ANALYSIS MAXIMUM AND MINIMUM PRINCIPLES ON HARMONIC FUNCTIONS WITH KILLED BROWNIAN MOTION

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Abstract.

The intention of paper uniform lower and upper bounds for positive finite element approximations to semi linear elliptic equations in several space dimensions subject to mixed Dirichet – Neumann boundary conditions are derived . The discrete maximum principle also holds for degenerate diffusion coefficients. The proofs are based on local maxima lecture truncation technique and on a variation formulation . Both methods are settled on careful estimates on truncation.

Key words. Maximum and minimum principle, Dirichlet –Neumann boundary operators , variable exponent storing week operators maximum and minimum principle, variation principle positivity- preserving approximation, harmonic functions of killed Brownian motion.

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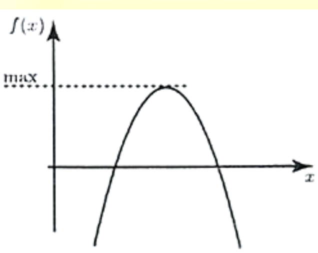
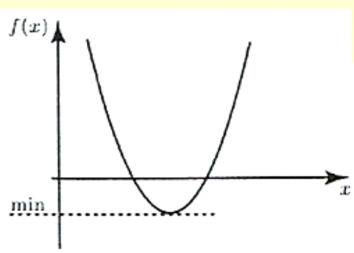
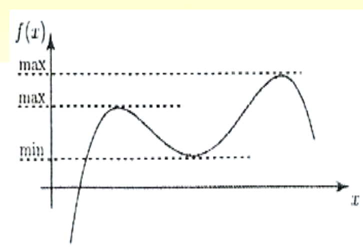
I. INTOUDUTION

The function satisfies a partial differential equation of elliptic type (with on undifferentiated term) then the maximum of the function must occur on the boundary of the region. This note concerns applications of this maximum principle . The first topic treated concerns the ratio of functions which satisfy principle . In results that the ratio obeys the same maximum principle . In particular the maximum principle applies to the ratio of harmonic functions. The last mentioned result is used to obtain maximum principles involving inharmonic functions, on the maximum principle states that a non-constant harmonic function cannot attain a maximum (or minimum) at an interior point of domain Ω are bounded by its maximum and minimum values on the boundary .Such that maximum principle estimates have many users, but they are typically a valuable only for scalar equations not systems of PDE . For example the maximum and minimum values , a point on the graph of a function $f(x)$ is a local maximum if, in its immediate neighbourhood , the function generates lower values to either side of the maximum than the value that it generates at the maximum point itself , as shown in fig (1) . A point on the graph of a function $f(x)$ is a local minimum if in its immediate neighbourhood, the function generates higher values to either side of the the minimum that the value that it generates at the minimum itself as shown in fig (2) . It is possible for a function to have more than one maximum or minimum point as shown fig (3) . The points where a graph takes its maximum or minimum values may also be referred to as turning points. In gradient a tangent drawn at a turning point will be parallel to the horizontal x axis and the gradient at such a point will therefore be zero. This is illustrated in fig (4) . Since gradient can be determined by differentiation , it follows that at either a maximum or minimum point on the graph of $y = f(x)$, the gradient function dy/dx will be equal to zero. On the graph of $y = f(x)$ at a maximum or minimum point $(dy/dx) = 0$







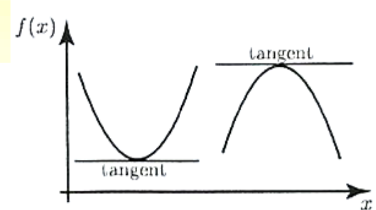


fig.(1): a local maximum fig.(2):a local minimum fig.(3): tow max. and one min. fig.(4):tangent at max. and min.

It is well-known that solutions of linear and non-linear PDE as well as (minimum , maximum) of variation integrals often satisfy the strong maximum principle a bounded non-constant continuous solution u can not attain its maximum or minimum in $Sup_B u \le C \inf_B u$, for all small balls $B \subset \Omega$. In fact inequality implies the set $\{u = 0\}$ is open and the continuity of u guarantees that the set $\{u \ge 0\}$ is open .if we are only interested on the minimum principle, it is enough to consider super-solutions or quasisu-perminimizers and similar reasoning since non-negative lower semi-continues representies satisfy the harnack inequality. The boundednces and symmetry of solution, the bounds for the first eigen-value for quantities of physical interst (maximum strees, the tensional stiffness , electrostatic capacity charge density etc.) , the necessary conditions of solvability for some boundary value problems , etc, the one dimensional maximum principle represents a generalization of the following simple result , let the smooth function u satisfy the inequality $u'' \ge 0$ in $\Omega = (\alpha, \beta)$. Then the maximum of u in Ω occurs on $\partial\Omega = \{\alpha, \beta\}$ on the boundary of Ω i,e $\max_\Omega u = \max_\Omega \{u(\alpha), u(\beta)\}$. And the n-dimensional case we treat the n-dimensional variants of results presented in some possible extensions for nonlinear equations and for equations of higher order as well as their applications , we consider the linear operator (summation convention is assumed , i.e summation from 1 to n is understood on repeated indices as that $Lu = a^{i,j}(x)u_{i,j} + b^i(x)u_i + c(x)u$, $a^{i,j}(x) = a^{j,i}(x)$. The maximum principle for sub-harmonic functions goes to for operators more general than the Laplace operator was proved in two dimensions , the generalized maximum principle $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfy the inequality $Lu \equiv \Delta u + c(x)u \ge 0$ where $c \ge 0$ in Ω .The maximum principles that we have presented above are valid only for the class $C^2(\Omega) \cap C^0(\bar{\Omega})$ the results are valid for classical solutions , we may consider operators L of the divergence form $\Omega \in R^n$.

II.MAXIMUM AND MIMINMUM PRINCIPLES ON HARMOIC FUNCTION

2.1 Maximum and minimum principles

The function satisfies a partial differential equation of elliptic type (with on undifferentiated term) then the maximum of the function must occur on the boundary of the region. This note concerns applications of this maximum principle . The first topic treated concerns the ratio of functions which satisfy principle

Theorem 2.1.1

Supposes that Ω is a connected open set and if $U \in C^2(\Omega)$ if U continuous . a subset F is relatively closed in Ω, $F = \tilde{F} \cap \Omega$ where $\tilde{F}$ is closed in R^n, if a $x \in F$ *and* $\beta_r(x) \subset \Omega$, then the mean value in equality for sub harmonic function implies that

(1) $$\int_{\beta_r(x)} [U(y) - U(x)] dy = \int_{\beta_r(x)} U(y) dy - U(x) \geq 0$$

Since X attains its maximum at X ,we have $U(y) = U(x) \leq 0$, for all $y \in \Omega$ and it follows that $U(y) = U(y) in \beta_r(x)$. Therefore F is open as well as closed . Since Ω is not connected, then U is constant in any connected component of Ω that contains an interior , and F is nonempty we must have $F = \Omega$ so U is constant in Ω . If Ω is not connected , then U is constant in any connected component of Ω that contains an interior point where U attains a maximum value .

Example 2.1.2

The function $U(x) = |x|^2$ sub harmonic in R^n at attains a global minimum in R^n at origin but it does not attain global maximum in any open set $\Omega \subset R^n$ it does of course attain a maximum on any bounded closed set $\overline{\Omega}$ $\overline{\Omega}$ but the attainment of maximum at a boundary point instead of an interior point does not imply the sub harmonic function in constant it follows immediately that sub harmonic function satisfy a minimum principle and harmonic function a maximum and minimum principle .

Theorem2.1.3 Harmonic Function is Maximum and Minimum

Suppose that Ω is a connected open set and $U \in C^2(\Omega)$, if U harmonic and attains a global minimum or maximum in Ω .then U is constant

Proof:

Any super harmonic function U that attains minimum Ω is constant since, $-U$ is sup harmonic and attains a maximum a harmonic function is both sub harmonic .

Example 2.1.4 Harmonic Function

The function $U(x, y) = x^2 - y^2$ is harmonic in R^n it's the real part of the analytic function $f(z) = z^2$ it has critical point at 0 meaning that $D_u = 0$,this critical point is a saddle –point however not an extreme value not also that.

(2) $$\int_{\beta r(0)} U\,dx\,dy = \frac{1}{2\pi}\int_0^{2\pi} (\cos^2\phi - \sin^2\phi)\,d\phi = 0$$

as required by mean value property , one consequence of this property is that any non constant harmonic function is an open mapping meaning , that it maps opens set to open set this not true of smooth function such as $x \to |x|^2$ that. extreme value

Theorem 2.1.5 Bounded Harmonic Function

Suppose that Ω is a bounded, connected open set in R^n and $U \in C^2(\Omega) \cap C(\bar{\Omega})$ is harmonic in Ω then.

$$\max_{\Omega} U = \max_{\partial\Omega} U \text{ and } \min_{\Omega} U = \min_{\partial\Omega} U$$

Proof :

Since U is continuous and $\bar{\Omega}$ is compact , U attain its global maximum and minimum on $\bar{\Omega}$, if U attains a maximum or minimum value at interior point then U is constant by otherwise both extreme values are attained in the boundary .In either cases the result follows let given a second of this theorem that does not depend on the mean value property .Instated we us argument based on the non-positivity of the second derivative at an interior maximum . In the proof we need to account for the possibility of degenerate maxima where the second derivative in zero . For $\varepsilon \ge 0, let\, U^{\varepsilon}(x) = U(x) + \varepsilon|x|^2$. Then $\Delta U^{\varepsilon} = 2n \quad \varepsilon \ge 0$,since U is harmonic .if U^{ε} attained a local maximum at an interior point then $\Delta U^{\varepsilon} \le 0$ by the second derivative test . thus U^{ε} no interior maximum, and it attains its maximum on the boundary .If, $|x| \le R,\ for\ all\ x \in \Omega$, if follows that.

(3) $$Sup_{\Omega} U \le Sup_{\Omega} U^{\varepsilon} \le Sup_{\partial\Omega} U^{\varepsilon} \le Sup_{\partial\Omega} U + \varepsilon\, R^2$$

letting $\varepsilon \to 0^+$,we get that $Sup_{\Omega} U \le Sup_{\partial\Omega} U$ .An application for the same a grummet to u given in, $\inf_{\Omega} U \le \inf_{\partial\Omega} U$ .and the result follows . Sub harmonic function satisfy a maximum principle $\min_{\bar{\Omega}} U = \min_{\partial\Omega} U$,while sub harmonic function satisfy a minimum principle

$\min_{\overline{\Omega}} U \le U \le \min_{\partial\Omega} U$ for all $x \in \Omega$. Physical terms, this means for example that the interior of abounded region which contains no heat sources on heat sources or sinks cannot be hotter that the maximum temperature on the boundary or colder than the minimum temperature on the boundary .The maximum principle given a uniqueness result for (Dirichlet problem) for the poison equation .

Theorem 2.1.6 Dirichlet Problem Function

Suppose that Ω is a bounded connected open set in R^n and $f \in C(\Omega), g \notin C(\partial\Omega)$ are given function then is at most one solution of the Dirichlet problem with $U \in C^2(\Omega) \cap C(\overline{\Omega})$.

Proof :

Suppose that $U_1, U_2 \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfy equation $-\Delta U = f \ in \ \Omega$, $U = g \ on \ \partial\Omega$, Let $V = U_2$ then, $V \in C^2(\Omega) \cap C(\overline{\Omega})$ is harmonic Ω and, $V = 0 \ on \ \partial\Omega$,the maximum principle implies that $V = 0 \ in \ \Omega$ so $U_1 = U_2$ and solution is unique .

2.2 The Maximum Principle and Uniqueness

Are our solution formulas the only solution of the heat equation with the specified initial and or boundary condition by linearity. This amounts to asking whether $U = 0$ is the only date 0 the answer is yes . We shall prove this using the (maximum principle) .The maximum principle this is an elementary for - reaching fact about solutions of linear parabolic equation. Here is the simplest version . Let D be the bounded domain suppose $f_t - \Delta f \le 0$, $\forall x \in D$ and $0 \le t \le T$ Then the maximum of f in the maximum closed cylinder $\overline{D} \times [0, T]$ is a chive either at the (initial boundary) $t = 0$ at the (spatial boundary) $x \in \partial x$. Notice the asymmetry between the initial boundary $t = 0$, (where f can easily easily a chive its maximum) and the finial boundary $t = T$ (where f does not achieve its maximum except in trivial case when f is constant) . This asymmetry reflects once again time has (preferred n when solving a parabolic P. D .E)

2.3 Function Space Elliptic Operators and Maximal Principle

We start with some results on solve space .Let $x = (x_1, x_2, \ldots\ldots, x_n)^T$ and $y = (y_1, y_2, \ldots\ldots, y_n)^T$ be two points in the $n-$dimensional Euclidean space R^n we set $x.y = \sum_{i=1}^{n} x_i . y_j$ and $|x| = \sqrt{(x, x)}$ an n-tupelo of nonnegative integers $\alpha = (\alpha_1, \alpha_2, \ldots\ldots \alpha_n)$ is called Elliptic operators – and we define.

$|\alpha| = \sum_{i=1}^{n} \alpha_i$, $x^\alpha = \left(x^\alpha{}_1, x^\alpha{}_2, \ldots\ldots, x^\alpha{}_n\right)$, $x = (x_1, x_2, \ldots\ldots, x_n)$

$D_k = \frac{\partial}{\partial x_k}$, $D = (D_1, D_2, \ldots., D_n)$, $D^\alpha = D_1^{\alpha 1}, \ldots\ldots, D_n^{\alpha n} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha 1}, \ldots.., \partial x_n^{\alpha}}$.

let Ω be fixed bounded domain (i.e.) open. And connected subset of R^n with smooth boundary $\partial\,\Omega$. For a nonnegative integer, m .We denote by $C^m(\Omega)$, $rep\, C^m(\bar{\Omega})$ the set of all $m-times$ continuously differentiable real-value or complex valued function in Ω (resp. $\bar{\Omega}$)and by $C_0^m(\Omega)$,the subspace of $C^m(\Omega)$ consisting of those function which have compact support in Ω for $U \in C^m(\Omega)$ *and* $1 \le p \le \infty$ we define $\| U \|_{m,p} = \left(\int_\Omega \sum_{|\alpha| \le m} |D^\alpha U|^p dx \right)^{1/p}$ Also for p=2 and $U, V \in C^m(\Omega)$ we can define .

(4) $$(U,V)_m = \left(\int_\Omega \sum_{|\alpha| \le m} |D^\alpha U|^p dx \right)^{\frac{1}{p}}$$

Let $C_p^m(\Omega)$ be the subset of $C^m(\Omega)$ consisting of those function U for which $\|U\|_{m,p} \le \infty$ define $W^{m,p}(\Omega)$ *and* $W^{m,p}(\Omega)$ to be the completions in the norm $\| . \|_{m,p}$ *of* C_p^m *and* $C_0^m(\Omega)$ respectively . It well known that $W_0^{m,p}, W^{m,p}$ are Bench space and $W_0^{m,p}(\Omega) \subset W^{m,p}(\Omega)$ we will also let $H^m(\Omega) = W^{m,2}(\Omega)$, $H_0^m(\Omega) = W_0^{m,2}(\Omega)$ The space $H^m(\Omega)$ *and* $H_0^m(\Omega)$ are Hilbert space with the scalar product $(\cdot,\cdot)$ given by space $W^{m,p}(\Omega)$ consist of function $U \in L^p(\Omega)$,whose derivatives $D^\alpha\,U$ in the sense of distribution of order $|\alpha| \le m$ are in $L^p(\Omega)$.The space $W^{m,p}(\Omega)$ in the subspace of elements of $W^{m,p}(\Omega)$ which vanish in some generalized sense on $\partial\Omega$,it can be shown that if $U \in C^{m-1}(\bar{\Omega}) \cap W_0^{m,p}(\Omega)$.Then U and the first $m-1$ normal derivatives vanish on $\partial\Omega$ then $U \in W^{m,p}(\Omega)$ following result describing various relation a the above spaces, are well known and will be used throughout the remainder of this section.

Theorem 2.3.1

The following relation among $W^{m,p}(\Omega), C^m(\Omega)$ *and* $L^p(\Omega)$ hold .

(a) $W^{m,p}(\Omega) \subset W^{mr}(\Omega)$ *and* $1 \le r \le p$.

(b) the impending in continuous. (b) $W^{m,r}(\Omega) \subset W^{j,p}(\Omega)$ *if* $1 \le r, p \le \infty$ and, m are integers such that $0 \le j \le m$ *and* $\frac{1}{p} \ge \frac{1}{r} \ge \frac{j}{n} - \frac{m}{n}$ and the imbedding is compacts .(c)

$W^{m,p}(\Omega) \subset L^{\frac{np}{n-mp}}(\Omega)$ *if* $mp \le n$ there exist consist constant C_1 such that

$\| U \|_{\frac{np}{n-mp}} \le C^1 \| U \|_{m,p} \quad for \; U \in W^{m,p}(\Omega)$. **(d)** $W^{m,p}(\Omega) \subset C^k(\Omega)$ $if \; 0 \le k \le m - \left(\frac{n}{p}\right)$ and there exists consist C_2 such that $\sup \{ |D^{\alpha}(x)|, |\alpha| \le k, x \in \overline{\Omega} \} \le C_2 \|U\|_{m,p} \quad for \; U \in W^{m,p}(\Omega)$.

(f) (Poincare inequality) there exists constant $C = C(\Omega)$ such that in $\inf_{k \in R} \|U + K\| \le C(\Omega) \|\nabla U\|_{0,2}$ for all $U \in H^1(\Omega)$ (Not that this inequality holds even if Ω in Lipchitz only) . For any $\sigma = K + \eta \ge 0$ where k is a nonnegative integer and $\eta \in (0,1), C^k(\Omega)$ and such that the derivatives of $D^{\alpha}U \; of \; order \; |\alpha| = k$ .Satisfy a uniform Holder condition with η ,The norm this space is defined as $\|U\|_{C^{\sigma}(\Omega)} = \|U\|_{C^k(\Omega)} + \sum_{|\alpha|=k} |D^{\alpha}U|_{\eta}$

$| V |_{\eta} = \sup_{x,y} \in \Omega, \quad x \ne v \frac{|V(x) - V(y)|}{|x-y|^{\eta}}$, Consider now the following linear parabolic equation .

(5)

$$\frac{\partial}{\partial t} U = A(x, D)u \qquad t \ge 0 \quad x \in \Omega \quad , \qquad B(x, D)U = 0 \qquad t \ge 0 \quad x \in \partial\Omega$$

for all some $\alpha \in (0,1)$ and $A(0,1)$ is strongly elliptic (i.e) there exists a constant $\alpha \ge 0$ so that $\sum_{i,j}^{n} a_{i,j}(x) z_i z_j \ge d \sum_{i-1}^{n} z_i^2$,$(z_1, ..., z_n)^T \in R^n$, the boundary condition in either of (Dirichlet type) (i.e.) $B(x, D)U = U(x)$ or (Neumann type) (i.e.)

$$B(x, D)U = \gamma(x)u(x) + \sum_{j=1}^{n} \left[\sum_{k=1}^{n} a_{i,j}(x) n_k(x) \right] \frac{\partial U(x)}{\partial x_j}.$$

2.4 Maximum Principle for Strong Solutions

In This sub section we treat the extension of the classical maximum principle in to strong solution in particular to solutions in space $W_{IOC}^{2,n}(\Omega)$ Recall that an operator L of the form elliptic in domain Ω if the coefficient matrix $A = [a^{i,j}]$ is positive every where in Ω .For such operators we will let D denote the determinate of a and set $D^* = D\left(\frac{1}{n}\right)$ so that D^* is geometric mean of the eigenvalues of A and $0 \le \lambda \le D^* \le \Lambda$ where as λ, Λ dent respectively the minimum and maximum eigenvalues of A . our condition on the coefficient of L and in homogeneous term f in the equation will now take the from .

(6) $\quad | b | / D^* \;, \; f / D^* \in L^n(\Omega) \qquad C \le 0 \quad in \quad \Omega$

The following weak maximum principle at A, D.

Theorem 2.4.1

Let $LU \geq f$ is a bounded domain Ω and $U \in C^0(\bar{\Omega}) \cap W^{2,n}_{IOC}(\Omega)$, Then

$Sup_{\Omega} U \leq Sup_{\partial\Omega} U^+ + C\|f/D^*\|_{L^n(\Omega)}$ that functions $W^{2,n}_{IOC}(\Omega)$ are at least continuous in (Ω) if U is not also assumed continuous on $(\bar{\Omega})$ in hypotheses can be modified by replacing .

(7) $$Sup_{\Omega} U^+ \quad by \quad \lim Sup_{\partial\Omega} U^+$$

Proof:

Contact set and normal mapping. If U is an arbitrary continuous function on (Ω) we define the upper constant set U denoted $\Gamma^+ or \Gamma^+_U$ to be the subset of (Ω) where graph of U lies below a support hyper plane in R^n that is $\Gamma^+ = \{y \in \Omega \mid U(x) \leq U(y) + p(x-y) \mid \}$ *for all* $x \in \Omega$. For some $p = p(y) \in R^n$ clearly U is a concave function in (Ω) if and only if $\Gamma^+ = \Omega$ when $U \in C^1(\Omega)$ we must have $p = DU(y)$ any support hyper plane must then be tangent hyper plane to the graph of U further more when $U \in C^2(\Omega)$ the Hessian matrix $D^2U = [D_{ij} U]$ is non positive on Γ^+ in general the set Γ^+ is closed relative to (Ω) for an arbitrary function .we define the normal mapping $X(y) = X_U(y)$.Of point $y \in \Omega$ to the set of (slopes) of support hyper planes of y lying above the graph of u that is $X(y) = \{p \in R^n, \mid U(x) \leq U(y) + p(x-y)\}$ *for all* $x \in \Omega$ clearly $X(y)$ in nonempty if and only if $y \in \Gamma^+$ furthermore when $U \in C^1(\Omega)$ then $X(y) = DU(y)$ *on* Γ^+ that is X is the gradient vector field of U on Γ^+ as a useful example of a non differentiable function U let us take $\beta = \beta_R(z)$ to be a ball and U to be Ω the function when graph is a cone base Ω and vector (z, a) for some positive $a \in R$, $U(x) = a\left(1 - \frac{|x-z|}{R}\right)$.

Lemma 2.4.2

For $U \in C^2(\Omega) \cap C^0(\bar{\Omega})$ we have $Sup_{\Omega} U \leq Sup_{\partial\Omega} U + \frac{d}{W_n^{\frac{1}{n}}}\left(\int_{\Gamma^+} |\det D^2U|\right)^{(1/n)}$ where $d = \dim \Omega$

lemma 2.4.3

For $U \in C^2(\Omega) \cap C^0(\bar{\Omega})$ we have $Sup_{\Omega} U \leq Sup_{\partial\Omega} + \frac{d}{nW^{n\,(1/n)}} \left\| \frac{a_{i,j}\, D_{i,j}\, U}{D^*} \right\|_{L^n(\Gamma^+)}$

lemma 2.4.4 Nonnegative Integerable Function on R^n

Let g be nonnegative locally integerable function on R^n then for any $U \in C^2(\Omega) \cap C^0(\bar{\Omega})$ we have $\int_{\beta\tilde{M}} g \leq \int_{\Gamma^+} g(DU) \left|\det D^2 U\right| \leq \int_{\Gamma^+} g(DU) \left(\frac{-a_{ij} D_{ij} U}{n D^*}\right)^n$ Where $\tilde{M} = (Sup_{\Omega} U - Sup_{\partial\Omega} U)d \quad , \ d = dam\ \Omega$

lemma 2.4.5

Let g be nonnegative, locally integerable function on R^n then for any $U \in C^2(\Omega) \cap C^0(\bar{\Omega})$ we have $\int_{X_U(\Omega)} g \leq \int_{\Gamma^+} g(DU) \left|\det D^2 U\right| \quad , \quad X_k(\Omega) \subset X_U(\Omega)$.

Theorem 2.4.6 Elliptic is Uanqunecs

Let L be the elliptic in bounded domain Ω and satisfy $|b|/D^*$, $f/D^* \in L^n(\Omega)$ suppose that U and V are a function in $W_{IOC}^{2,n}(\Omega) \cap C^0(\bar{\Omega})$ satisfying $L_U = L_V$ in Ω $U = V$ on $\partial\Omega$ then $U = V$ in Ω

Definition 2.4.7 L^p Estimates Preliminary Analysis

The basic L^p estimates of this is via interpolation .In this we develop some preliminary analysis .A cube decomposition procedure also necessary for Holder estimates the Marci kinesics interpolation theorem that is applied in next L^p.

Lemma 2.4.8

Let K_0 be a cube in L^p , f nonnegative integral function defined in K_0 and (t, a) positive number satisfying $\int_{K_0} f \leq t|K_0|$ by bisection of the edges of K_0 we subdivide into Z^n , congruent sub cubes K with disport interiors .Those sub cubes K which satisfy $\int_{K_0} f \leq t|K|$ are similarly subdivided and the process repeated indefinitely .Let φ denote the set of sub cube K thus obtained that $\int_{K_0} f \leq t|K|$ and for each $K \in \varphi$ denote by $\tilde{K}$ the subcube whose subdivision given K , $|\tilde{K}|/|K| = 2^n$ we have for any $K \in \varphi$ furthermore setting $F = \bigcup_{K \in \varphi} K$ and $G = K_0 - F$ we have $f \leq t$ in G .For the point wise estimate also need to consider the set $F = \bigcup_{K \in \varphi} \tilde{K}$ satisfies $\int_F f \leq |\tilde{F}|$ in particular of f in the characteristic function X_Γ of measurable subset Γ of K_0

(8) $$|\Gamma| = |\Gamma \cap \tilde{F}| \leq t|\tilde{F}|$$

2.5 Pontryagin's Maximum Principle and Minima Problem

The main purpose of this section is to derive a counterpart of the Pontryagin maximum principle valid for certain maximum problems .Our problems is to minimize the function

(9) $$H(x) = Sup_t\, F\left(t, x, \frac{dx}{dt}\right)$$

Under given boundary condition. The a admissible function are absolutely continuous vector functions, such problems have been treated by D .S and the author .

(10) $$F \equiv \left\| A(t)\left(\frac{dx}{dt} + B(t)x + C(t)\right) \right\|$$

Where $A(t)$ and $B(t)$ are matrix function and $C(t)$ is vector function the author has treated the case x a scalar for fairly general nonlinear function F .

2.6 Statement of The Problem and Theorem

Let us state the problem in detail by D .We denote a region in R^{n+1} and points in D are written (t, x) where has (n) component by (t_0, X_0) and (t, X) we denote two points in D with $(t_0 \le X_1)$ the class F of dismissible section $x = x(t)$ with graph in D ,defined for $T_0 \le t \le T_1$ absolutely continuous there and satisfying $x(t_0) = x_0, x(t_1) = x_1$ *for* $x(t) \in F$ we define the functional $H(x)$ by .

(11) $$H(x) = \left(Sup_{t \in E}\; F\left(t, x(t), \frac{dx(t)}{dt}\right)\right)$$

And it will be proved in a lemma that $Sup_{t\in E}\, F(.,.,.) = Sup_{t\in E}\, F(.,.,.)$ for any $x(t) \in F$ a longhouse function F which we are interested (i,e) satisfying condition . Further, we writ $M_0 = \inf_{x\in F} H(x)$ and it will be seen below that $M_0 \ge -\infty$ we impose that following conditions $F(t, x, z)$.

(a) $F(t, x, z) \in C^1(D \times R^n)$.

(b) For any fixed $(t, x) \in D$ the function $\mu(z) \equiv F(t, x, z)$ is strictly convex in z , Further there exists a mapping $W = D \to R^n$ such that $F(t, x, w(t, x)) = \min_{z\in R^n} F(t, x, z)$ for all $(t, x) \in D$. Also $w \in C^1(D)$.

III.HARMONIC FUNCTIONS OF KILLED BROWNIAN MOTION

Suppose that X_i and T_i are two independent processes, where X_i is Brownian motion in R^d and T_i is an $\alpha/2$ stable subordinator starting at zero $0 \le \alpha \le 2$ .It is well known that $Y^{\alpha}(t) = X_{T_i}$ is a rotationally invariant α stable process whose generator is $-(-\Delta)^{\alpha/2}$, the fractional power of the negative Laplacian . The potential theory corresponding to the process Y_α is the Riesz potential theory of order α. Suppose that D the killed process $Y_\alpha^D(t)$ has been extensively studied in recent years and deep properties have been obtained. Let $\Delta|_D$ be the Dirichlet Laplacian in D. The fractional power $-(\Delta_D)^{\alpha/2}$ of the negative Dirichlet Laplacian is a very useful object in analysis and partial differential equations.

3.1 Notation and setting

Let X_i be the Brownian motion in R^d, which runs twice as fast as the standard d-dimensional Brownian motion and let T_i be an $\alpha/2$ stable subordinator stating at zero $0 \le \alpha \le 2$ we assume that X and T are independent we are going to use P_x and E_x to stand for the probability and expectation to the Brownian motion, $(P_t)_{t\ge 0}$ to stand for the Brownian of $u_t^{\alpha/2}(S)$ to denote the density of T_i, let $D \subset R^d$ be a bounded domain, and let X_i^D be the Brownian motion killed upon exiting D we define now the subordinate killed motion Z_t^D by subordinating X_i^D via the $\alpha/2$ stable. More precisely, let $Z_t^D(t) = X^D(T_t)$, $t \ge 0$ then $Z_t^D(t)$ is a symmetric Hunt process on D if we use $(P_t)_{t\ge 0}$ and G^D to denote the semi group and potential operator of X^D respectively then the semi group.

(12) $$Q_1^x f(x) = \int_0^\infty p_s^D f(x) u_t^{\alpha/2}(s)\, ds$$

3.2 Maxima and Minima Lecture

Example 3.2.1

A Texas based company called (Hamilton,s wares) sells baseball bats at a fixed price c. A field researcher has calculated that the profit the company makes selling the bats at the price c is $p(c) = \frac{-1}{2000}c^4 + \frac{1}{5}c^3 - \frac{51}{2}c^2 + 1150c$ at what price should the company sell their bats to make the most money?

Intuitively what would we have to do solve this problem ? We wish to know at what point c is this function $P(c)$ is maximized . we do not have many tools as moment to solve this problem so let,s try to graph the function and guess at where the value should be.

3.3 Absolute global maxima

Definition 3.3.1

Let f be function defined on an interval I containing c.We say that f has an absolute maximum (or a global maximum) value on I at c at $f(x) \le f(c)$ for all x contained in I. Similarly , we say that f has an absolute minimum (or a global minimum) value on I at c if $f(x) \ge f(c)$ for all x contained in I. Those points together are known as absolute global extreme.

Example 3.3.2

$f(x)=x^2+1$ for $x\in(-\infty,\infty)$ remember this notation means for x living in the interval from negative infinity to infinity . This can also be written as $x\in R$ or in words as for all real x ,this function has an absolute minimum of 1 at the point $x=0$ but no absolute maximum on the interval .

Example 3.3.3

$f(x)=x^2+1$ for $x\in(-2,2)$ remember closed brackets means we include the endpoints in our interval this function has an absolute minimum of I at the point $x=0$ and a absolute maximum of $f(\pm 2)=(\pm 2)^2+1=5$ at the points $x=2$ and $x=-2$.

Example 3.3.4

$f(x)=x^2+1$ for $x\in(0,2)$ remember open brackets means we omit the endpoint in our interval .

Example 3.3.5

$f(x)=x^3$ for $x\in(-\infty,\infty)$, this function has no absolute minimum and no absolute maximum .

3.4 Extreme value theorem

A function have an a absolute maximum and minimum , these examples seen to suggest that if we have a closed interval then we're in business.

Example 3.4.1

Consider the function .fro the graph, its clear that this function has no absolute minimum or absolute maximum but $f(x)$ is defined on all of $[0,2]$ the problem with this example is that the function is not continuous .

$$f(x)=\begin{cases} x & if \quad 0\le x\le 1 \\ 1.5 & if \quad x=0,1,2 \\ -x+4 & if \quad 1\le x\le 2 \end{cases}$$

(13)

Theorem 3.4.2 Extreme value

Let $f(x)$ be a continuous function defined on a close interval , then $f(x)$ has an absolute maximum and an absolute minimum on that interval .

[Notice]: that this says nothing about uniqueness. Remember the example $f(x)=x^2+1$ for $x\in[-2,2]$ has two points where the absolute maximum was obtained . Also note that functions that are not continuous and defined on a closed interval can still have extreme.

Example 3.4.3

Consider the following function on $[-1,1]$ as function $f(x)$, this function is not continuous at 0 however it has a global minimum of 0 of -3 because at all non-zero points this function is sturdily positive.

$$f(x)=\begin{cases} x & if \quad x\neq 0 \\ -3 & if \quad x=0 \end{cases}$$

(14)

Definition 3.4.4

Let I be an open interval on which a function f is defined and suppose that $c \in I$ .We say that c is a local maximum value of f if $f(x) \le f(c)$ for all x contained in some open interval of I .Similarly we say that c is a local minimum value of f if $f(x) \ge f(c)$ for all x contained in some open interval I . These points together are known as local extreme.

[Note] : Your textbook uses any arbitrary interval, but requires c to be an interior point.

[Note] : Global extreme of a function that occur on an open interval contained in our domain are also local extreme.

Theorem 3.4.5 Fermat's or local extreme point theorem

If a function $f(x)$ has a local minimum or maximum at the point c and $f'(c)$ exists, then $f'(c) = 0$

Example 3.4.6

We look at $f(x) = |x|$. Notice that this function is not differentiable $x = 0$ but since $f(x) = |x| \succ 0 = f(0)$ we see that it has a local minimum at 0 (and in fact this is a global minimum).

Definition 3.4.7

A critical point is a point c in the domain of f where $f'(c) = 0$ or $f'(c)$ fails to exist . In fact all critical points are candidates for extreme but it is not true that all critical points are extreme.

Example 3.4.8

Consider the function $f(x) = x^3$ .We saw before that this function has no maximum or minimum . However $f'(x) = 3x^2$ and $f(0) = 3(0)^2 = 0$ so the point $x = 0$ is a critical point of f that is not an extreme.

3.5 Algorithm for finding global minima and maxima

Let f be a continuous function on a closed interval $[a,b]$ (so that our algorithm satisfies the conditions the conditions of the extreme value theorem :

(a) Find all the critical points of (a,b), that is the points $x \in (a,b)$ where $f'(x)$ is not defined or where $f'(x) = 0$ (usually done by setting the numerator and denominator to zero) call these points $x_1, x_2, ..., x_n$.

(b) Evaluate $f(x_1), ..., f(x_n), f(a), f(b)$ that is evaluate the function at all the critical points found from the previous step and the two end point values.

(c) The largest and the smallest values found in the previous step are the global minimum and global maximum values.

Example 3.5.1

Compute the absolute maximum and minimum of $3x - 4x + 2 \ on \ [-1,2]$.

Solution

Our function is continuous (and in fact differentiable) everywhere . Hence we $f'(x) = 6x - 4$ setting $f'(x) = 0$ and solving yields $0 = f'(x) = 6x - 4 \Rightarrow 4 = 6x \Rightarrow 2/3 = x$. Now we evaluate f at $x = 2/3, -1$ and 2 (that is the critical points and the end points) we get that .

(15)

$$f\left(\frac{2}{3}\right) = 3\left(\frac{2}{3}\right)^2 - 4\left(\frac{2}{3}\right) + 4 = \frac{2}{3}\ ,\ f(-1) = 3(-1)^2 - 4(-1) + 2 = 9,\ f(2) = 3(2)^2 - 4(2) + 2 = 0$$

From this , we see that the absolute maximum is 9 obtained at $x = -1$ ans the absolute minimum is $(2/3)$ obtained at $x = (2/3)$.

Example 3.5.2

Compute the critical points of $f(x) = 5x^{2/3}$.

Solution

We compute the derivative $f'(x) = \frac{10}{3} x^{\frac{-1}{3}}$,Now we check when the derivative is 0 and when it is undefined This function is never 0 but happens to be undefined at 0 which is a point in our in domain . Hence the critical points are just $x = 0$.

Example 3.5.3

Let finish off with our first example .We compute the global maximum on $[0,200]$ of $P(c) = \frac{-1}{2000}c^4 + \frac{1}{5}c^3 - \frac{51}{2}c^2 + 1150c$ the function is continuous and differentiable everywhere so we can apply the algorithm. Taking the derivative yields $P'(c) = \frac{-1}{500}c^3 + \frac{3}{5}c^2 - 51c^2 + 1150$ setting this to zero then solving (using a computer) yields $c = 35.08903837\,1, 94.40553426, 169.7040820$ Evaluating the function at these points and the end points 0 and 200 yields $p(0) = 0$. From this we can see that the maximum occurs $c = 169.70$ and given a proof of 23545.89 dollars.

(16)

$$\begin{aligned} p(0) &= 0 \\ p(35.89038371) &= 16843.48591 \\ p(94.40553426) &= 9860.6282 \\ p(169.7040820) &= 23545.8859 \\ p(200) &= 1000 \end{aligned}$$

I. CONCLUSION

(a) The Ω is a connected open set and if $U \in C^2(\Omega)$ if U continuous . a subset F is relatively closed in Ω , $F = \tilde{F} \cap \Omega$ where $\tilde{F}$ is closed in R^n , if a $x \in F$ *and* $\beta_r(x) \subset \Omega$, then the mean value in equality for sub harmonic function implies that conclusion $\int_{\beta_r(x)} [U(y) - U(x)]dy = \int_{\beta_r(x)} U(y)dy - U(x) \geq 0$.

(b) The Ω is a bounded, connected open set in R^n and $U \in C^2(\Omega) \cap C(\bar{\Omega})$ is harmonic in Ω then.

$\max_{\Omega} U = \max_{\partial\Omega} U$ and $\min_{\Omega} U = \min_{\partial\Omega} U$

(c) Ω in hypotheses can be modified by replacing $Sup_{\Omega}\ U^{+}$ *by* $\lim Sup_{\partial\Omega} U^{+}$

(d) The f on an interval I containing c we say that f has an absolute maximum (or a global maximum) $f(x) \le f(c)$. Similarly f has an absolute minimum $f(x) \ge f(c)$ for all x contained in I .

Appendix

Appendixes, if needed, appear before the acknowledgment.

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