# "APPLICATIONS OF FRACTIONAL DIFFERENTIAL EQUATIONS"

**Dr. Pooja Khurana***

**Abstract**

The study focuses on the uses of fractional differential equations (FDEs) in modelling physical phenomena. Also some applications of the theory of fractional calculus and some applications of fractional calculus in science and engineering have been reviewed. Merits and demerits of the simplest fractional calculus viscoelastic models, which manifest themselves during application of such models in the problems of forced and damped vibrations of linear and nonlinear hereditarily elastic bodies, propagation of stationary and transient waves in such bodies, as well as in other dynamic problems, are demonstrated with numerous examples. As this takes place, a comparison between the results obtained and the results found for the similar problems using viscoelastic models with integer derivatives is carried out. The concept of a fractional differential equation has been introduced. A brief survey of the possible used of fractional differential equations is presented. Specific examples from recent applications enable to understand the variety of fields in which fractional calculus can be of use.

Keywords: fractional differential equations, viscoelasticity, single FDEs, multi FDEs, Distributed Order Differential Equations

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## Introduction

Fractional calculus is a field of mathematical study that grows out of the traditional definitions of calculus integral and derivative operators in much the same way fractional exponents is an outgrowth of exponents with integer value. Free damped vibrations are often used in Rheology for the determination of dynamic mechanical properties of linear viscoelastic materials. The concept of distributed order differential equations (DODEs) is introduced. Several uses of linear and non-linear single and multi-term FDEs have been discussed.

## Review of Literature

Bagley & Torvik (1983) The study gives analytical solutions to the various fractional differential equations with which we are concerned and explain why numerical approximations are necessary.

Bagley & Torvik (1984) Generalized constitutive relationships for viscoelastic materials are suggested in which the customary time derivatives of integer order are replaced by derivatives of fractional order. To this point, the justification for such models has resided in the fact that these are effective in describing the behavior of real materials. In the work, the fractional derivative is shown to arise naturally in the description of certain motions of a Newtonian fluid. The study claims this provides some justification for the use of adhoc relationships which include the fractional derivative. An application of such a constitutive relationship to the prediction of the transient response of a frequency-dependent material is included.

Bagley & Torvik (1986) A mathematical model of the viscoelastic phenomenon employing derivatives of fractional order is examined in light of its consistency with thermodynamic principles. In particular, the development of constraints on parameters of the model ensure that the model predicts a nonnegative rate of energy dissipation and a nonnegative internal work. These constraints lead the model to predict realistic sinusoidal response as well as realistic relaxation and creep responses. Coupled with the established properties of the model, these attractive characteristics enhance its credibility for engineering analyses.

Dalir & Bashour (2010) Different definitions of fractional derivatives and fractional Integrals (Differintegrals) are considered. By means of them explicit formula and graphs of some special functions are derived. In the paper we consider different definitions of fractional derivatives and integrals (differintegrals). For some elementary functions, explicit formula of fractional drevative and integral are presented.

Debnath (2003) The paper deals with recent applications of fractional calculus to dynamical systems in control theory, electrical circuits with reactance, generalized voltage divider, viscoelasticity, fractional-order multi poles in electromagnetism, electrochemistry, tracer in fluid flows, and model of neurons in biology. Special attention is given to numerical computation of fractional derivatives and integrals.

Grootenhuis (1970) The transmission through the structure of vibrational energy can then be reduced by introducing viscoelastic materials at discrete points such as at the edges of panels. The requirements for edge type damping are a low stiffness and a high loss factor.

Koeller (1984) The connection between the fractional calculus and the theory of Abel's integral equation is shown for materials with memory. Expressions for creep and relaxation functions, in terms of the Mittag-Leffler function that depends on the fractional derivative parameter β, are obtained. These creep and relaxation functions allow for significant creep or relaxation to occur over many decade intervals when the memory parameter, β, is in the range of 0.05–0.35.

Rossikhin & Shitikova (1997) The aim of the article is to collect together separated results of research in the application of fractional derivatives and other fractional operators to problems connected with vibrations and waves in solids having hereditarily elastic properties, to make critical evaluations, and thereby to help mechanical engineers who use fractional derivative models of solids in their work.

**Introduction and Applications of Fractional Differential Equations**

**Objectives of the study**

In ordinary differential equations, if the values of a problem's derivatives are known at a particular instance, the value of the solution at that instance can be determined. In fractional differential equations previous values of the solution and the derivatives are required to obtain a solution at a particular instance. The memory effect of the convolution in the fractional integral gives the equation increased expressive power. The study allows the modelling of a variety of physical behaviour, such as viscoelasticity, diffusion, viscoplasticity, as well as more abstract concepts such as mappings using tensor fields.If multiple instances of derivatives of different orders are added,composite behavior can be obtained. These multi-term and distributed order equations have recently been the subject of models from a variety of industrial fields.

**Single-term FDEs and Multi-term FDEs**

The single-term non-linear FDE is defined as:

$$D^{\alpha}y(t) = f(t, y(t)),$$

The non-linear multi-term equation is defined as:

$$D^{\alpha_N}y(t) = f(t, y(t), D^{\alpha_1}y(t), \ldots, D^{\alpha_{N-1}}y(t))$$

equipped with initial conditions,

$$y^{(k)}(0) = y_0^{(k)}, k = 0, 1, \ldots, \lceil \alpha_N \rceil - 1.$$

The linear multi-term FDE is defined as:

$$\left[D^{\alpha_N} + b_{N-1}D^{\alpha_{N-1}} + \ldots + b_1 D^{\alpha_1} + b_0 D^0\right] y(t) = g(t),$$

with same initial conditions defined above.

**Viscoelastic Damped Vibration**

In the aerospace industry, to stop unwanted motion in gas turbine fan blades, viscoelastic damping is applied. Hartley and Lorenzo illustrate how fractional calculus can be used to model the damping required. Figure 1.1 shows a spring-mass-visco damped dynamic system. The required function requires zero initializations, thus the equations describing figure 1.1 are given by:

$$F_1 = k_1 D^{q_1}(x_0 - x_{m1})$$

$$F_1 = k(x_{m1} - x_i)$$

$$F_2 = -k_2 D^{q_2}(x_0 - x_{m2})$$

$$F_2 = k_3 D^{q3}(x_{m2} - x_i)$$

$$F_1 + F_2 = mD^2 x_0$$

Where

m, = mass

F = force

x = displacements

k = damping coefficients

The q values can vary between 0 and 1 representing a continuum from a spring of infinite stiffness to a. dashpot. For any given nass and force, the displacement can be controlled by manipulating the q values.

The above is a vector single-term FDE.

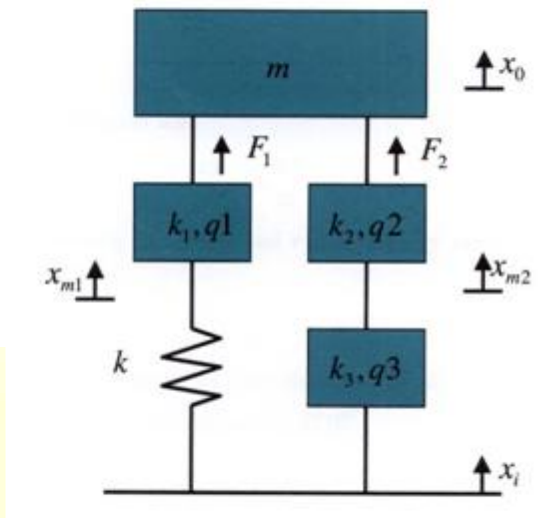


Figure 1: Showing Viscoelastic Damped Vibrations

**Viscoplasticity**

Viscoplasticity (A. Freed, Informal communication, 2004) is used as a model for rate-dependent plasticity. This is important for (high-speed) transient plasticity calculations. Viscoplasticity influences the stresses via the plastic strains

$$\sigma = E(\epsilon - \epsilon^p)$$

$$\frac{d\epsilon^p}{dt} = [f(\sigma - \beta)](\sigma - \beta)$$

α stress

€ strain

$€^P$ plastic strain

E modulus

β back stress (internal or hidden variable)

Example of f:

$$f = a|\sigma - \beta|^n$$

where α, $\eta$ are constants.

β is governed by evolution law,

$$\frac{d\beta}{dt} = h\frac{d\epsilon^p}{dt} - r(\beta)$$

where h is a hardening parameter and r is a recovery function which can be thermal or dynamic.

For thermal recovery

$$r(\beta) = r_0\beta^4.$$

The power 4 of the β term comes from physics.

This results in the non-linear single-term FDE.

**Bagley-Torvik Equation**

The motion of a rigid plate, mass (M) and area (S), connected by a massless spring of stiffness (K), immersed in a Newtonian fluid, was originally proposed by Bagley and Torvik.

The section follows the exposition of Podlubny.

Now let p be the fluid density, μ be the viscosity and v(t,z) be the transverse velocity, which is a function of time t and the distance from the plate, z. The displacement of the plate, y, is described by,

$$MD^2y(t) = g(t) - Ky(t) - 2S\sigma(t, 0).$$

where

$$\sigma(t,z) = \sqrt{\mu\rho}D^{1/2}v(s,z).$$

Therefore,

$$AD^2y(t) + BD^{3/2}y(t) + Cy(t) = g(t), \quad y(0) = y'(0) = 0,$$

where, $A = M$, $B = 2S\sqrt{\mu\rho}$, and $C = K$.

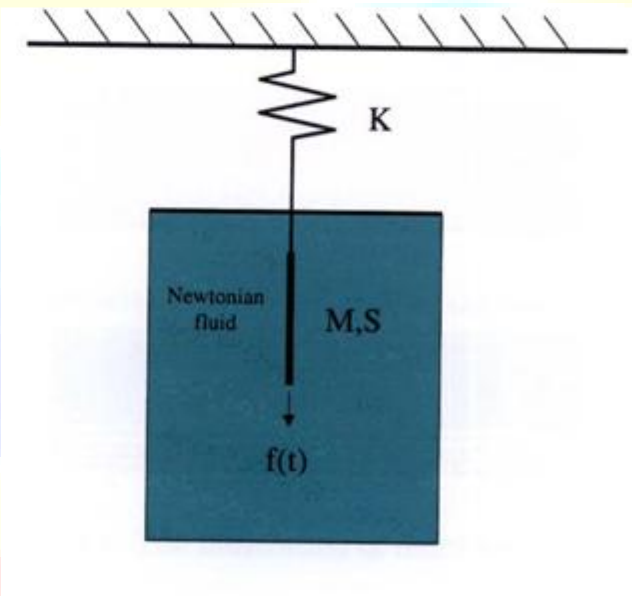


Figure 2: Showing an Immersed Plate In Newtonian Fluid

**Distributed Order Differential Equations**

Diethelm and Ford states that a distributed order differential equation (DODE) has the form,

$$\sum_{n=1}^{N} \lim_{\epsilon_n \to 0} \int_{\epsilon_n}^{1} b_n(\alpha) I^{\alpha} u^{(n)}(t) d\alpha + \sum_{n=0}^{N} c_n u^{(n)}(t) = g(t).$$

The concept of a distributed order differential equation (DODE) can be thought of as generalizing a multi-term equation. The study considers a tile made up of layers of different viscoelastic components, each layer can be modelled individually by a fractional derivative, and any thickness or density issues by appropriate constants. In conjunction the tile can be modelled by a multi-term fractional differential equation comprising of the individual fractional derivatives. Consider a tile with the particular property that the fractional derivatives of sequential layers possess orders which are monotonic. As progressively more layers have been introduced, illustrated in Figure 1.3, in the limiting case, as the thickness of layers approaches 0, a DODE is generated.

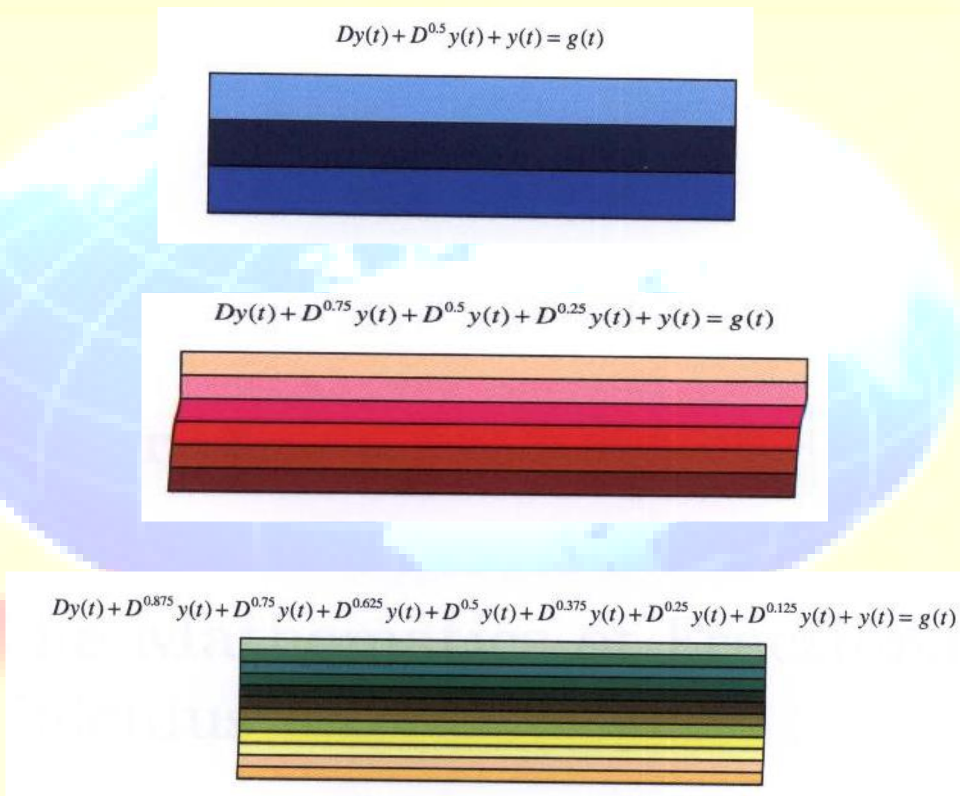


Figure 3: Showing Tile illustration of multi-term problem

In practice the engineering of such a tile is unlikely. The same material properties can be produced however, by applying a temperature gradient to a tile made up of thermo-viscoelastic material.

**Objectives**

In ordinary differential equations, if the values of a problem's derivatives are known at a particular instance, the value of the solution at that instance can be determined. In fractional differential equations previous values of the solution and the derivatives are required to obtain a solution at a particular instance. The memory effect of the convolution in the fractional integral gives the equation increased expressive power. The study allows the modelling of a variety of physical behaviour, such as viscoelasticity, diffusion, viscoplasticity, as well as more abstract concepts such as mappings using tensor fields.If multiple instances of derivatives of different orders are added,composite behavior can be obtained. These multi-term and distributed order equations have recently been the subject of models from a variety of industrial fields.

**Conclusions**

The study implies brief insight into the many applications of fractional differential equations. The concept of a distributed order differential equation has been introduced. It has been shown that how DODEs can also be used to model super-diffusion and diffusion. It is shown that the fractional calculus constitutive equation allows for a continuous transition from the solid state to the fluid state when the memory parameter varies from zero to one. Bending vibration of flat plates is controlled using patches of active constrained layer damping (ACLD) treatments. Each ACLD patch consists of a visco-elastic damping layer which is sandwiched between two piezo-electric layers. The first layer is directly bonded to the plate to sense its vibration and the second layer acts as an actuator to actively control the shear deformation of the visco-elastic damping layer according to the plate response. With such active/passive control capabilities the energy dissipation mechanism of the visco-elastic layer is enhanced and the damping characteristics of the plate vibration are improved.

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