ANALYSIS MAXIMUM AND MINIMUM PRINCIPLES ON MAXIMUM RIEMANNIAN MANIFOLDS

Mohamed M.Osman*

Abstract .

In this paper uniform upper and lower continuous function $U \in (M)$ on manifolds spaces with curvature bounds on (M) and applications compact Riemannian boundary $(u, v) \in M^{\pm}$ is complete with Ricci and we prove is spectral and Direct computatations on spectrum R .

Key words. Maximum and minimum principle, Dirichlet –Riemannian boundary operators Maximum principles , variable exponent storing week operators curvature , computation , variation principle positivity- hyper surfaces on Riemannian manifolds approximation, spectral and Direct on spectrum manifolds .

* Department of mathematics faculty of science, University of Al-Baha – Kingdom of Saudi Arabia

I. INTRODUCTION

The studying minimal immersions into cones of the Euclidean space, introduced a global version of maximum principle on any Riemannian manifold $(M,\langle\cdot\rangle)$ whose sectional curvature is bounded from below and also provided examples where it fails to hold. When stated for Laplace , Direct compute operator and with the obvious meaning of the symbols, the global maximum principle sounds as follows for every bounded above $u \in C^2(M)$ there is a sequence $\{x_n\} \subset M$ such that for each n .

(1) $$u(x_n) \geq \sup u - (1/n), |\nabla u|(x_n) \leq (1/n), \Delta u(x_n) \leq (1/n).$$

The purpose of this project is to provide a maximum for manifolds partial differential operators which would be assessable to an able student with potter and Weinberger maximum and minimum principle on R.M by grasp of subject sub manifolds before proceeding to the more general computation of the spectrum, isopectral manifolds.

II. ANALYSIS MAXIMUM PRINCIPLE FOR RIEMANNIAN MANIFOLDS

2.1 Maximum principle on Riemannian manifolds

In this section we consider Viscosity solutions to second order partial differential equation on Riemannian manifolds . We prove maximum principles for solutions to (Dirichlet problem) on a compact Riemannian Manifold with boundary , using a different method, we generalize maximum principles of omori to a Viscosity version. We also prove maximum principle.

Definition 2.1.1 Upper and Lower Semi continuous Functions

We use the following notations

(a) USC(M) : is Upper semi- continuous function on (M) .

(b) LSC(M) : is Lower semi- continuous function on (M)

Definition 2.1.2 A viscosity Sub solutions on M

A Viscosity sub solution of, $F = 0$ on M is a function $U \in USC(M)$ such that .

(a) is a Viscosity super solution of $u \in LSX(M)$.

(b) $F(x,u,p,X) \geq 0\ for all\, x \in M\ and\ (p,X) \in J^{2,-}u(x)$, u is a Viscosity solution of $F = 0$ on M if it is both a Viscosity sub solution and a Viscosity sup solution of $F = 0$ on M.

We can similarly define , $\bar{J}^{2,+}u(x)$, $\bar{J}^{2,-}u(x)$ as

$$J^{2,+}u(x) = \{(p,X) \in T^*_{x0}(M \times S^2) T^*_{x0}(M) \Rightarrow (x_0, u(x_0)), p(X)$$
$$as \lim t(x_k, u(x_k, p_k, X_k) \in J^{2,+}u(x_k) in the top \log y of\ f(M)$$

(2) $$J^{2,-}u(x) = \{(p,X) \in T^*_{x0}(M \times S^2) T^*_{x0}(M) \Rightarrow (x_0, u(x_0)), p(X)$$
$$as \lim t(x_k, u(x_k, p_k, X_k) \in J^{2,-}u(x_k) in the top \log y of\ f(M)$$

Theorem 2.1.3 Riemannian Manifold with Sectional Curvature

Let M be a connected Riemannian Manifold with sectional curvature bounded below by $-K^2$. Given tow points $x,y\in M$ with $d(x,y)\le \min\{i(x), i(y)\}, \gamma:[0,l]\to M$ the unique geodesic of unit sped with $\gamma(l)=y$. Denote by $p_\gamma(t):T_{\gamma(0)}M\to T_{\gamma(t)}M$ the parallel transport along γ Then any tow vector V_1 and V_2 satisfying $\langle V_1,\gamma^-(0)\rangle=\langle V_2,\gamma^-\rangle=0$, the Hessian of square of distance function φ on $M\times M$ satisfies.

(3) $$D^2\varphi(V_1,V_2),(V_1,V_2)\le 2l\,k\langle V_1,V_2\rangle+\coth kl\langle V_1,V_2\rangle-2lk\left[\frac{1}{\sinh kl}\langle V_2,p_\gamma(l)V_1\rangle\right]$$

particularly $D^2\varphi(V_1,p_\gamma(t)V_1),(V_1,p_\gamma(t)V_2)\le 4l\,k\,|V_1|^2+kl\coth\left[\frac{kl}{2}\right]$ Before proving the give some remarks.

Remake 2.1.4 Hyperbolic Space $H^n(-k^2)$

When M is the hyperbolic space $H^n(-k^2)$, we can also write the $D^2\varphi$ on the subspace

(4) $$\{\gamma'(0)\}^\perp\times\{\gamma'(l)\}^\perp\subset T_xM\times T_yM$$

Remark 2.1.5 Curvature Bounded

Not that we allow K .to be an imaginary and in case that the curvature is bounded below by a positive constant, we can get a corresponding estimate.

Remark 2.1.5

We also a version similar to laplacian comparison , when we have a Ricci curvature lower bound. We will that is useful in applications.

Lemma 3.1.6

Let M be a compact Riemannian Manifold with or without boundary $u\in USC(M), v\in LSC(M)$ and $\mu_\alpha=\sup\left\{u(x)-v(y)-\frac{\alpha}{2}d(x,y)^2, x,y\in M\right\}$ For $\alpha\ge 0$, Assume that $\mu_\alpha\le+\infty$ for large α and (x_α,y_α) satisfy

(5) $$\lim_{\alpha\to\infty}\left[\mu_\alpha-\left(u(x_\alpha)-v(y_\alpha)-\frac{\alpha}{2}\right)d(x_\alpha,y_\alpha)^2\right]=0$$

Lemma 2.1.7 M_1, M_2 Riemannian Manifold Boundary

Let M_1,M_2 be Riemannian manifold with or with or without boundary $u_1\in USC(M_1), u_2\in LSC(M_2)$ and $\varphi\in C^2(M_1\times M_2)$ Suppose $(\hat{x},\hat{y})\in M_1\times M_2$ is local maximum of $u_1(x)-u_2(y)-\varphi(x,y)$ then for any $\varepsilon\ge 0$ there exist $X_1\in S T_{\hat{x}}^* M_1$ and $X_2\in S^2 T M_2$ such that .

(6) $$(D_x\varphi(x,y), X_i)\in\bar{J}^{2,+}u_1(\hat{x}_i)\quad for\, i\ne 1$$

Theorem 2.1.8 Compact Riemannian Manifolds

Let M be compact Riemannian Manifold with or without boundary ∂M , $f(M) \in C(F, R)$ proper function satisfying. $\beta(-s) \le F(x, r, p, X) - F(x, r, p, X)$ $for\ r \ge s$ There exist a function $W : [0, \infty) \to [0, \infty)$ satisfying $W(t) \ge 0\ when\ t \ge 0$ and $W(0+) = 0$.

2.2 Viscosity Maximum Principle on Complete Riemannian manifolds

Maximum principle for C^2. function on complete Riemannian manifold were already developed gradient estimate for C^2. function some in equalities and proved a maximum principle which is well known and has many important applications in geometry , In this will generalize both maximum principle to non-differentiable function , the approach is quite different m even for C^2. function we present here a new proof for maximum principle.

Theorem 2.2.1 Curvature Bounded in Riemannian Manifold

Let M be complete Riemannian manifold with sectional curvature bounded below by a constant $-K^2$ let $u \in USC(M)$ and $v \in LSC(M)$ be tow functions satisfying. $\mu_0 := \sup_{x \in M} [u(x) - v(x)] \le +\infty$ Assume that u and v are bounded from above and below respectively and there exists a function $W : [0, \infty) \to [0, \infty)$ satisfying $W(l) \ge 0$. when $l \ge 0$. and $W(0, +) = 0$ such that . $u(x) - u(y) \le w(d(x, y))$ Then for each $\varepsilon \ge 0$ there exist $x_\varepsilon, y_\varepsilon \in M$, such that $(p_\varepsilon, X_\varepsilon) \in \bar{J}^{2,+} u(x_\varepsilon)$, $(q_\varepsilon, Y_\varepsilon) \in \bar{J}^{2,+} v(y_\varepsilon)$ such that $u(x) - u(y) \ge \mu_0 - \varepsilon$ .And such that .

(7) $$d(x_\varepsilon, y_\varepsilon) \le \varepsilon \ , \left| p_\varepsilon - q_\varepsilon \circ P_\gamma(l) \right| \le \varepsilon \ , \ X_\varepsilon \le Y_\varepsilon \circ p_\gamma(l) + \varepsilon \ P_\gamma(l)$$

Where $l = d(x_\varepsilon, y_\varepsilon)$ and $p_\gamma(l)$ is the parallel transport along the shortest geodesic connecting x_ε and y_ε.

Proof :

We divide the proof into tow parts :

[1]: without loss of generality , we assume that $\mu_0 \ge 0$. Otherwise we replace u by $u - \mu_0 + 1$ for each $\alpha \ge 0$ we take $\hat{x}_\alpha \in M$ such that . $u(x_\alpha) - v(x_\alpha) + w\left(\sqrt{\frac{\mu_0}{\alpha}}\right) \ge \mu_0$.

[2] : We apply to $\varphi_\alpha(x, y) = \frac{\alpha}{2} d(x, y)^2 + \frac{\lambda_\alpha}{2} d(x_\alpha, x)^2 + \frac{\lambda_\alpha}{2} d(x_\alpha, y)^2$. We have for any $\delta \ge 0$ there exist $X_\alpha \in ST^*_{x_\beta} M$ and $Y_\alpha \in ST^*_{y_\alpha} M$ such that . $(D_x \varphi_\alpha(x_\alpha, y_\alpha), X_\alpha) \in \hat{J}^{2,+} u(x_\alpha)$ and $(-D_y \varphi_\alpha(x_\alpha, y_\alpha), X_\alpha) \in \hat{J}^{2,-} u(y_\alpha)$ and the block diagonal matrix satisfies .

(8) $$-\left(\frac{1}{\delta} + \|\Lambda_\alpha\|\right) I \le \begin{pmatrix} X_\alpha & 0 \\ 0 & -Y_\alpha \end{pmatrix} \le \Lambda_\alpha + \delta \Lambda_\alpha^2$$

Theorem 2.2.2 A Complete Riemannian manifold with Ricci

Let M be a complete Riemannian Manifold with Ricci curvature bounded below by a constant be two functions satisfying $-(n-1)k^2$, $let\ u \in USC\ (M)\ v \in LSC(M)$,

$\mu_0 := \sup_{x\in M} \left(u(x) - v(x)\right) \leq +\infty$. Assume that u and v are bounded from above and below respectively and there exists a function $w : R_+ \rightarrow R_+$ with $w(0) = w(0+) = 0$ such that $u(x) - u(y) \leq w(d(x, y))$. Then for each $\varepsilon \geq 0$ there exist $x_\varepsilon, y_\varepsilon \in M$,

$(p_\varepsilon, X_\varepsilon) \in \bar{J}^{2,+}u(x_\varepsilon)$, $(q_\varepsilon, Y_\varepsilon) \in \bar{J}^{2,+}v(y_\varepsilon)$ such that $d(x_\varepsilon, y_\varepsilon) \leq \varepsilon$ $\left|p_{\varepsilon - q_\varepsilon} \circ P_\gamma(l)\right| \leq \varepsilon$, $trX_\varepsilon + \varepsilon$ where $l = d(x_\varepsilon, y_\varepsilon)$ and $P_\gamma(l)$ is the parallel transport along shortest geodesic connecting x_ε *and* y_ε.

Proof :

Let D_α be the convex neighborhood of $\hat{x}_\alpha$ chosen in the proof of the last theorem and $-k_\alpha^2$ is the lower bound of curvature in D_α .We can assume the diameter of D_α is small such that both $\|P_\alpha\|$ and $\|Q_\alpha\|$ are bounded by 2α and $2\lambda_\alpha$ respectively , By any orthonormal bass $(e_1, e_2, \ldots\ldots\ldots\ldots, e_n)$ at x_α with $e_1 = \gamma'(0)$.

(9)
$$\sum_{i=1}^{n}\left\langle X_\alpha e_i, e_i\right\rangle - \sum_{i=1}^{n}\left\langle Y_\alpha P_y(l_\alpha)e_i\right\rangle \leq 2(n-1)k_\alpha\, l_\alpha \frac{\sinh kl_\alpha/2}{\cosh kl_\alpha/2} + 4(n-1)\delta\left(\alpha^2 + \lambda_\alpha^2\right)$$

Here we may change base if necessary since we are computing the traces of X_α and Y_α respectively therefore we have $\alpha \geq 0$ such that when $\alpha \geq \alpha_1$.

(10)
$$\left(trX_\alpha - trY_\alpha\right) \leq \frac{\varepsilon}{2} \leq \varepsilon$$

The rest of the proof is the same as that of the preceding theorem .

Corollary 2.2.3 Complete Manifolds with Ricci Curvature Bounded

Let M be complete Riemannian Manifold with Ricci curvature bounded below by a constant $-(n-1)k^2$ and f *a* C^2 function on M bounded from below then for any $\varepsilon \geq 0$ there exist a point $x_\varepsilon \in M$ such that

$$f(x_\varepsilon) \leq \inf f + \varepsilon \ , \ \left|\nabla f\right|(x_\varepsilon) \leq \varepsilon \ , \Delta f(x_\varepsilon) \geq -\varepsilon$$

Proof :

Let $u = \inf f$ *,and* $v = f$ *.w* can be chose to be a linear function . it is straightforward to verify that all conditions in the theorem are satisfied .

2.3 Maximum Principle for Inhomogeneous Elliptic Inequalities on Complete Riemannian Manifolds

In this section we introduce the main notation ,form now on denotes a smooth complete Riemannian Manifold, with metric tensor $g \in C^\infty(m, T^\infty, m \otimes T^*, m)$.

Definition 2.3.1 Two Bundles is Manifold

The fibered product bundle of two bundles (E, π_1, m) and (F, π_2, m) is the manifold.

(11)
$$E \times mF = \left\{(e, f) \in E \times F : \pi_1(e) = \pi_2(f)\right\}$$

With the induced vector bundle structure.

Lemma 3.3.2 If $u : \Omega \rightarrow R$ is lipschitz function , then $u \in H_{loc}^{I,p}(\Omega)$ for every $p \geq 1$

2.4 Maximum Principle for Inhomogeneous Inequalities

In this , we extent to a Riemannian setting results concerning Maximum principle for solution of inhomogeneous elliptic inequalities for the Euclidean case , we cover their results, but at the some time we establish more precise a prior estimates. The p-laplacian Beltrami operator defined for a smooth function u as (12)

$$\Delta_p u = div\left(|\nabla u|^{p-2} \nabla u\right), \quad p \geq 1$$

Or mean curvature operator given by but they also apply to more general and sophisticated differential operators on a Riemannian Manifolds, which are elliptic a cording to the new definition of elliptic proposed , from now on we assume Ω to be a bounded and smooth domain of m , so that $\overline{\Omega}$ is a smooth manifold with boundary . However can also treat the case of Ω , $\overline{\Omega}$ is a smooth manifold with boundary , condition $u \leq M$ on $\partial\Omega$ is replaced by .

(13) $$\lim_{|x|\to\infty} \sup u(x) \leq M$$

While by $u \leq M$ on $\partial\Omega$ we mean that for every $\delta \geq 0$ there exists a neighborhood of $\partial\Omega$ in which $u \leq M + \delta$ in fact u will be assumed only of class $H^{I,p}_{Ioc}(\Omega)$ so that it have n trace on $\partial\Omega$.We consider inequalities of the form $div A(x,u,\nabla u) + B(x,u,\nabla u) \geq 0 \quad in \quad \Omega$.Where divergence and gradient are taken with respect to the Riemannian structure .We assume that $A: T(\Omega\times_\Omega R) \to T\Omega$ where $T(\Omega\times_\Omega R)$ stands for $T\Omega\times_\Omega(\Omega\times R)$ as already mentioned , and $A(x,z,\zeta) \in T_x m$ for all $x \in \Omega$, $z \in R$ and $\zeta \in T_x m$ while B is a real function defined in $T(\Omega\times_\Omega R)$, We also suppose that there exist $p \geq 1$, $a_1 \geq 0 \ \forall \ a_2, b_1, b_2, b \geq 0$ such that for all $(x,z,\zeta) \in T\times_\Omega R$ there hold $\langle A(x,z,\zeta),\zeta\rangle \geq a_1|\zeta|^p - a_2|z|^p - a^p$ and $B(x,z,\zeta) \leq a_1|\zeta|^{p-1} + b_2|z|^{p-1} + b^{p-1}$ If $p=1$ of course, by a rescaling argument it is enough to consider only the case $a_1 = 1$ so without loss of generality we assume $a_1 = 1$ is throughout .

Definition 2.3.1 Weak Solution

A (weak) solution of (5.36) is function $H^{I,p}_{Ioc}(\Omega)$ such that $A(u,\nabla u) \in L^1_{Ioc}(\Omega, T\Omega)$, $B(\nabla u) \in L^{p'}_{Ioc}(\Omega)$, where $p' = p/(p-1)$ if $p \geq 1$ and $p' = \infty$ if $p = 1$ and such that .

(14) $$\int_\Omega A(u,\nabla u)\nabla\phi \, dm \leq \int_\Omega B(\nabla u)\phi \, dm$$

Theorem 2.3.2 Semi – Maximum Principle

Let u be p-regular solution of satisfying as . Assume also that $u \leq M$ on $\partial\Omega$ for some constant $M \geq 0$. Then $u^+ \in L^\infty(\Omega)$ and there exists a universal constant $C = C(n, p, |\Omega|) \geq 0$ such that .

$u \leq M + C\left(\|u^+\|_p + a + b + k\right) in \quad \Omega$, where $k = k(a_1, b_1, b_2)$ is given by.

(15) $$k = \left\{ \left[b_1 + (a_2 + b_2)^{1/p} \right]^{n/p} + \left(a_2^{1/p} + b_2^{1/(p-1)} \right) M \right. \quad if\ p \geq 0$$

The same result can be given if we depend on the x-variable with some regularity precisely denoting simply by $\|f\|$ the norm in $L^q(\Omega)$ of f and q is assigned ,we have the following generalization.

Theorem 2.3.3 Maximum Principle

Let U be p-regular solution in Ω where A and B satisfy (with $b_1 = b_2 = 0$, suppose $u \leq M$ on $\partial\Omega$ for some constant $M \geq 0$ and if $p = 1$ we also assume that $a_2 + b \leq |\Omega|^{\frac{1}{n}} (1-\delta) S$

Defection 2.3.4 Maximum Principles Riemannian

For a sub harmonic function on f on Riemannian manifold M if there exist a pints in M at which attains the this property is to give a certain condition for a sub harmonic function to be constant , when we give attention to the fact relative t these maximum principles.

Definition 2.3.5Liouvile's

(a) Let f be a sub harmonic function on R^n, if it is bounded then it is constant.**(b)** Let f be a harmonic functions on R^n , $m \geq 3$. If it is bounded then it is constant . We are interested in Riemannian analogues of Liouvile,s theorem compared with these Last tow theorems we give attention to the fact that there is an essential difference between base manifold . In fact one is compact and the other is complete and an compact , we consider have a family of Riemannian manifold (M, g) at the global situations it suffices to consider a bout the family of complete Riemannian manifold of course , the subclass of compact Riemannian manifolds. (M, g): is complete Riemannian manifold since a compact Riemannian manifold .

Theorem 2.3.6 Complete Riemannian Manifold

A let M be complete Riemannian manifold whose Ricci curvature is bounded from below , if C^2- nonnegative function f satisfies Where Δ denotes the Laplacian on M , then f vanishes identically, the purpose of this theorem is t prove the following (Leadville Type) theorem in a complete Riemannian manifolds similar to theorem in a complete Riemannian manifold similar to give anther proof of (Nishikawas theorem) . In this note main theorem is as follows .

Theorem 2.2.5 Riemannian Manifold whose Ricci is Bounded

Let M be a complete Riemannian manifold whose Ricci curvature is bounded from blew , if C^2- nonnegative function f satisfies $\Delta f \leq C_0 f_n$. Where C_0 is any positive constant and n is any real number greater f vanishes identically .

Theorem 2.2.6 Ricci Riemannian Manifold

Let M an n-dimensional Riemannian manifold whose Ricci curvature is bonded from below on M , Let G be a C^2- functions bounded from below on M , then for any $\varepsilon \geq 0$, there exists a point P such that

(16) $|\nabla G(P)| \le \varepsilon \quad , \quad \Delta G(P) - \varepsilon \ and \ \inf G + \varepsilon \ge G(p)$

Proof :

In this section we prove the theorem stated in introduction first all in order prove theorem , then our theorem is directly obtained as a corollary of this property and hence Nishikawas theorem is also a direct consequence of this (Nishikawas one)

Theorem 2.2.7 Manifold and Ricci Curvature

Let M be a complete Riemannian manifold whose Ricci Curvature is bounded from blew , Let F be any formula of the variable F with constant coefficients such that $F(f) = (C_0 f^n + C_1 f^{n-1} + \dots\dots\dots + C_k f^{n-k}) + C_{k+1}$ Where $n \ge 1$, $1 \ge n-k \ge 0$ and $C_0 \ge C_{k+1}$ if a C^2 - nonnegative function f satisfies .

(17) $$\Delta f \ge F(f)$$

Then we have Where f_1 denotes the super mum f the given function f .

Proof :

From the assumption there exists a positive number a which satisfies $C_{k+1} \le a^n C_0$ For the constant a given above the function $G(f)$ with respect to 1-variable f is defined by $(f+a)^{\frac{1-n}{2}}$, n is the maximal degree of the f , then it is easily seen that G is the C^2 - function so that it is bounded from appositive by the constant $a^{\frac{1-n}{2}}$ and bounded from below by 0 , By the simple calculating we have

(18) $$\nabla G = -\frac{n-1}{2} G^{\frac{n+1}{n-1}} \nabla f$$

Hence we get by using the above equation $\frac{1-n}{2} G \frac{2n}{n-1} \Delta f = G\Delta G - \frac{n+1}{n-1} |\nabla G|^2$ Since the Ricci curvature is bounded from below by the assumption and the function G defined above satisfies the condition that it is bounded from below , we can apply the theorem (3.27) to the function G . Given any positive number ε there exist a point P at which it satisfies (31) and (32) , (33) the following relationship at P .

(19) $$\frac{1-n}{2} G(P)^{\frac{2n}{n-1}} \Delta(f) \ge -\varepsilon G(P) - \frac{n+1}{n-1}\varepsilon^2$$

Can be derived , where $G(P)$ denotes $G(f\varphi)$ thus for any convergent sequence e $G_0 = \inf G$, by taking a sub sequence , if necessary because the sequence is bounded and therefore each term $G(P_m)$ of the sequence satisfies equation we have $G(P_m) \to G_0 = \inf G$ and the assumption $n \ge 1$.An the other hand it follows from we have

(20) $$\frac{1-n}{2} G(P_m)^{\frac{2n}{n-1}} \Delta(P_m) \ge -\varepsilon_m G(P_m) - \frac{n+1}{n-1}\varepsilon_m{}^2$$

And the right side of the a above inequality converges to zero because the function G is bounded by choosing the constant a it satisfies $C_{k+1}a^{-n} \le C_0$, A accordingly there is a positive number δ such that $\frac{1-n}{2} C_{k+1}\ a^{-n} \le \delta \le \frac{n+1}{2} C_0$, C_0 is the constant coefficient of the maximal degree of function F so for a given such that $a\delta \ge 0$, we can take a sufficiently large integer m such that

(21) $$\frac{1-n}{2} G(P_m)^{\frac{2n}{n-1}} F(f(P_m)) \ge -\delta$$

Where we have used the assumption equation (3.2) of the theorem (3.2.6) and equation (3.4) so this inequality together with the definition of $G(P_m)$ Yield $F(f(P_m)) \le \frac{2\delta}{n-1}(f+a)(P_m)^n$

Remark 2.2.8

Suppose that a nonnegative function f satisfies the condition we can directly yield $\nabla f^{\ n-1} = (n-1) f^{\ n-2} \nabla f$

(22) $$\Delta f^{\ n-1} = (n-1)(n-2)\ f^{n-3}\ \nabla(f\ \nabla\ f) + (n-1) f^{n-2} \Delta\ f$$

we define a function h by $f^{\ n-1}$, if $n \ge 2$ then it satisfies $\Delta h \ge (n-1)\ C_0 h^2$ Thus concerning the theorem in the case $n \ge 2$ the condition (2.7) is equivalent $1 \le n \le 2$ where C_1 is a positive constant

III. GEOMETRIY MAXIMUM PRINCIPLES FOR HYPESURFACES IN LORTZIAN AND RIEMANNIAN MANIFOLDS

3.1 Geometrid Maximum and principle Riemannian manifolds

The version of the analytic principle given by:

(a) U_0 is lower semi – continuous and $M[U_0] \le H_0$ in the sense of support function.

(b) U_1 is upper – semi – continuous and $M[U_0] \ge H_0$ in the sense function with a one – sided Hessian bound .

(c) $U_1 \le U_0$ in Ω and $U_1 = U_0$ is locally a $C^{1,1}$- function in Ω finally if a^{ij} and b are locally $C^{k+2,\alpha}$ function in Ω. In particular if a^{ij} and b are smooth is $U_1 = U_0$, $\Omega \subset R^n$ is specially natural in Lorentzian setting as C^0 space like hyper surfaces in definition $S_{\eta,r} = \{p : d(p, \exp(r,\eta)) = r\}$, them $S_{\eta,r}$ contains $\pi(\eta)$ and neighborhood of $\pi(\eta)$ is smooth , at $\pi(\eta)$ pointing unit normal $r \ge 0$ and $k \subset T(M)$ can a lows be locally represented as a graphs also applies to hyper surfaces in Riemannian manifolds that can be represented locally as graphs. We first state our conventions on the sign of the second fundamental form and the mean curvature to fix choice of signs a Lorentzian manifold $(M.g)$.

Definition 3.1 Space time and Spacelike

A subset $N \subset M$ of that spacetime $(M.g)$ is C^0 spacelike hypersurface , if for each $p \in N$, there is a neighborhood U of p in M so that $N \cap V$ is causal and edge less in U .

Remark 3.1.1

In This definition not that if $D(N \cap U, U)$ is the domain of dependence of in U, then $D(N \cap U, U)$ is open in M and $U \cap U$ is a Cauchy hyper surface is globally hyperbolic thus by replacing U by $D(N \cap U, U)$ we can assume the neighborhood U is the last definition is globally hyperbolic and that $U \cap U$ in a Cauchy surface in U In particular a C^0 space like hyper surface is a topological. Let $(M.g)$ be a spacetime and let N_0 and N_1 be two C^0 space like hyper surfaces in $(M.g)$ which meet at a point q. Say that no is locally to the future of N_1 near q iff for some neighborhood U of P in which N_1 is a causal and edgeless $N_0 \cap U \subset J^+(N_1, \cup)$ where $J^+(N_1, \cup)$ is causal future of N_1 in U .

Definition 3.1.2 Saclike hyper surface is Space-time

(a) Let N be a C^0 space like hyper surface in the space-time $(M.g)$ and H_0 a constant then N has mean curvature $\leq H_0$, in the sense of support hyper surfaces for all $q \in N$, $\varepsilon \geq 0$ there is C^2 future support hyper surface $S_{q,\varepsilon}$ to N at q and the mean curvature of $S_{q,\varepsilon}$ at q satisfies $H_q^{S_{q,\varepsilon}} \leq H_0 + \varepsilon$.

(b) N has mean curvature $\geq H_0$ in the sense of support hyper surfaces with one- sided Hessian bounds for all compact sets $K \subseteq N$ there is compact set $K^\wedge \subseteq T(M)$ and constant $C_k \geq 0$, such that for all $q \in N$ so that .

the future pointing unit normal $n^{P(q,\varepsilon)}$ and second fundamental form $h^{P(q\varepsilon)}$ of $P_{q,\varepsilon}$ satisfy .

(23) $$H_q^{P(q,\varepsilon)} \geq H_0 - \varepsilon \quad , \quad h_q^{P(q,\varepsilon)} \geq -C_{kg} |_{P_{q,\varepsilon}}$$

Proposition 3.1.3

Let $(M.g)$ be a space-time $r_n \geq 0$ and $K \subset T(M)$ a compact set of future pointing time like unit vectors. Assume that there is a δ so that for all $\eta \in k$, the geodesic $\gamma_\eta(t) = \exp(t_\eta)$ maximizes the Lorentzian distatance on the interval $[0, r_0 + \delta]$ for each $\eta \in k$, and $r_0 \geq 0$, let $\pi(\eta)$ be the base point of r_n and set .

(24) $$S_{\eta,r} = \{P : d(P, \exp(r,\eta)) = r\}$$

Theorem 3.1.4

Let N_0 and N_1 be C^0 spacesurlike hyper surfaces in a specimen $(M.g)$ which meet at a point q_0, such that N_0 is locally to future of N_1, near q_0. Assume for some constant.

(a) No has mean curvature $\leq H_0$ in the sense of support hyperspaces .

(b) N_1 has mean curvature $\geq H_0$ in the sense of support hyperspaces with one- sided Hessian bounds, then $U_0 = U_1$ near q_0 ,(i,e)there is a neighborhood of q_0 such that $N_0 \cap 0 = N_1 \cap 0$, Moreover is smooth space like hyper surface with mean curvature H_0 .

Remark 3.1.5

If $(M.g)$ the metric only has finite differentiability , say g g is $C^{k,\alpha}$ with $k \geq 2$, and $0 \leq \alpha \leq 1$ then since the function a^{ij} and b in the definition of the mean curvature operator H, depend on the first derivatives of the metric , they are of class . Thus the regularity part, implies hyper surface $N_0 \cap 0 = N_1 \cap 0$ in the statement of the last is $C^{k+1,\alpha}$.

3.2 Reduction to Analytic Maximum Principle

Let $(M.g)$ be an an-dimesinal pastime and let ∇ be metric connection of metric g then near any point q of M there is a coordinate system $(x^1, x^2, \ldots, x^n)$ so that the metric takes the form.

(25) $$g = \sum_{A.B=1}^{n} (g_{A,B})\, dx^A\, dx^B = \sum_{i,j}^{n-1} (g_{i,j})\, dx^i dx^j - (dx^n)^2$$

And so that $\partial / \partial x^n$ is future pointing time like unit vector . (To construct such coordinates choose smooth spaccelike hypersurface) S in M passing through g and let $(x^1, x^2, \ldots, x^{n-1})$ be is as required . Let f be a function defined near the origin in R^{n-1} with $f(0) = 0$ the define a map F_f form a neighborhood of the origin in R^{n-1} to M so that the coordinate system $(x^1, x^2, \ldots, x^n)F_f$ is given by. $F_f(x^1, x^2, \ldots, x^{n-1}) = (x^1, x^2, \ldots, x^{n-1})$, this parameterizes a smooth hypersurface N_f through x_0 and moreover every smooth spacelike hypersurface x_0 is uniquely parameterized in this manner for unique f satisfying .

(26) $$1 - \sum_{i,j=1}^{n-1} g^{ij} (D_i f D_j) , \qquad f \geq 0$$

This is exactly the condition that the image of F_f is space like when the image is space like set .

$$X_i = (\partial/\partial x^i) + D_i f (\partial/\partial x^n),\ W = \left(1 - \sum_{i,j}^{n-1} g^{i,j} D_i f D_j\right)^{1/2},\ n = \frac{1}{w}\left(\partial/\partial x^n + \sum_{i,j}^{n-1} g^{i,j} D_i f (\partial/\partial x^j)\right)$$

Then $(x_1, x_2, \ldots, x_n)$ is a basis for the tangent space to image of N_f and n is the future pointing timelike unit normal to N_f. Now tedious calculation shows that the second fundamental form h of is given by .

(27) $$h(X_i, X_j) = \frac{1}{w}(D_{ij} f + \Gamma_{ij}^m - V_{ij})$$

Where Γ_{ij}^k are the christoffel symbols and V_{ij} are by

Solving for the Hessian of f in terms of the second fundamental form of N_f given .

(28) $$D_{ij} f = Wh(X_i, X_j) - \Gamma_{ij}^n + V_{ij}$$

The induced metric on N_f has its components in the coordinated system $(x^1, x^2, \ldots, x^{n-1})$ given by

(29) $$G_{ij} = g(X_i, X_j) = g_{ij} - D_i f D_j f$$

Let $[G^{ij}] = [G_{ij}]^{-1}$ then the mean curvature of N_f

$$H = \frac{1}{n-1}, h = \frac{1}{n-1}\sum_{i,j=1}^{n-1} G^{ij} h(X_i, X_j) = \frac{1}{(n-1)W}\sum_{i,j=1}^{n-1} G^{ij}(D_{ij} f + \Gamma_{ij}^n - V_{ij})$$ Where

$x = (x^1, x^2, \ldots x^{n-1})$, $[G^{ij}(x, f, Df)] = [g_{ij}(x, f) - D_i f(x) D_j f(x)]^{-1}$ and $V_{ij}(x, f, Df)$ is

$a^{ij}(x, f, Df) = \frac{1}{(n-1)w} G^{ij}(x, f, Df)$ and

(30) $$b(x, f, Df) = \frac{1}{n-1}\sum_{i,j=1}^{n} G^{i,j}(\Gamma_{ij}^n - V_{ij})$$

Therefore if $H(f)$ is the mean curvature of N_f then the operator $f :\to H(f)$ is quasi-linear .

Lemma 3.2.1 Curvature Tangent Bundle

Let $U_\alpha(\Omega_\alpha) \subset K$ where K is compact , then there is a compact subset $\hat{K}$ of the tangent bundle $T(M)$ that contains the set $U_\alpha\{h_\alpha(x) : x \in \Omega_\alpha\}$ if and only if there is a $\rho \geq 0$ so that for all α the lower bound $W_\alpha(x) \geq \rho_0$ hold for $x \in \Omega_\alpha$, Moreover if this lower bound holds and $0 \leq \rho \leq \rho_0$ there is bound $|f_\alpha| \leq \beta$ and if $U = U_{P,B,k} \subset R^{n-1} \times R \times R^{n-1}$ is defined by .

(31) $$U = U_{\rho,B,K} = \{(x^1, x^2, \ldots, x^{n-1}, r)(p_1, p_2, \ldots, p_{n-1}) = (x, r, p)$$

$x \in \beta$, $|r| \leq \beta$, $\sum_{ij=1}^{n-1} g^{i,j}(x, r) P_i P_j \leq 1 - \rho^2$ then for any $x \in k$ the fiber functions f_α are U admissible over and , finally the mean curvature operator H is uniformly elliptic on U .

3.3 Geometric Maximum Principle for Riemannian Manifolds

We now fix our sign conventions on the imbedding invariants of smooth hypersurfaces in Riemannian manifold (M, g). It will be convenient to assume that our hyper surfaces are the boundaries of open sets.An this a lways true Locally it is not a restriction by ∇ let $D \subset M$ be connected open set and let $N \subset \partial D$, be part of all ∂D is smooth , let $_n$ be the outward pointing unit normal along N then the second fundamental form of N symmetric bilinear form defined on the tangent space to N by $h^N(X,Y)=\langle \nabla_x, \nabla_n \rangle$ The mean curvature of N is then

$H^N = \frac{1}{n-1} \big|_{g,N}$ and $h^N = \frac{1}{n-1} \sum_{i=1}^{n-1} h^N(e_1, e_j)$ where $(e_1, e_2, ..., e_{n-1})$ is local orthogonal from for $T(N)$ this is the sign convention so that for the boundary S^{n-1} of the unit ball β^n in R^n the second fundamental form $h^N = -g \big|_{S^N}$ is negative definite the mean curvature is $H^{s^{(n-1)}} = -1$.

Definition 3.3.1 Hypersuface on Curvature $\geq H_0$

Let U be an open set in the Riemannian manifold (M, g) then :

(a) ∂U has mean curvature $\geq H_0$ in the sense of contact hypersurfaces iff for all $q \in \partial U$ and $\varepsilon \geq 0$ there is an open set D of M with $\bar{D} \subseteq \bar{U}$ and $q \in \partial D$ near q is a C^2 hypersuface of M and at point $q, H_q^{\partial D} \geq H_0 - \varepsilon$

(b) ∂U has mean curvature $\geq H_0$ in the sense contact hypersurface is constant $C_k \geq 0$ so that for all $q \in k$ and $\varepsilon \geq 0$ there is open set D of M with $\bar{D} \subseteq \bar{U}$ and $q \in \partial D$ the of ∂D near q, $H_q^{\partial D} \geq H_0 - \varepsilon$ and also .

(32) $$H_q^{\partial D} \geq -C_{k\,g} \big|_{\partial D} .$$

The Hyper-surfaces of manifolds as Let $M \subset K$ be any hyper-surface of quaternion manifold (K, Q),we define $H \subset TM$ to be the maximal Q- invariant distribution on M , if f is any defining function for M .

(a) If f is any defining function for M , i,e $M = f^{-1}(0)$ and $df \big|_M \neq 0$ then .

$$H = \{X \in TM : df(J_1 X) = df(J_2 X) = df(J_3 X) = 0\}$$

This H is always a smooth co-dimension 3-disribution on M .

(b) we say that a hyper-surface M of quaternion manifold $(K, Q = \{J_1, J_2, J_3\})$ is a QC - hyper-surface if :

$$\nabla df(X, X) \neq 0 \;,\; X \in H \;, X = 0$$

$$\nabla df(JX, JY) = \hat{\nabla} df(X, Y) \;,\; X, Y \in H \;, s = 1,2,3$$

Where $H \subset TM$ is the maximal Q-invariant distribution on $M, \hat{\nabla}$ is any torsion – free quaternion of (K, Q) and f is any defining function for M , for example the field of quaternions $H = Sup_R \{1, i, j, k\}$ where $i^2 = j^2 = -k^2 = -1$ and $i.j = -j, i = k$. Consider the flat quaternionic manifold $K = H^{n+1}$ whith its standard quaternionic structure $Q = Span\{J_1, J_2, J_3\}$,

$J_1(x) = -x.i$, $J_2(x) = -x.j$, $J_3(x) = -x.k$ is a torsion free quaternionic connection $\hat{\nabla}$ we take the flat connection here . It clearly holds $\hat{\nabla}\times Q\subset Q$, $Let\ x=(q_1,q_2,....,p)\in H^n\times H$ we have the following three basic of QC hyper-surfaces $H^n\times H$,

$M_1:\Sigma_{a=1}^{n}|q_1|^2+\mathrm{Re}(p)=0$, $M_2:\Sigma_{a=1}^{n}|q_a|^2-|p|^2=-1$, $M_3:\Sigma_{a=1}^{n}|q_a|^2+|p|^2=1$ the sphere.

Theorem 3.3.1 Geometric Maximum Principle for Riemannian Manifolds

Let (M,g) be a Riemannian manifold $U_0,U_1\subset M$ open sets and let H_0 be a constant , assume that .

(a) $U_0\cap U_1=0$

(b) ∂U_0 has mean curvature $\geq H_0$ in the sense of contact hypersurfaces.

(c) ∂U_1 has mean curvature $\geq H_0$ in the sense of contact hypersurfaces with a one sided Hessian bound .

(d) there is a point $p\in\dot{U}_0\cap\overline{U}$ and a neighborhood N of p that has coordinates $(x^1,x^2,....,x^n)$ cantered at p so that for some $r\geq 0$ the image of these coordinates is the box $(x^1,x^2,....,x^n):|x^i|\leq r$ and there are Lipschitz continous and there are Lipschitz continuous function $U_0,U_1:(x^1,x^2,....,x^{n-1}):|x^i|\leq r$, $(-r,r)$, so that $U_0\cap N$ are given by .

(33)
$$U_0,N=(x^1,x^2,...,x^n):x^n\geq U_0(x^1,x^2,....,x^{n-1})$$
$$U_1,N=(x^1,x^2,...,x^n):x^n\geq U_1(x^1,x^2,....,x^{n-1})$$

This implies $U_0\equiv U_1$ and U_0 is smooth function , therefore $\partial U_0\cap U_1=\partial U_1\cap N$ is a smooth embedded hypersurface with constant mean curvature H_0 (with respect to the outward normal to U_1).

Definition 3.3.2 Lorentzian Mainfolds

Let (M,g) be a Lorentzian manifold and let $q\geq 0$, then (M,g) is globally hyperbolic of order q if and only if M is strongly causal and $x\leq y$, $d(x,y)\leq\frac{\pi}{q}$ implies that $C(x,y)$ is compact where $c(x,y)$ is set of causal curves connecting x and y .

Corollary 3.3.3 Lorentzian Maximal Diameter Theorem

Let (M,g) be connected Lorentzian manifold which is globally hyperbolic of order 1. and assume that $Ric(T,T)\geq(n-1)$ for any time like unit vector T if M a timelike geodesic segment $\gamma:\left[\frac{-\pi}{2},\frac{\pi}{2}\right]\to M$ of length π connecting x and y , then $D=(z:x\leq z\leq y)$ is isometric to $(S_1^n(-1),g_S)$.Moreover if M contains a time like geodesic $\gamma=(-\infty,\infty)\to M$ such that each segment $\gamma|_{[t,t+\pi]}$ is maximizing then (M,g) is isometric to $(S_1^n(-1),g_g)$. Moreover if

M contains a time like geodesic $\gamma = (-\infty,\infty) \to M$ such that each segment $\gamma |_{[t,t+\pi]}$ is maximizing then (M,g) is isometric to universal anti-de sitter space $R^n(-1)$.

Definition 3.3.4 Asymptote Curvature]

Let $\gamma = (\pi/2, \pi/2) \to M$ be line in M, and let $s \in (\pi/2, \pi/2)$ for $P \in \gamma(s)$, let α_S be a maximal geodesic connecting P and $\gamma(s)$, if there is sequence and timelike unit vector V such that $S_K \to \pi/2$, $P \le \gamma(s_K)$, and $\alpha_{S_K}(0) \to V \in T(M)_P$ then the maximal geodesic starting at P in the direction V is called an asymptote to γ and V.

Definition 3.3.5 Timelike Lines I

A strip is a totally geodesic immersion f of $(\pi/2, \pi/2 \times s)$ is a timelike line for each $S \in I$. We will denote by S the space $(-\pi/2, \pi/2)\times I$, $-dt^2 + \cos^2(t)dt^2$ into M for some interval I so that $f |_{[-\pi/2, \pi/2 \times s]}$ is time like line for each $S \in I$.

Lemma 3.3.6 Parallel Lines γ_1 and γ_2

If γ_1 and γ_2 are parallel lines, then $I(\gamma_1) = I(\gamma_2)$, and the Busman function b_1^t and b_2^t of γ_1 and γ_2 through x and parallel to γ_1.

lemma 3.3.7 Lorentzian Productmtric

Let (N, g_N) be a Riemannian manifold of dimension at least three, set $M = R \times N$ and give M the lorentzian productmtric $g = -dt^2 + g_N$ let $R_{A,B,C,D}$ be the curvature tensor of (M,g) as tensor.

3.4 The Spectrum of the Palladian in Riemannian Manifolds

To any compact Riemannian manifold (M,g) is boundary we associate second- order (P.D.E), the Laplace operator Δ is defined by : $\Delta(f) = -div(grad\ f)$ For $f \in L^2(M,g)$. We also sometimes write Δ_g for Δ if we want to emphasize which metric the Laplace operator is associated with the set of eigenvalues of Δ is called the spectrum of Δ or of M which we will write as space Δ (or space (M,g), they form a discrete sequence $0 = \lambda_1 \le \lambda_2 \le, \ldots, \le \lambda_n$ for simplicity, we will assume that M is connected. This will for example imply that the smallest eigenvalue λ_0. Occurs with multiplicity.

Definition 3.4.1

If L is liner operator defined on T_pM , then the spectrum of L is the set of eigenvalues of L .It is denoted by space (L) . We take the Laplace operator Δ defined as $\Delta = -(d\delta + \delta d)$, where δ is adjoin of d in spectral geometry we consider the following two equations :**(a)** Does the spectrum of M determine the geometry of M .**(b)** Does the geometry of M determine the spectrum of M .

Definition 3.4.2 Sequences be Spectra

Sequences occur can as the spectra of manifolds a version of this question. Has been answered what finite sequences can occur as the initial part of spectra of manifolds . If M is a closed connected manifold of dimension greater that or Equal p preassigned finite sequence $0 = \lambda_1 \leq \lambda_2, ..., \leq \lambda_k$ is Sequence of first $K+1$ eigenvalues of Δ_g for some choice of the metric g on M . In particular , this means that for closed connected manifolds of 3-dimension or Greater , there are no restrictions on the multiplicities of the eigenvalues λ_i for $i \geq 0$. In 2-dimension , there are some restrictions on the multiplicities of the eigenvalues. Let M be a closed connected 2-manifold with Euler characteristic $\chi(M)$, and let m_j be the multiplicity of the $j-th$ eigenvalue $j \geq 0$ of the laplacian operator associated to a metric on M then :**(a).** If M is the unit sphere, then $m_j \leq 2_j + 1$. **(b).** If M is the real projective plane , then $m_j \leq 2_j + 3$ **(c).** If M is the torus , then $m_j \leq 2_j + 4$. **(d).** If M is the klen bottle , then $m_j \leq 2_j + 3$ **(f)** If , $\chi(M) \leq 0$ then $m_j \leq 2_j + 2\chi(M) + 3$. **[Note]** : For finite sequences $0 = \lambda_0 \leq \lambda_1 \leq \lambda_2, ..., \leq \lambda_n$ however the result by-Colin de derriere holds – even in 2-dimension .

Definition 3.4.3 Estimates on the first Eigenvalue

The geometry of a manifold affects more that the multiplicities of the eigenvaluees . Here we will focus on bounds on the first non-zero eigenvalue λ_1 imposed by the geometry . the first lower bound is due to lichnzeowicz .

Theorem 3.4.4 Ricci Tensor

Let (M, g) be a closed Riemannian manifold of dimension $n \geq 2$ and let Ric be its Ricci tensor field if Ricci $(X, X) \geq (n-1)k \geq 0$. For some constant $k \geq 0$, and for all $X \in T(M)$, then $\lambda_1 \geq nk$.

Theorem 3.4.5

Let (M,g) be a closed Riemannian manifold , if Ricci $(X,X) \geq (n-1)k \geq 0$ .For some nonnegative constant k and for all $X \in T(M)$ then $\lambda_1 \geq \frac{(n-1)k}{4} + \frac{\pi^2}{D^2(M)}$. It is in general much easier to given upper bounds on λ_1 that it is give lower bounds . The basic result in this area is a comparison theorem due to a complete Riemannian n- manifold whose Ricci curvature is $\geq (n-1)k, k$, is some const.

Theorem 3.4.6 Ricci Curvature

If M is a compact n-manifold with Ricci curvature $\geq (n-1)(-k)k \geq 0$, then $\lambda_1 \leq \frac{(n-1)^2}{4}k + \frac{c_2}{D^2(M)}$ Where c_2 is positive constant depending only on n .

Definition 3.4.7 Geometric Implications Of The spectrum

The spectrum does not in general determine the geometry of a manifold Neverthless earthiness , some geometric information can be extracted from the spectrum . In what follows , we define a spectral invariant to be any thing that is completely determined by the spectrum .

Definition 3.4.8 Invariants From The Heat Equation

Let M be a Riemannian manifold . A heat kernel or alternatively fundamental solution to the heat equation , is a function $K:(0,\infty)\times(M\times)M \to M$. That satisfies $K(t,x,y)$ *is* C^1 *in* (t) and C^2 in x and y . $\frac{\partial K}{\partial t} + \Delta_2(K) = 0$ where Δ_2 is the Laplacian with respect to the second variable . $\lim_{t\to 0^+} \int_M K(t,x,y)\ f(y)\,dy = f(x)$ For any compactly supported function f on M .The heat kernel exists and unique for Riemannian manifold , its importance stems from the fact that the solution to the heat equation .

(34) $$\frac{\partial u}{\partial t} + \Delta(u) = 0, u:(0,\infty)\times M \to R$$

Where Δ is Laplacian with respect to second variable , with initial condition $u(0,x) = f(x)$ is given by:

(35) $$u(t,x) = \int_M K(t,x,y)\ f(y)\,dy$$

If λ_i in spectrum of M and ζ_i are the associated eigenfunctions (normalized so thst they form an orthonormal basis of $L^2(M)$ then we can write .

(36) $$K(t,x,y)=\Sigma_i e^{-\lambda_i t}\zeta_i(x)\zeta_i(y)$$

From this it clear that the heat trace $Z(t)=\int_{t\to 0} K(t,x,x)=\sum_{M} e^{-\lambda_i t_j}$ a spectral invariant . The heat trace has an asymptotic expansion as $t\to 0^+ . Z(t)=(4\pi t)^{\dim M/2}\sum_{j=1}^{\infty} a_j t^j$. Where the a_j are integrals over M of universal homogenous polynomials in the curvature and covariant derivatives. The first few of these are :

(37) $$a_0 = vol(M), a_1=\frac{1}{6}\int_M S, a_2=\int_M \left(S^2-2|Ric|^2-|Rm|^2\right)$$

Where S is the scalar curvature , Ric. is the Ricci tensor , R.m. is the curvature tensor . the dimension the volume and total scalar curvature are thus completely determined by spectrum . If M is a surface then the Gauss Bonnet theorem implies that the Euler characteristic of M is also a spectral invariant . Amore in depth study of the heat trace can yield more information of dimension $n\le 6$ and if M has same spectrum as the n-sphere S^n with the standard metric (resp . RP^m) then M is in fact isometric to S^n (resp . RP^m) more on this can be found .

Definition 3.4.9 Isospectral Manifolds

As was alluded to earlier, geometry is not in general a spectral in variant two manifolds are said to be isospectral if they have the same spectrum . Of non isometric isospectral manifolds was found too distinct but isospectral manifolds .

Definition 3.4.10 Direct Computation of The Spectrum

The first of those is straightforward: direct computation . it rarely possible to explicitly compute the spectrum of a manifold were actually discovered via this method . Milnor's example mentioned above consists of two isospectral factory-quotients of Euclidean space by lattices of full rank being one of full rank being one of the few examples of Riemannian manifolds whose spectra can be computed explicicitly spherical space forms – quotients of spheres by finite groups of orthogonal transformations acting without fixed points form another class of examples of manifolds isospectral for the Laplaction acting on p-forms for $p\le k$ but not for the Laplaction acting on p-forms for $p\le k+1$ (recall that a lens space is spherical space form where the group is cyclic .

Theorem 3.4.11

Let $m\Gamma_1$ and $m\Gamma_2$ be compact discrete subgroup of a lie group G , and let g be a left invariant metric on G if $m\Gamma_1$ and $m\Gamma_2$ are representation equivalent then .

(38) $$Spec(m_1 / G, g) = Spec(m_2 / G, g).$$

IV. CONCLUSION

(a) The M_1, M_2 be Riemannian manifold with or with or without boundary $u_1 \in USC(M_1)$, $u_2 \in LSC(M_2)$ and $\varphi \in C^2(M_1 \times M_2)$, do not replicate the abstract as the conclusion. A conclusion might elaborate on the importance of the work or suggest applications and extensions.

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[14] J.Ansgar staudingerweg 9.55099 mainz,Germany e-mail:juengel@mathematik.uni-mainz.de ,U.Andreas institute fiir mathematic,MA6-3,TU Berlin strabe des 17.juni. 136,10623 Berlin ,Germany, e-mail: unterreiter@math.tu-berlin.de.

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Authors

First Author :

Dr. : Mohamed Mahmoud Osman- (phd)

Studentate the University of Al-Baha –Kingdom of Saudi Arabia

Al-Baha P.O.Box (1988) – Tel.Fax : 00966-7-7274111

Department of mathematics faculty of science

1- Email: mm.eltingary@hotmail.com

2- Email : Moh_moh_os@yahoo.co

Tel. 00966535126844