{2\sigma} A(D_XZ)A(Y)$$

Equations (1.3) (a), (2.7) and (2.8) imply

(2.9) $$(X\sigma)g(\overline{Y}, \overline{Z}) - 2(X\sigma)e^{\sigma} A(Y)A(Z) - (e^{\sigma} - 1)\{(D_XA)(Y)A(Z) + (D_XA)(Z)A(Y)\} = {}`H(X, Y, Z) + {}`H(X, Z, Y) - (e^{\sigma} - 1)\{A(H(X, Y))A(Z) + A(H(X, Z))A(Y)\}$$

Writing two other equations by cyclic permutation of X, Y, Z and subtracting the third equation from the sum of the first two equations and using symmetry of $`H$ in the first two slots, we get

(2.10) $$2`H(X, Y, Z) = -2e^{\sigma}\{(X\sigma)A(Y)A(Z) + (Y\sigma)A(Z)A(X) - (Z\sigma)A(X)A(Y)\} - (e^{\sigma} - 1)[A(Z)\{(D_XA)(Y) + (D_YA)(X) - 2A(H(X, Y))\} + A(X)\{(D_YA)(Z) - (D_ZA)(Y)\} + A(Y)\{(D_XA)(Z) - (D_ZA)(X)\}]$$

This gives

(2.11) $2H(X, Y) = -2e^{\sigma}[(X\sigma)A(Y)T + (Y\sigma)A(X)\, T - ({}^{-1}G\nabla\sigma)A(X)A(Y)] - (e^{\sigma} -$
1DXAY+DYAX−2AHX,YT+AXDYT+AYDX T−AX(−1G∇AY−AY(−1G∇A)(X)]

Substitution of (2.11) into (2.5) (a) gives (2.6).

3. Affine connection

Let B be an affine connection in M_n, then

(3.1) $$B_XY = D_XY + H(X, Y)$$

And torsion tensor S of B is given by

(3.2) $$S(X, Y) = H(X, Y) - H(Y, X)$$

Let us define

(3.3) (a) $`H(X, Y, Z) \stackrel{\text{def}}{=} g(H(X, Y), Z)$ and (b) $`S(X, Y, Z) \stackrel{\text{def}}{=} g(S(X, Y), Z)$, Then

(3.4) $`S(X,Y,Z) = `H(X,Y,Z) - `H(Y,X,Z)$

Theorem 3.1 Let B be an affine connection in M_n satisfying

(3.5) (a) $(d`F)(X,Y,Z) = 0$

(b) $`S(X,Y,\overline{Z}) + `S(Y,Z,\overline{X}) + `S(Z,X,\overline{Y}) = 0$, Then

(3.6) $(B_X`F)(Y,Z) + (B_Y`F)(Z,X) + (B_Z`F)(X,Y) = 0.$

Proof. We have

$$X\,(`F(Y,Z)) = (B_X`F)(Y,Z) + `F(B_XY,Z) + `F(Y,B_XZ)$$

$$= (D_X`F)(Y,Z) + `F(D_XY,Z) + `F(Y,D_XZ)$$

Writing two other equations by cyclic permutations of $X,Y,Z,$ adding these and using (3.1), (3.3) (a), we get

$$(d`F)(X,Y,Z) = (B_X`F)(Y,Z) + (B_Y`F)(Z,X) + (B_Z`F)(X,Y) - `H(X,Y,\overline{Z}) - `H(Y,Z,\overline{X}) - `H(Z,X,\overline{Y}) + `H(X,Z,\overline{Y}) + `H(Y,X,\overline{Z}) + `H(Z,Y,\overline{X})$$

Using (3.5) (a) and (3.5) (b) in above equation, we get (3.6).

Theorem 3.2 Let B be an affine connection in M_n satisfying

(3.7) (a) $(B_X`F)(Y,Z) = 0$

(b) $`H(X,Y,\overline{Z}) + `H(Z,X,\overline{Y}) = `H(Z,Y,\overline{X})$, Then

(3.8) (a) $`F$ is closed if $`H(X,Y,\overline{Z}) + `H(Y,Z,\overline{X}) + `H(Z,X,\overline{Y}) = 0$ and

(b) An L- Contact manifold is an L-Cosymplectic if $`H(X,Y,\overline{Z}) = 0$

Proof. We have

$$X\,(`F(Y,Z)) = (B_X`F)(Y,Z) + `F(B_XY,Z) + `F(Y,B_XZ)$$

$$= (D_X`F)(Y,Z) + `F(D_XY,Z) + `F(Y,D_XZ)$$

From which, we obtain

(3.9) (a) $(D_X`F)(Y,Z) = `H(X,Z,\overline{Y}) - `H(X,Y,\overline{Z})$

Similarly, we can write

(b) $(D_Y`F)(Z,X) = `H(Y,X,\overline{Z}) - `H(Y,Z,\overline{X})$ and

(c) $(D_Z`F)(X,Y) = `H(Z,Y,\overline{X}) - `H(Z,X,\overline{Y})$

Adding these and using (3.7) (b), we get

$$(D_X`F)(Y,Z)+(D_Y`F)(Z,X) + (D_Z`F)(X,Y) = 0$$

Consequently, $`F$ is closed. (3.8) (b) follows from (1.5), (3.7) (a) and (3.7) (b).

Let us note one more obvious fact.

Theorem 3.3 Let B be an affine connection in M_n satisfying

(3.10) (a) $(B_X`F)(Y,Z) = 0$

(b) $`H(X,Y,\overline{Z}) - `H(Z,X,\overline{Y}) = `S(X,Z,\overline{Y}) = 0$, Then

An L- Contact manifold is an L-Cosymplectic manifold. Also $`F$ is closed if

$$`S(X,Y,\overline{Z}) + `S(Y,Z,\overline{X}) + `S(Z,X,\overline{Y}) = 0,$$

Theorem 3.4 Let B be an affine connection in M_n satisfying

(3.11) $`H(X,\overline{Y},\overline{Z}) + `H(X,\overline{Z},\overline{Y}) = 0$, Then

(3.12) (a) $(B_X g)(\overline{Y},\overline{Z}) = 0$

(b) $g(B_X\overline{Y},\overline{Z}) = g(D_X\overline{Y},\overline{Z}) - `H(X,\overline{Z},\overline{Y})$

Proof. We have

$$X\,(g(\overline{Y},\overline{Z})) = (B_X g)(\overline{Y},\overline{Z}) + g(B_X\overline{Y},\overline{Z}) + g(\overline{Y},B_X\overline{Z})$$

$$= g(D_X\overline{Y},\overline{Z}) + g(\overline{Y},D_X\overline{Z})$$

Using (3.1), (3.3) (a) and (3.11), we get (3.12) (a). (3.12) (b) follows from (3.1), (3.3) (a) and (3.11).

Theorem 3.5 Let B be an affine connection in M_n satisfying

(3.13) (a) $(B_X`F)(\overline{Y},\overline{Z}) = 0$

(b) $`H\left(X,\overline{Y},\overline{\overline{Z}}\right) + `H\left(X,\overline{Z},\overline{\overline{Y}}\right) = 0$, Then

an L- Contact manifold is an L-Cosymplectic if $`H\left(X,\overline{Y},\overline{\overline{Z}}\right) = `H\left(X,\overline{Z},\overline{\overline{Y}}\right)$

Proof. Inconsequence of (3.13) (a), we have

$$X\,(`F(\overline{Y},\overline{Z})) = `F(B_X\overline{Y},\overline{Z}) + `F(\overline{Y},B_X\overline{Z})$$

$$= (D_X`F)(\overline{Y},\overline{Z}) + `F(D_X\overline{Y},\overline{Z}) + `F(\overline{Y},D_X\overline{Z})$$

Result follows from (3.1), (3.3) (a) and (3.13) (b).

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