ASYMPTOTIC PROPERTIES OF MLE IN STOCHASTIC DIFFERENTIAL EQUATIONS WITH RANDOM EFFECTS INTHE DRIFT COEFFICIENT

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ABSTRACT:

In this paper, a class of statistical modelsis proposed, where random effects are inserted into onedimensional stochastic differential equations (SDEs) model; SDE defined N independent stochastic processes$(X_i(t), t \in [0, T_i]), i = 1, \dots, N$. The drift term dependent on a random variableϕ_i , we have been discussedtheparametric estimation of the density of the random effect$\phi_i$within two kinds of mixed models. Anadditive and multiplicativerandom effect are successively considered, when ϕ_i has exponential distribution. We obtained an expression of the exact likelihood and proved the consistency and asymptotic normality of the maximum likelihood estimators (MLE's).

Keywords: Asymptotic normality, consistency, maximum likelihood estimator, mixed effects stochastic differential equations.

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1. Introduction

The Stochastic differential equations have been widely used in financial economics, biological sciences and physical sciences; all models involve unknown parameters or functions, which need to be estimated from observations of the process. Models including random effects enjoy an increasing popularity. Maximum likelihood estimation *(MLE)* of the parameters of the random effects, is generally not possible because of the likelihood function is unavailable in most cases, except in (Ditlevsen et al. andPicchini et al. [6,8]),they used a special case of a mixed-effects Brownian motion with drift and find the likelihood function which leading to explicit parameters estimators. Many authors proposed approximations of the likelihood, Laplace's approximation (Wolfinger and Vonesh, [10, 11]),approximation by Hermit polynomials (Aït-Sahalia,[1]) and (Beal and Sheiner [4]) approximation based on linearization. Delattre et al.[5] consider the special case (multiple case) in the drift where $b(x,\phi_i)$ is linear in ϕ_i ($b(x,\phi_i) = \phi_i\, b(x)$), and proved the strong consistency and the asymptotic normality of the (*MLE)* when ϕ_i has Gaussian distribution. Maitra et al [7]used alternative route for proving consistency and asymptotic normality in SDE (independent identical and independent non-identical) setup involving random effects. Alsukaini et al.[3] discussed and proved asymptotic properties of ϕ_i (ϕ_i has exponential and Gaussian distribution) in the diffusion term where $\sigma(x,\phi_i)$ is nonlinear in ϕ_i ($\sigma(x,\phi_i) = \sigma(x)/\phi_i$). Alkreemawi et al. [2] considered the additive case in the drift where $b(x,\phi_i)$ is linear in $\phi_i(b(x,\phi_i) = \phi_i + b(x))$, and studied asymptotic properties of ϕ_i when ϕ_i has Gaussian distribution.

Delattre , et al.[5] Studied the stochastic differential equations $(SDE's)$ of the form

$$dX_i(t) = b(X_i(t),\phi_i)dt + \sigma\big(X_i(t)\big)dW_i(t), \text{with } X_i(0) = x^i, i = 1,\dots,N \quad (1)$$

Here, the stochastic process $(X_i(t), t \geq 0, i = 1,\dots,N)$ is assumed to be continuously observed on the time interval $[0,T_i]$ with $T_i > 0$ known, and $(x^i,\ \mathrm{i} = 1,\dots,N)$ are known initial values of the ith process.Processes $(W_1,\dots,W_N)$are independent standard Brownian motions, $(\phi_1,\dots,\phi_N)$ are independently and identically distributed (iid) random variables with common distribution $g(\varphi,\theta)d\nu(\varphi)$ for allθ, $g(\varphi,\theta)$ is a density with respect to a dominating measure on $\mathbb{R}^m$, where $\mathbb{R}$ is the real line and m is the dimension. $(\phi_1,\dots,\phi_N)$and$(W_1,\dots,W_N)$are independent. Here θ is an unknown parameter belong to a set $\Theta \subset \mathbb{R}^m$ which be estimated. The drift function$b(x;\varphi)$ is a known real-valued function defined on$\mathbb{R} \times \mathbb{R}^m$, the diffusion coefficient $\sigma(x)$ is a known real-valued function on $\mathbb{R}$. And proved the strong consistency and the asymptotic normality of the (*MLE)* of θ when the model includes a Gaussian one-dimensional and a Gaussian multi-dimensional random effect.

In this paper, according to model (1), we discussed two special cases. First, an additive random effect in the drift:

$$dX_i(t) = (\phi_i + b(X_i(t)))dt + \sigma(X_i(t))dW_i(t), X_i(0) = x^i, i = 1, \dots, N \quad (2)$$

Second, a multiplicative random effect in the drift:

$$dX_i(t) = \phi_i b(X_i(t))dt + \sigma(X_i(t))dW_i(t), X_i(0) = x^i, \qquad \mathrm{i} = 1, \dots, N \quad (3)$$

We assume the functions b and σ are known real functions, and $\phi_i$has exponential distribution. We have to use the sufficient statistics as mentioned in [5] and investigate a sufficient statistics as in [2], we find an explicit likelihood function in each case and (*MLE)* of parameters of ϕ_i. The consistency and the asymptotic normality of (*MLE)* of $\theta$are proved.

This paper is organized as follows, in section.2; we introduce the assumptions and notations. The explicit likelihood function and a specific distribution for the random effect and the asymptotic properties of MLE in additive case of random effect are introduced in section 3. In section 4, we study the consistency and asymptotic normality of the MLE in multiplicative case of random effect.

2. Assumptions and notations

Consider the stochastic processes $(X_i(t), t \geq 0), i = 1, \dots, N,)$ which defined by (1). The processes $(W_1, \dots, W_N)$ and the random variables $(\phi_1, \dots, \phi_N)$ are defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We depended on the assumptions (H1, H2, and H3) in [5], in order to derive the likelihood function of our observations. Consider the filtration $(\mathcal{F}_t,\ t \geq 0)$ defined by$\mathcal{F}_t = \sigma(\phi_i, W_i(s), s \leq t, i = 1, \dots, N)$. As$\mathcal{F}_t = \sigma(\phi_i, W_i(s), s \leq t) \vee \mathcal{F}_t^i$, with $\mathcal{F}_t^i = \sigma(\phi_i,$ $\phi_j, W_j(s), s \leq t, i \neq j)$ independent of W_i, each process W_i is a $(\mathcal{F}_t,\ t \geq 0)$-Brownian motion and the random variables $\phi_i$are $\mathcal{F}_0$-measurable.

(H1)I - The function $(x, \varphi) \to b(x, \varphi)$ is C^1 on $\mathbb{R} \times \mathbb{R}^m$, and such that:

$$\exists K > 0, \forall (x, \varphi) \in \mathbb{R} \times \mathbb{R}^m, \qquad b^2(x, \varphi) \leq K(1 + x^2 + |\varphi|^2),$$

II - The function $\sigma(.)$ is C^1 on $\mathbb{R}$ and

$$\forall x \in \mathbb{R}, \qquad \sigma^2(x) \le K(1+x^2)$$

By (H1), for all φ, the stochastic differential equation

$$dX_i^\varphi(t) = b\big(X_i^\varphi(t),\varphi\big)dt + \sigma\left(X_i^\varphi(t)\right)dW_i(t), \qquad X_i^\varphi(0) = x^i \qquad (4)$$

Admits a unique solution process $(X_i^\varphi(t), t \ge 0)$ adapted to the filtration $(\mathcal{F}_t, t \ge 0)$, and we introduce the distribution $Q_\varphi^{x^i,T_i}$ on (C_{T_i}, C_{T_i}) of $(X_i^\varphi(t), t \in [0,T_i])$ given by (4), where C_{T_i} denote the space of real continuous functions $(x(t), t \in [0,T_i])$defined on$[0,T_i]$. On$\mathbb{R}^m \times C_{T_i}$, let $P_\theta^i = g(\varphi,\theta)d\nu(\varphi) \otimes Q_\varphi^{x^i,T_i}$ denote the joint distribution of $\big(\phi_i\,,\ X_i(.)\big)$ and let Q_θ^i denote the marginal distribution of $(X_i(t), t \in [0,T_i])$ on (C_{T_i}, C_{T_i}).

(H2) For $i = 1,\dots,N$ and for all φ, φ'

$$Q_\varphi^{x^i,T_i}\left(\int_0^{T_i} \frac{b^2\big(X_i^\varphi(t),\varphi'\big)}{\sigma^2\left(X_i^\varphi(t)\right)}dt < +\infty\right) = 1.$$

(H3) For $f = \frac{\partial b}{\partial \varphi_j}, j = 1,\dots,m,$ there exist $c > 0$ and some $\gamma \ge 0$ such that

$$\sup_{\varphi \in \mathbb{R}^m} \frac{|f(x,\varphi)|}{\sigma^2(x)} \le c(1+|x|^\gamma)$$

We assume $\varphi \in \mathbb{R}$ and$b$, σ are known functions.As mentioned in models(2) and (3), we simplify (H1, H2) and assume that b, σ are C^1 and have linear growth, which implies Proposition 1 in [5].We assume that $\int_0^{T_i} b^2(X_i(t),\phi_i)/\sigma^2(X_i(t))\,ds, Q_\varphi^{x^i,T_i}$ $a.s$ for all φ, and for $i = 1,\dots,N$, $T = T_i$, $x^i = x$, so that the observed processes $(X_i(t), t \in [0,T_i], i = 1,\dots,N)$ are iid. Let us introduce the statistics:

$$U_i = \int_0^T \frac{b(X_i(t))}{\sigma^2(X_i(t))}dX_i(s)\,, V_i = \int_0^T \frac{b^2(X_i(t))}{\sigma^2(X_i(t))}ds \qquad (5)$$

As mentioned in [5], the following statistics as in [2]:

$$Y_i = \int_0^T \frac{1}{\sigma^2(X_i(t))}dX_i(s), \quad Z_i = \int_0^T \frac{1}{\sigma^2(X_i(t))}ds\,, \quad D_i = \int_0^T \frac{b(X_i(t))}{\sigma^2(X_i(t))}ds \qquad (6),$$

Which are well defined under (H1, H2).

By an analogue way of propositions (2 and 3) [5], the following results are used:

$$\frac{dQ_\varphi^{x^i,T_i}}{dQ^i} = L_{T_i}(X_i,\varphi) = exp\left(\int_0^{T_i} \frac{b(X_i(s),\varphi)}{\sigma^2\big(X_i(s)\big)}dX_i(s) - \frac{1}{2}\int_0^{T_i} \frac{b^2(X_i(t),\varphi\,)}{\sigma^2\big(X_i(t)\big)}ds\right),$$

$$\frac{dQ_\theta^i}{dQ^i}(X_i) = \int_{\mathbb{R}^m} L_{T_i}(X_i,\varphi)g(\varphi,\theta)\,dv(\varphi) = \lambda_i(X_i,\theta)$$

and the likelihood function is,

$$\Lambda_N(\theta) = \prod_{i=1}^{N} \lambda_i(X_i,\theta). \quad (7)$$

3. Additive random effect in the drift

In this section, we study model (2) and obtained

$$L_{T_i}(X_i,\varphi) = exp\left(\varphi(Y_i - D_i) - \frac{\varphi^2}{2}Z_i + \left(U_i - \frac{1}{2}V_i\right)\right),$$

Then,

$$\lambda_i(X_i,\theta) = \int_{\mathbb{R}^m} g(\varphi,\theta)exp\left(\varphi(Y_i - D_i) - \frac{\varphi^2}{2}Z_i + \left(U_i - \frac{1}{2}V_i\right)\right)dv(\varphi) \quad (8)$$

3.1 Maximum likelihood estimation

For finding the Maximum likelihood estimation of θ, we assume that the random effects in model (2) has exponential distribution$exp(\beta)$, and set $\theta = \beta \in \mathbb{R}^+$ for the unknown parameter to be estimated.

Proposition 3.1.1. Assume that $g(\varphi,\beta)dv(\varphi)$ be an exponential distribution with unknown parameter β. Then

$$\lambda_i(X_i,\beta) = \beta\sqrt{\frac{2\pi}{Z_i}}exp\left[\frac{(Y_i - D_i - \beta)^2}{2Z_i}\right]exp\left[U_i - \frac{1}{2}V_i\right]$$

The conditional distribution under P_θ^i of $\phi_i$given $X_i$is the distribution

$$N\left(\frac{(Y_i - D_i - \beta)}{Z_i}, \frac{1}{Z_i}\right).$$

Proof: The joint density of (ϕ_i, X_i) with respect to $d\varphi \otimes dQ^i$

$$exp\left(\varphi(Y_i - D_i) - \frac{\varphi^2}{2}Z_i + \left(U_i - \frac{1}{2}V_i\right)\right) \times \beta exp(-\beta\varphi),$$

developing the exponent yields:

$$E_i = -\frac{Z_i}{2}\left[\varphi^2 - \frac{2}{Z_i}\varphi(Y_i - D_i - \beta)\right] + \left(U_i - \frac{1}{2}V_i\right). \quad (9)$$

After some elementary algebra, we obtain

$$E_i = -\frac{1}{2\sigma_i^2}(\varphi - m_i)^2 + \left[\frac{(Y_i - D_i - \beta)^2}{2Z_i}\right] + \left[U_i - \frac{1}{2}V_i\right],$$

where

$$m_i = (Y_i - D_i - \beta)/Z_i, \qquad {\sigma_i}^2 = 1/Z_i$$

And the conditional distribution of $\phi_i$given $X_i$is $N(m_i, {\sigma_i}^2)$.

Then

$$\lambda_i(X_i, \beta) = \beta\sqrt{\frac{2\pi}{Z_i}}exp\left[\frac{(Y_i - D_i - \beta)^2}{2Z_i}\right]exp\left[U_i - \frac{1}{2}V_i\right]$$

Thus, the logarithm of the likelihood function (7) is explicitly given by

$$\mathcal{L}_N(\beta) = \sum_{i=1}^{N} log\sqrt{\frac{2\pi}{Z_i}} + \sum_{i=1}^{N} log\beta + \sum_{i=1}^{N}\frac{(Y_i - D_i - \beta)^2}{2Z_i} + \sum_{i=1}^{N}\left[U_i - \frac{1}{2}V_i\right] \qquad (10)$$

Hence, the derivative of the log-likelihood (10) as,

$$\frac{\partial}{\partial\beta}\mathcal{L}_N(\beta) = \sum_{i=1}^{N}\left(\frac{1}{\beta} - \frac{(Y_i - D_i - \beta)}{Z_i}\right) = 0 \ ,$$

And the maximum likelihood estimator of β

$$\hat{\beta}_N = \frac{\sum_{i=1}^{N}(Y_i - D_i)/Z_i + \sqrt{(\sum_{i=1}^{N}(Y_i - D_i)/Z_i)^2 - 4N\sum_{i=1}^{N}1/Z_i}}{2\sum_{i=1}^{N}1/Z_i}. \qquad (11)$$

For studying properties(consistency and asymptotic normality) of maximum likelihood estimators of $\theta = \beta$ we need investigate properties of the following random variable:

$$\psi_i(\theta) = \left(\frac{1}{\beta} - \frac{(Y_i - D_i - \beta)}{Z_i}\right), \qquad (12)$$

And so,

$$\frac{\partial}{\partial\beta}\mathcal{L}_N(\theta) = \sum_{i=1}^{N}\psi_i(\theta), \qquad (13)$$

Proposition 3.1.2. For$\theta = \beta \in \mathbb{R}^+$, then the expectation$(E_\theta)$ of $\psi_1(\beta)$ with respect to P_θ equal to zero and$var_\theta(\psi_1(\theta)) = E_\theta(\psi_1^2(\theta))$.

Proof: Since $\lambda_1(X_1, \beta) = \frac{dQ_\theta^1}{dQ^1}$, $Q_\theta^1 = \int_{C_T}\lambda_1(X_1, \beta)dQ^1 = 1$, where C_T is the space of real continuous functions$(x(t), t \in [0, T_i], i = 1, \dots, N)$ defined on $[0, T]$. By interchange derivation with respect to β and integration with respect to Q_τ^1, we have $\int_{C_T}(\partial\lambda_1/\partial\beta)dQ^1 = 0.$

Since $$\frac{\partial\lambda_1}{\partial\beta} = \sqrt{\frac{2\pi}{Z_1}} exp\left[\frac{(Y_1-D_1-\beta)^2}{2Z_1}\right] exp\left[U_1 - \frac{1}{2}V_1\right]$$

$$-\frac{(Y_1-D_1-\beta)}{Z_1}\beta\sqrt{\frac{2\pi}{Z_1}} exp\left[\frac{(Y_1-D_1-\beta)^2}{2Z_1}\right] exp\left[U_1 - \frac{1}{2}V_1\right]$$

$$= \frac{\lambda_1}{\beta} - \lambda_1\frac{(Y_1-D_1-\beta)}{Z_1} = \lambda_1\left(\frac{1}{\beta} - \frac{(Y_1-D_1-\beta)}{Z_1}\right)$$

$$= \lambda_1\psi_1(\theta).$$

Hence

$$\int_{C_T}\frac{\partial\lambda_1}{\partial\beta}dQ^1 = \int_{C_T}\lambda_1\psi_1(\theta)dQ^1 = E_\theta(\psi_1(\theta)) = 0,$$

and

$$var_\theta(\psi_1(\theta)) = E_\theta(\psi_1^2(\theta)) - \left(E_\theta(\psi_1(\theta))\right)^2 = E_\theta(\psi_1^2(\theta))$$

3.2 Convergence

Proposition 3.2.1. For all β under Q_θ, as N tends to infinity and from Proposition 3.1.2, the random variable $\frac{1}{N}\frac{\partial}{\partial\beta}\mathcal{L}_N(\theta)$ converges in distribution to $N_2(0, I(\theta))$ where $I(\theta)$ is fisher information, and the random variable $-\frac{1}{N}\frac{\partial^2}{\partial\beta^2}\mathcal{L}_N(\theta)$ converges in probability to $I(\theta) = var_\theta(\psi_1(\theta)) = E_\theta(\psi_1^2(\theta))$ (the covariance of $\psi_1(\theta)$).

Proof: The second derivative of function $\mathcal{L}_N(\theta)$ as,

$$\frac{\partial^2}{\partial\beta^2}\mathcal{L}_N(\theta) = \sum_{i=1}^{N}\frac{\beta^2 - Z_i}{\beta^2 Z_i}$$

To conclude, we use the simple law of large numbers, the standard central limit theorem and proposition 3.1.2

3.3 Strong consistency of MLE

In addition to previous assumptions we need the following assumptionsand the following theorem (theorems 7.49 and 7.54 of Schervish (1995)) [9]before verification of strongconsistency and asymptotic normality of the maximum likelihoodestimator of the parameter β:

(H4) The function $b(.)/\sigma(.)$ is not constant. Under Q, the random variable (D_1, Y_1, Z_1) admits a density $f(d,y,z)$ with respect to the Lebesgue measure on $\mathbb{R}\times\mathbb{R}^+$ which is jointly continuous and positive on an open ball of $\mathbb{R}\times\mathbb{R}^+$.

(H5) The parameter set Θ is a compact subset of $\mathbb{R}\times\mathbb{R}^+$.

(H6) The true value θ_0 belongs toΘ^o.

(H7) The matrix $I(\theta_0)$is invertible.

(H8)i-$b(.)$and $\sigma(x)$areC^1on $\mathbb{R}$satisfying $b^2(x)\le K(1+x^2)$and$\sigma^2(x)\le K(1+x^2)$ for all $x\in\mathbb{R}$, for some $K>0$.

ii- Almost surely for each $i\ge 1$,

$$\int_0^{T_i}\frac{b^2(X_i(s))}{\sigma^2(X_i(s))}ds<\infty.$$

Theorem 1 [9]Let$\{x_n\}_{n=1}^{\infty}$be conditionally$i.i.d$given$\theta$with density$f_1(x|\theta)$with respect to a measureνon a space$(\chi^1,\mathcal{B}^1)$. Fix$\theta_0\in\Omega$, and define, for each$M\subseteq\Omega$and$x\in\chi^1$

$$Z(M,x)=\inf_{\gamma\in M}\log\frac{f_1(x|\theta_0)}{f_1(x|\gamma)}$$

Assume that for each $\theta\ne\theta_0$, there is an open set $N_\theta$such that$\theta\in N_\theta$andthat$E_{\theta_0}Z(N_\theta,X_i)>-\infty$. Also assume that$f_1(x|.)$is continuous at$\theta$for every θ, *a.s.* $[P_{\theta_0}]$. Then, *if*$\hat{\theta}_n$*is* the *MLE of* θcorresponding to$n$observations. It holds that$\lim_{n\to\infty}\hat{\theta}_n=\theta_0$ *a.s.* $[P_{\theta_0}]$.

Proposition 3.3.1 Under conditions (H7) and (H8). Then the *MLE is* strongly consistent, in other words$\lim_{n\to\infty}\hat{\beta}_n=\beta_0$ *a.s.* $[P_{\theta_0}]$.

Proof. To verify strong consistency of MLE in our case (additive random effect), we note that for any x, $f_1(x|\theta)=\lambda_1(x,\theta)=\lambda(x,\theta)$, which is clearly continuous in θ. Also for every $\theta\ne\theta_0$we compute

$$\log\frac{f_1(x|\theta_0)}{f_1(x|\theta)}=\log\frac{\beta_0}{\beta}+\frac{(Y_1-D_1-\beta_0)^2}{2Z_1}-\frac{(Y_1-D_1-\beta)^2}{2Z_1}$$

$$=\log\frac{\beta_0}{\beta}+(\beta-\beta_0)\frac{(Y_1-D_1)}{Z_1}+({\beta_0}^2-\beta^2)\frac{1}{2Z_1}$$

$$\ge-\log\frac{\beta_0}{\beta}-|\beta-\beta_0|\left|\frac{(Y_1-D_1)}{Z_1}\right|-\frac{1}{2}\left|{\beta_0}^2-\beta^2\right|\left|\frac{1}{Z_1}\right|,(14)$$

by using lemma 3.1.1 in [2], we note that $E_{\theta_0}\left|\frac{(Y_1-D_1)}{Z_1}\right|$ and $E_{\theta_0}\left|\frac{1}{Z_1}\right|$are finite, and by using **H7,** follows that $E_{\theta_0} Z(N_\theta, X_i) > -\infty$, where $N_\theta = (\underline{\beta}, \bar{\beta})$. Hence $\lim_{n\to\infty} \hat{\beta}_n = \beta_0$ a.s.$[P_{\theta_0}]$.

3.3 Asymptotic normality of MLE

Proposition 3.4.1 Assume (H1 − H2) and (H4 − H7) The MLE satisfies, as N tends to infinity, $\sqrt{N}(\hat{\theta}_N - \theta_0.) \to_D N_2(0, I^{-1}(\theta_0))$.

Proof. For N tends to infinity, $\hat{\theta}_N$ is an interior point of Θ and $|\hat{\theta}_N - \theta_0| < \delta$, where $\delta > 0$, Since MLE $\hat{\theta}_N$ is maximizer of $\mathcal{L}_N(\theta) = (1/N)\sum_{i=1}^N logf(X_i|\theta)$ we have $\mathcal{L}'_N(\hat{\theta}_N) = 0$, then use the Taylor expansion of $\mathcal{L}'_N(\hat{\theta}_N)$ about θ_0

$$0 = \mathcal{L}'_N(\hat{\theta}_N) = \mathcal{L}'_N(\theta_0) + \mathcal{L}''_N(\bar{\theta}_N)(\hat{\theta}_N - \theta_0),$$

where $\bar{\theta}_N$ is a point on the line segment between $\hat{\theta}_N$ and θ_0. Rewriting this expansion, we have

$$\sqrt{N}(\hat{\theta}_N - \theta_0) = -\frac{\sqrt{N}\mathcal{L}'_N(\theta_0)}{\mathcal{L}''_N(\bar{\theta}_N)} \qquad (15)$$

Since θ_0 is the maximizer of $\mathcal{L}_N(\theta)$, we have:

$$\mathcal{L}'_N(\theta_0) = E_{\theta_0} l'(X, \theta_0) = 0,$$

where $l'(X_i, \theta_0) = (log\lambda(x_i, \theta))'$. And so,

$$E_{\theta_0} l''(X_i, \theta_0) = E_{\theta_0} \frac{\partial^2}{\partial \beta^2} log\lambda(x_i, \theta) = -I(\theta_0),$$

the numerator in (15)

$$\sqrt{N}\mathcal{L}'_N(\theta_0) = \sqrt{N}\left(\frac{1}{N}\sum_{i=1}^N l'(X_i, \theta_0) - 0\right)$$

$$= \sqrt{N}\left(\frac{1}{N}\sum_{i=1}^N l'(X_i, \theta_0) - E_{\theta_0} l'(X_1, \theta_0)\right) \to^D N\left(0,\ var_{\theta_0}\left(l'(X_1, \theta_0)\right)\right) \quad (16)$$

Converge in distribution by CLT. From the denominator in (15) we have for all θ, $\mathcal{L}''_N(\theta) = \frac{1}{N}\sum_{i=1}^N l''(X_i, \theta_0)$ converge to $E_{\theta_0} l''(X_1, \theta_0)$ by the law of large numbers.

Since $\bar{\theta}_N \in [\hat{\theta}_N - \theta_0]$ and $\hat{\theta}_N \to \theta_0$, we have $\bar{\theta}_N \to \theta_0$. So we have

$$\mathcal{L}''_N(\bar{\theta}_N) \to E_{\theta_0} l'(X_1, \theta_0) = -I(\theta_0),$$

hence $-\frac{\sqrt{N}\mathcal{L}'_N(\theta_0)}{\mathcal{L}''_N(\bar{\theta}_N)} \to^D N\left(0, \frac{var_{\theta_0}\left(l'(X_1, \theta_0)\right)}{(I(\theta_0))^2}\right)$ and since

$$var_{\theta_0}\left(l'(X_1, \theta_0)\right) = E_{\theta_0}\left(\left(l'(X_1, \theta_0)\right)^2\right) - \left(E_{\theta_0} l'(X_1, \theta_0)\right)^2$$
$$= I(\theta_0) - 0.$$

By substituting in (15), we have:

$$\sqrt{N}\big(\hat{\theta}_N - \theta_0.\big) \to_D N_2\big(0, I^{-1}(\theta_0)\big).$$

4. Multiplicative random effect in the drift

In this section, we have studying model (3) within random effect ϕ_i which has exponential distribution, then we will obtains

$$L_{T_i}(X_i, \varphi) = exp\left(\varphi U_i - \frac{\varphi^2}{2} V_i\right),$$

Then,

$$\lambda_i(X_i, \theta) = \int_{\mathbb{R}^m} g(\varphi, \theta) exp\left(\varphi U_i - \frac{\varphi^2}{2} V_i\right) dv(\varphi).$$

4.1 Maximum likelihood estimation

Clearly, the Maximum likelihood estimation of $\theta = \alpha \in \mathbb{R}^+$ will discussing in this section for an unknown parameter to be estimated.

Proposition 4.1.1. Suppose that $g(\varphi, \theta)dv(\varphi)$ be an exponential distribution with an unknown parameter α. Then

$$\lambda_i(X_i, \theta) = \alpha \sqrt{\frac{2\pi}{V_i}} exp\left[\frac{(U_i - \alpha)^2}{2V_i}\right]$$

Proof: The joint density of (ϕ_i, X_i) with respect to $d\varphi \otimes dQ^i$

$$exp\left(\varphi U_i - \frac{\varphi^2}{2} V_i\right) \times \alpha exp(-\alpha\varphi),$$

Developing the exponent yields:

$$E_i = -\frac{V_i}{2}\left[\varphi^2 - 2\varphi \frac{(U_i - \alpha)}{V_i}\right] \quad (17)$$

After some elementary algebra, we obtain

$$E_i = -\frac{1}{2{\sigma_i}^2}(\varphi - m_i)^2 + \left[\frac{(U_i - \beta)^2}{2V_i}\right],$$

where

$$m_i = (U_i - \beta)/V_i, \qquad {\sigma_i}^2 = 1/V_i$$

And the conditional distribution of $\phi_i$given $X_i$is $N(m_i, {\sigma_i}^2)$. Then

$$\lambda_i(X_i, \theta) = \alpha \sqrt{\frac{2\pi}{V_i}} exp\left[\frac{(U_i - \alpha)^2}{2V_i}\right]$$

Thus, the logarithm of the likelihood function (7) is explicitly given by

$$\mathcal{L}_N(\theta) = \sum_{i=1}^{N} log\sqrt{\frac{2\pi}{V_i}} + \sum_{i=1}^{N} log\alpha + \sum_{i=1}^{N} \frac{(U_i - \alpha)^2}{2V_i} \qquad (18)$$

Hence, the derivative of the log-likelihood (18) are

$$\frac{\partial}{\partial\beta}\mathcal{L}_N(\theta) = \sum_{i=1}^{N}\left(\frac{1}{\alpha} - \frac{(U_i - \alpha)}{V_i}\right) = 0,$$

And the maximum likelihood estimator of α

$$\hat{\alpha}_N = \frac{\sum_{i=1}^{N} U_i/V_i + \sqrt{(\sum_{i=1}^{N} U_i/V_i)^2 - 4N\sum_{i=1}^{N} 1/V_i}}{2\sum_{i=1}^{N} 1/V_i} \qquad (19)$$

For studying properties(consistency and asymptotic normality) of maximum likelihood estimators of $\theta = \alpha$ we need investigate properties of the following random variable:

$$\xi_i(\theta) = \left(\frac{1}{\alpha} - \frac{(U_i - \alpha)}{V_i}\right), \qquad (20)$$

And so,

$$\frac{\partial}{\partial\alpha}\mathcal{L}_{\mathrm{N}}(\theta) = \sum_{i=1}^{\mathrm{N}} \xi_i(\theta), \qquad (21)$$

Proposition 4.1.2. For$\theta = \alpha \in \mathbb{R}^+$, then the expectation$(E_\theta)$ of $\xi_1(\theta)$ with respect to P_θ equal to zero and$var_\theta(\xi_1(\theta)) = E_\theta(\xi_1^2(\theta))$.

Proof: By an analogous way to proof of Proposition 3.1.2, and we need compute $\partial\lambda_1/\partial\alpha$

$$\frac{\partial\lambda_1}{\partial\alpha} = \sqrt{\frac{2\pi}{V_1}} exp\left[\frac{(U_1 - \alpha)^2}{2V_1}\right] - \frac{(U_1 - \alpha)}{V_1}\alpha\sqrt{\frac{2\pi}{V_1}} exp\left[\frac{(U_1 - \alpha)^2}{2V_1}\right]$$

$$= \frac{\lambda_1}{\alpha} - \lambda_1\frac{(U_1 - \alpha)}{V_1} = \lambda_1\left(\frac{1}{\alpha} - \frac{(U_1 - \alpha)}{V_1}\right)$$

$= \lambda_1\xi_1(\theta),$

hence

$\int_{C_T} \frac{\partial\lambda_1}{\partial\alpha} dQ^1 = \int_{C_T} \lambda_1\xi_1(\theta) dQ^1 = E_\theta(\xi_1(\theta)) = 0,$

and

$$var_\theta(\xi_1(\theta)) = E_\theta(\xi_1^2(\theta)) - \left(E_\theta(\xi_1(\theta))\right)^2 = E_\theta(\xi_1^2(\theta))$$

4.2 Convergence

Proposition 3.2.1. For all α underQ_θ, as N tends to infinity and from Proposition 4.1.2, the random variable $\frac{1}{N}\frac{\partial}{\partial\alpha}\mathcal{L}_N(\theta)$ converges in distribution to $N_2(0, I(\theta))$ where $I(\theta)$ is fisher information, and the random variable $-\frac{1}{N}\frac{\partial^2}{\partial\alpha^2}\mathcal{L}_N(\theta)$ converges in probability to $I(\theta) = var(\xi_1(\theta)) = E_\theta(\xi_1^2(\theta))$ (the covariance of$\xi_1(\theta)$).

Proof: The second derivative of function$\mathcal{L}_N(\theta)$,

$$\frac{\partial^2}{\partial\alpha^2}\mathcal{L}_N(\theta) = \sum_{i=1}^{N}\left(\frac{1}{V_i} - \frac{1}{\alpha^2}\right)$$

To conclude, we use the simple law of large numbers, the standard central limit theorem and proposition 4.1.2

4.3 Strong consistency of MLE

In addition to previous assumptions we need the assumption (H9) (H4 in [5]) and theorem 1 in section.3.3before verification ofstrongconsistency and asymptotic normality of the maximum likelihood estimator of the parameter α:

(H9) The function $b(.)/\sigma(.)$ is not constant. Under Q, the random variable (U_1, V_1) admits a density $f(u,v)$ with respect to the Lebesgue measure on $\mathbb{R}\times(0,+\infty)$ which is jointly continuous and positive on an open ball of $\mathbb{R}\times(0,+\infty)$.

Proposition 3.3.1 Under conditions(H7)and(H8). Then the *MLE* is strongly consistent, in other words$\lim_{n\to\infty}\hat{\alpha}_n = \alpha_0$ *a.s.* $[P_{\theta_0}]$.

Proof. To verify strong consistency of MLE in our case (multiplicative random effect), we note that for any x, $f_1(x|\theta) = \lambda_1(x,\theta) = \lambda(x,\theta)$, which is clearly continuous in θ. Also for every $\theta \neq \theta_0$we compute

$$log\frac{f_1(x|\theta_0)}{f_1(x|\theta)} = log\frac{\alpha_0}{\alpha} + \frac{(U_1-\alpha_0)^2}{2V_1} - \frac{(U_1-\alpha)^2}{2V_1}$$

$$= log\frac{\alpha_0}{\alpha} + (\alpha-\alpha_0)\frac{U_1}{V_1} + ({\alpha_0}^2-\alpha^2)\frac{1}{2V_1}$$

$$\geq -log\frac{\alpha_0}{\alpha} - |\alpha - \alpha_0|\left|\frac{U_1}{V_1}\right| - \frac{1}{2}|\alpha_0{}^2 - \alpha^2|\left|\frac{1}{V_1}\right|, \qquad (22)$$

by using lemma 1 in [5], we note that $E_{\theta_0}\left|\frac{U_1}{V_1}\right|$ and $E_{\theta_0}\left|\frac{1}{V_1}\right|$are finite,and by using **H7,** follows that $E_{\theta_0}Z(N_\theta, X_i) > -\infty$, where $N_\theta = (\underline{\alpha}, \bar{\alpha})$.

Hence $\lim_{n\to\infty}\hat{\alpha}_n = \alpha_0$a.s.$[P_{\theta_0}]$.

4.4 Asymptotic normality of MLE

Proposition 3.4.1. Assume (H1, H2) and (H5) − (H8).TheMLE satisfies, as N tends to infinity, $\sqrt{N}(\hat{\theta}_N - \theta_0.) \to_D N_2(0, I^{-1}(\theta_0))$

Proof: By an analogous proof of proposition 3.4.1.

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