FINITE DIFFERENCE METHOD FOR THE BURGERS EQUATION

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ABSTRACT

Burgers equation: $u_t + uu_x = \lambda u_{xx}$ is a fundamental partial differential equation arising from fluid mechanics. It occurs in various areas of applied mathematics, such as modeling of gas dynamics and traffic flow. It was named after Johannes Martinus Burgers (1895 – 1981). J. M. Burgers did studies on the equation in 1940s principally as a model problem of the interaction between nonlinear and dissipative phenomena.

The equation arises in model studies of turbulence and shock wave theory. In physical application of shock waves in fluids, coefficient λ has the meaning of viscosity. For light fluids or gases the solution considers the inviscid limit as λ tends to zero.

The solution of the Burgers equation is classified into two categories: numerical solutions and analytic solutions. In both methods, the solutions have been valid for λ Є (0, 1).

In this paper we have solved the Burgers equation using finite difference methods where λ is not restricted to the interval (0, 1). In this work we have managed to solve the Burgers equation with λ Є $(0,\frac{10}{3})$. The methods involved developing a finite difference scheme, analyzing the scheme for stability and solving the resulting system of equations using Mathcad 2000 professional. It is our hope that this will be of great contribution to the mathematical knowledge in the application of the Burgers equation.

KEYWORDS: 1.EXPLICIT 2. SCHMIDT 3. DISCRETIZATION 4.SCHEME

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Symbols and Notations

H.O.T.	Higher order terms
O	Order of accuracy
λ	Viscosity constant
$\triangle x$	Small change in x
$\triangle t$	Small change in t
w.r.t.	With respect to
u	constant coefficient
u_x	Partial differential operator with respect to x
u_t	partial differentiation with respect to t
PDE	partial differential equation
ODE	ordinary differential equation
$\triangle$	Discriminant
FDE	Finite difference equation
u_{xx}	second order partial differential operator with respect to x

1 Introduction

Finite difference method is applied in numerical analysis especially in finite differential equations which aim at the numerical solution of ordinary differential equation (ODE) and partial differential equation (PDE) respectively [1].

The idea is to replace the derivatives appearing in the differential equation by finite differences that approximate them.

The accuracy of a finite difference calculation can be improved by suitable mesh reduction or by increasing the order of the truncation error [7]. Uniform mesh reduction improves accuracy but results in a significant increase in the number of equations which are inefficient from the point of view of computer storage and calculation time.

Finite difference methods are applied in computational science and engineering disciplines such as thermal engineering.

The paper describes the procedure for the solution of the burgers equation

$$u_t + uu_x = \lambda u_{xx} \tag{1}$$

Using finite difference method.

Equation (1) is used in the study of fluid dynamics and in engineering as a simplified model for turbulence, boundary layer behavior, shock wave formulation and mass transport. The equation has been studied and applied for many decades. Many different closed-form, series approximation and numerical solutions are known for particular sets of boundary conditions [2].

2.0 Preliminaries.

The finite difference technique consists of replacing the partial derivatives occurring in the partial differential equation as well as in the boundary and initial conditions by their corresponding finite difference approximations, and then solving the resulting linear system of equations by an iterative procedure[6].

The numerical values are obtained at the mesh points or nodal points as indicated in the figure 1 below.

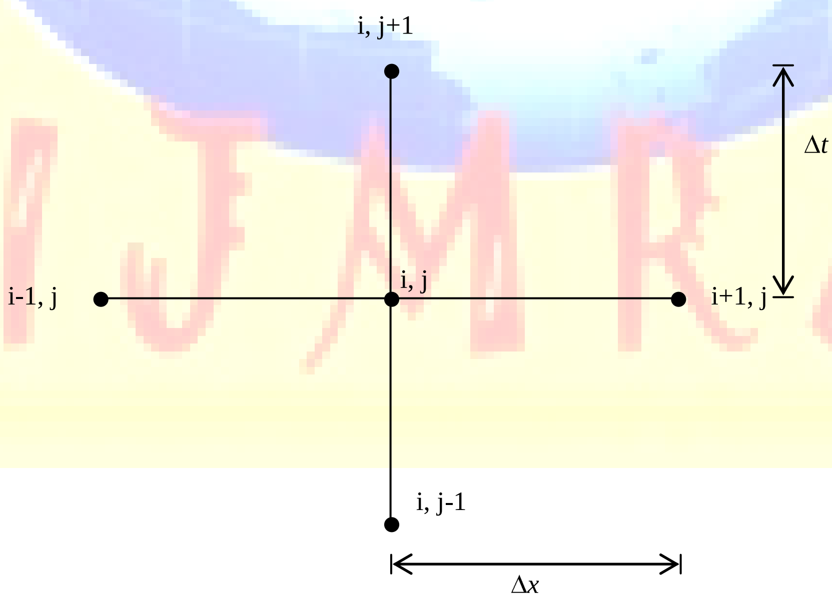

Fig 1: The inside node of a finite difference mesh

The subscript *i* represent *x* co-ordinate and *j* represent time, hence

$$u\ (x,t) \cong u(i\Delta x, j\Delta t) \cong\ u_{i,j} \tag{2}$$

The finite difference approximations to derivatives can be obtained from Taylor's series expansion using either the backward, forward or central difference approximations. The Taylor's series expansion of $u_{i+1,j}$ and $u_{i-1,j}$ about (i, j) will be

$$u_{i+1,j} = u_{i,j} + \left[\Delta x \frac{\partial u}{\partial x} + \frac{(\Delta x)^2}{2} \frac{\partial^2 u}{\partial x^2} + \frac{(\Delta x)^3}{6} \frac{\partial^3 u}{\partial x^3} \right]_{i,j} + H.O.T \tag{3}$$

and

$$u_{i-1,j} = u_{i,j} - \left[\Delta x \frac{\partial u}{\partial x} - \frac{(\Delta x)^2}{2} \frac{\partial^2 u}{\partial x^2} + \frac{(\Delta x)^3}{6} \frac{\partial^3 u}{\partial x^3} \right]_{i,j} + H.O.T \tag{4}$$

respectively.

The Taylor's series expansions of $u_{i,j+1}$ and $u_{i,j-1}$ about (i,j) will be

$$u_{i,j+1} = u_{i,j} + \left[\Delta t \frac{\partial u}{\partial t} + \frac{(\Delta t)^2}{2} \frac{\partial^2 u}{\partial t^2} + \frac{(\Delta t)^3}{6} \frac{\partial^3 u}{\partial t^3} \right]_{i,j} + H.O.T \tag{5}$$

and

$$u_{i,j-1} = u_{i,j} - \left[\Delta t \frac{\partial u}{\partial t} - \frac{(\Delta t)^2}{2} \frac{\partial^2 u}{\partial t^2} + \frac{(\Delta t)^3}{6} \frac{\partial^3 u}{\partial t^3} \right]_{i,j} + H.O.T \tag{6}$$

respectively

Solving for $\frac{\partial u}{\partial x}$ in equation (3) gives

$$\left(\frac{\partial u}{\partial x} \right)_{i,j} = \left[\frac{u_{i+1,j} - u_{i,j}}{\Delta x} - \frac{\Delta x}{2} \frac{\partial^2 u}{\partial x^2} - \frac{(\Delta x)^2}{6} \frac{\partial^3 u}{\partial x^3} \right] + H.O.T \tag{7}$$

This can be written as

$$\left(\frac{\partial u}{\partial x} \right)_{i,j} = \frac{u_{i+1,j} - u_{i,j}}{\Delta x} - O(\Delta x) \tag{8}$$

Where O (Δx) is the order of accuracy in small change in *x*.

Similarly,

$$\left(\frac{\partial u}{\partial x} \right)_{i,j} = \left[\frac{u_{i,j} - u_{i-1,j}}{\Delta x} + \frac{\Delta x}{2} \frac{\partial^2 u}{\partial x^2} - \frac{(\Delta x)^2}{6} \frac{\partial^3 u}{\partial x^3} \right] + H.O.T$$

can be written as

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i,j} - u_{i-1,j}}{\Delta x} - O(\Delta x) \qquad (9)$$

also, equations (5) and (6) can be written as

$$\left(\frac{\partial u}{\partial t}\right)_{i,j} = \frac{u_{i,j+1} - u_{i,j}}{\Delta t} + O(\Delta t) \qquad (10)$$

and

$$\left(\frac{\partial u}{\partial t}\right)_{i,j} = \frac{u_{i,j} - u_{i,j-1}}{\Delta t} - O(\Delta t) \qquad (11)$$

respectively. Where O (Δt) is the order of accuracy in a small change in *t*. Equations (8) and (10) are the forward difference approximations to the derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial t}$ respectively.

Subtracting equations (4) from (3) we have,

$$u_{i+1,j} - u_{i-1,j} = 2\Delta x \frac{\partial u}{\partial x} + \frac{(\Delta x)^2}{3} \frac{\partial^3 u}{\partial x^3} + - - -$$

Rearranging we have

$$\frac{\partial u}{\partial x} = \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} - \frac{(\Delta x)^2}{3} \frac{\partial^3 u}{\partial x^3} + - - -$$

Thus

$$\frac{\partial u}{\partial x} = \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} + O(\Delta x^2) \qquad (12)$$

Similarly subtracting equation (6) from (5) we have

$$u_{i,j+1} - u_{i,j-1} = 2\Delta t \frac{\partial u}{\partial t} + (\Delta t)^3 \frac{\partial^3 u}{\partial t^3}$$

Rearranging, we have,

$$\frac{\partial u}{\partial t} = \frac{u_{i,j+1} - u_{i,j-1}}{2\Delta t} + O(\Delta t^2) \qquad (13)$$

Thus equation (12) and (13) are the central difference approximations and they have a truncation error of orders $O(\Delta x^2)$ and $O(\Delta t^2)$ respectively.

Similarly, adding equations (3) to (4) to get

$$u_{i+1,j} + u_{i-1,j} = 2u_{i,j} + (\Delta x)^2 \frac{\partial^2 u}{\partial x^2}$$

rearranging, we have,

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2} = \frac{\partial^2 u}{\partial x^2} + \frac{(\Delta x)^2}{12}\frac{\partial^3 u}{\partial x^3} + O(\Delta x^3). \quad (14)$$

Thus;

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2} + O(\Delta x^2) \quad (15)$$

also, adding equations (5) to (6) we get

$$u_{i,j+1} + u_{i,j-1} = 2u_{i,j} + (\Delta t)^2 \frac{\partial^2 u}{\partial t^2}$$

Rearranging we have,

$$\left(\frac{\partial^2 u}{\partial t^2}\right)_{i,j} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(\Delta t)^2} + O(\Delta t^2). \quad (16)$$

Equations (15) and (16) are the central difference approximations to a second order partial derivative and they have a truncation error of orders $O(\Delta x^2)$ and $O(\Delta t^2)$ respectively.

In general the approximations of partial derivatives can be given as

$$\frac{\partial u}{\partial t} = u_t = \frac{u_{i,j+1} - u_{i,j-1}}{(2\Delta t)} + O(\Delta t^2) \quad (17)$$

$$\frac{\partial^2 u}{\partial x^2} = u_{xx} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2} + O(\Delta x^2) \quad (18)$$

$$u\frac{\partial u}{\partial x} = uu_x = u_{i,j}\left(\frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x}\right) \quad (19)$$

Higher order finite difference approximations can be obtained by taking more terms in Taylor series expansion.

3.0 Numerical solution

We considered numerical schemes in our discretization of the Burgers equation i.e. explicit method. The solution is based on the boundary and initial conditions shown below.

$u(x, 0) = Sin(\pi x) \quad 0 \leq x \leq 1, \quad t > 0$

$\mathrm{u}(0, \mathrm{t}) = \mathrm{u}(1, \mathrm{t}) = 0, \mathrm{t} \geq 0$

3.1 Explicit method

This is a finite difference method for solving partial differential equations. The approach in solving such equation is to replace the derivatives of the given differential equation by their finite difference approximations. In this process, we developed various finite difference formulae.

In this method, we expressed one unknown value at a given node in terms of the known preceding values. We replaced time derivative by Forward difference approximation and space derivatives by central difference approximations commonly known as Schmidt method.

3.2 Schmidt method

Solve the Burgers equation

$u_t + uu_x = \lambda u_{xx} \qquad 0 \leq x \leq 1, \quad t > 0$ Subject to boundary conditions

$\mathrm{u}(0, \mathrm{t}) = \mathrm{u}(1, \mathrm{t}) = 0, \mathrm{t} \geq 0$ And

Initial conditions: $u(x, 0) = Sin(\pi x)$

In this scheme, we discretize the Burgers equation by replacing u_t and u_x by forward difference while u_{xx} by central difference approximations to a second order. Thus (1) becomes

$$\frac{U_{i,j+1} - U_{i,j}}{k} + U_{i,j}\left[\frac{U_{i+1,j} - U_{i,j}}{h}\right] = \lambda\left[\frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{h^2}\right]$$

which simplifies to

$$U_{i,j+1} = U_{i,j} + \frac{k}{h} U_{i,j} this\left[U_{i,j} - U_{i+1,j}\right] + \frac{k\lambda}{h^2}\left[U_{i+1,j} - 2U_{i,j} + U_{i-1,j}\right]. \tag{20}$$

Let $\Delta x = 0.1$, $\Delta t = 0.002$

Thus,

$$r_1 = \frac{k}{h} = \frac{0.002}{0.1} = 0.02 \qquad and \qquad r_2 = \frac{k}{h^2} = \frac{(0.002)}{0.1^2} = 0.2$$

The numerical scheme to (20) becomes

$$U_{i,j+1} = U_{i-1,j} + 0.02U_{i,j}\left[U_{i-1,j} - U_{i+1,j}\right] + 0.2\lambda\,[U_{i+1,j} - 2U_{i,j} + U_{i-1,j}] \qquad (21)$$

Consider the scheme of computation shown below.

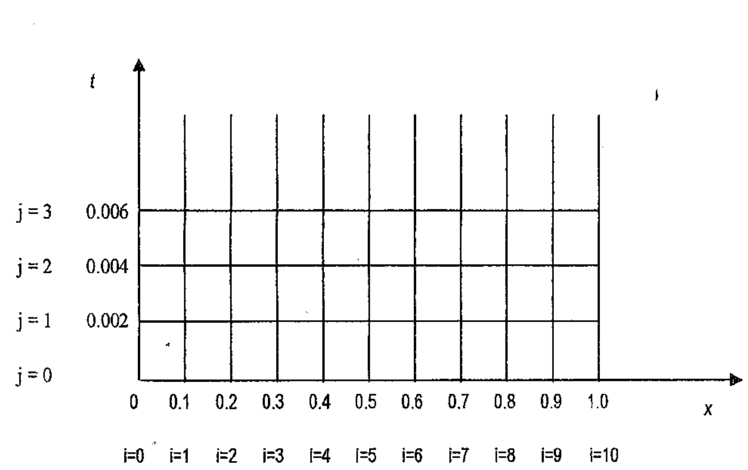

Fig.2: Scheme of Computation for explicit Schmidt method for Burgers equation.

At j=0, t=0 and the end values of U are given by the boundary and initial conditions.

$U_{1,0} = 0.3090$, $U_{2,0}$=0.5878, $U_{3,0}$=0.8090, $U_{4,0}$=0.9511, $U_{5,0}$=0.1.000, $U_{6,0}$=0.9511

$U_{7,0} = 0.8090$, $U_{8,0}$=0.5878 , $U_{9,0} = 0.3090$

$$U_{0,j} = U_{10,j} = 0$$

Set j=0 into Scheme (21) and vary $i = 1,2 \ldots\ldots\ldots\ldots\ldots\ldots..N-1$. That is

j=0

$i = 1 : U_{1,1} = U_{1,0} + 0.02U_{1,0}\left[U_{1,0} - U_{2,0}\right] + 0.2\lambda[U_{2,0} - 2U_{1,0} + U_{0,0}]$

$$i = \ 2 : U_{2,1} = U_{2,0} + 0.02U_{2,0}\left[U_{2,0} - U_{3,0}\right] + 0.2\lambda[U_{3,0} - 2U_{2,0} + U_{1,0}]$$

$$i = \ 3 : U_{3,1} = U_{3,0} + 0.02U_{3,0}\left[U_{30} - U_{4,0}\right] + 0.2\lambda[U_{4,0} - 2U_{3,0} + U_{2,0}]$$

$$i = \ 4 : U_{4,1} = U_{4,0} + 0.02U_{4,0}\left[U_{4,0} - U_{5,0}\right] + 0.2\lambda[U_{5,0} - 2U_{4,0} + U_{3,0}]$$

$$i = \ 5 : U_{5,1} = U_{5,0} + 0.02U_{5,0}\left[U_{5,0} - U_{6,0}\right] + 0.2\lambda[U_{6,0} - 2U_{5,0} + U_{4,0}]$$

$$i = \ 6 : U_{6,1} = U_{6,0} + 0.02U_{6,0}\left[U_{6,0} - U_{7,0}\right] + 0.2\lambda[U_{7,0} - 2U_{6,0} + U_{5,0}]$$

$$i = \ 7 : U_{7,1} = U_{7,0} + 0.02U_{7,0}\left[U_{7,0} - U_{8,0}\right] + 0.2\lambda[U_{8,0} - 2U_{7,0} + U_{6,0}]$$

$$i = \ 8 : U_{8,1} = U_{8,0} + 0.02U_{8,0}\left[U_{8,0} - U_{9,0}\right] + 0.2\lambda[U_{9,0} - 2U_{8,0} + U_{7,0}]$$

$$i = \ 9 : U_{9,1} = U_{9,0} + 0.02U_{9,0}\left[U_{9,0} - U_{10,0}\right] + 0.2\lambda[U_{10,0} - 2U_{9,0} + U_{8,0}]$$

Similarly set j=1 and vary i=1,2………………………………….N-1.

j=1

$$i = \ 1 : U_{1,2} = U_{1,1} + 0.02U_{1,1}\left[U_{1,1} - U_{2,1}\right] + 0.2\lambda[U_{2,1} - 2U_{1,1} + U_{0,1}]$$

$$i = \ 2 : U_{2,2} = U_{2,1} + 0.02U_{2,1}\left[U_{2,1} - U_{3,1}\right] + 0.2\lambda[U_{3,1} - 2U_{2,1} + U_{1,1}]$$

$$i = \ 3 : U_{3,2} = U_{3,1} + 0.02U_{3,1}\left[U_{3,1} - U_{4,1}\right] + 0.2\lambda[U_{4,1} - 2U_{3,1} + U_{2,1}]$$

$$i = \ 4 : U_{4,2} = U_{4,1} + 0.02U_{4,1}\left[U_{4,1} - U_{5,1}\right] + 0.2\lambda[U_{5,1} - 2U_{4,1} + U_{3,1}]$$

$$i = \ 5 : U_{5,2} = U_{5,1} + 0.02U_{5,1}\left[U_{5,1} - U_{6,1}\right] + 0.2\lambda[U_{6,1} - 2U_{5,1} + U_{4,1}]$$

$i = 6 : U_{6,2} = U_{6,1} + 0.02U_{6,1}[U_{6,1} - U_{7,1}] + 0.2\lambda[U_{7,1} - 2U_{6,1} + U_{5,1}]$

$i = 7 : U_{7,2} = U_{7,1} + 0.02U_{7,1}[U_{7,1} - U_{8,1}] + 0.2\lambda[U_{8,1} - 2U_{7,1} + U_{6,1}]$

$i = 8 : U_{8,2} = U_{8,1} + 0.02U_{8,1}[U_{8,1} - U_{9,1}] + 0.2\lambda[U_{9,1} - 2U_{8,1} + U_{7,1}]$

$i = 9 : U_{9,2} = U_{9,1} + 0.02U_{9,1}[U_{9,1} - U_{10,1}] + 0.2\lambda[U_{10,1} - 2U_{9,1,} + U_{8,1}]$

Also set j=2 and vary i=1,2……………………………………..N-1.

j=2

$i = 1 : U_{1,3} = U_{1,2} + 0.02U_{1,2}[U_{1,2} - U_{2,2}] + 0.2\lambda[U_{2,2} - 2U_{1,2} + U_{0,2}]$

$i = 2 : U_{2,3} = U_{2,2} + 0.02U_{2,2}[U_{2,2} - U_{3,2}] + 0.2\lambda[U_{3,2} - 2U_{2,2} + U_{1,2}]$

$i = 3 : U_{3,3} = U_{3,2} + 0.02U_{3,2}[U_{3,2} - U_{4,2}] + 0.2\lambda[U_{4,2} - 2U_{3,2} + U_{2,2}]$

$i = 4 : U_{4,3} = U_{4,2} + 0.02U_{4,2}[U_{4,2} - U_{5,2}] + 0.2\lambda[U_{5,2} - 2U_{4,2} + U_{3,2}]$

$i = 5 : U_{5,3} = U_{5,2} + 0.02U_{5,2}[U_{5,2} - U_{6,2}] + 0.2\lambda[U_{6,2} - 2U_{5,2} + U_{4,2}]$

$i = 6 : U_{6,3} = U_{6,2} + 0.02U_{6,2}[U_{6,2} - U_{7,2}] + 0.2\lambda[U_{7,2} - 2U_{6,2} + U_{5,2}]$

$i = 7 : U_{7,3} = U_{72} + 0.02U_{7,2}[U_{7,2} - U_{8,2}] + 0.2\lambda[U_{8,2} - 2U_{7,2} + U_{6,2}]$

$i = 8 : U_{8,3} = U_{8,2} + 0.02U_{8,2}[U_{8,2} - U_{9,2}] + 0.2\lambda[U_{9,2} - 2U_{8,2} + U_{7,2}]$

$i = 9 : U_{9,3} = U_{9,2} + 0.02U_{9,2}[U_{9,2} - U_{10,2}] + 0.2\lambda[U_{10,2} - 2U_{9,2} + U_{8,2}]$

Substituting the known values into the above equations for $\lambda = 0.1$ *and* $j = 0,1,2;$

We obtain the following tabulated values.

Table 1: Schmidt solution for the Burgers equation for λ=0.1

	x										
t	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	0.0	0.3090	0.5878	0.8090	0.9511	1.0000	0.9511	0.8090	0.5878	0.3090	0.0
0.002	0.0	0.3066	0.5840	0.8051	0.9483	0.9990	0.9519	0.8110	0.5899	0.3103	0.0
0.004	0.0	0.3043	0.5803	0.8012	0.9455	0.9980	0.9527	0.8130	0.5920	0.2902	0.0
0.006	0.0	0.3022	0.5766	0.7974	0.9427	0.9969	0.9535	0.8150	0.5940	0.2921	0.0

Similarly, for λ= 0.5 and j=0, 1, 2 the table below is computed.

Table 2: Schmidt solution for the Burgers equation for λ=0.5

	x										
t	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	0.0	0.3090	0.5878	0.8090	0.9511	1.0000	0.9511	0.8090	0.5878	0.3090	0.0
0.002	0.0	0.3043	0.5794	0.7988	0.9408	0.9912	0.9445	0.7968	0.5853	0.3079	0.0
0.004	0.0	0.2997	0.5713	0.7888	0.9307	0.9824	0.9372	0.7938	0.5820	0.3067	0.0
0.006	0.0	0.2953	0.5634	0.7790	0.9207	0.9736	0.9301	0.7903	0.5789	0.3054	0.0

Also for λ = 2.0 and j varied from 0,1,2 the table below is computed.

Table 3: Schmidt solution for the Burgers equation for λ= 2.0

	x										
t	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	0.0	0.3090	0.5878	0.8090	0.9511	1.0000	0.9511	0.8090	0.5878	0.3090	0.0
0.002	0.0	0.2952	0.5622	0.7751	0.9129	0.9619	0.9165	0.7809	0.5680	0.2988	0.0
0.004	0.0	0.2823	0.5382	0.7429	0.8765	0.9250	0.8829	0.7533	0.5485	0.2887	0.0
0.006	0.0	0.2703	0.5155	0.7125	0.8416	0.8895	0.8502	0.7263	0.5294	0.2788	0.0

Similarly for $\lambda = 3.0$ and j varied from 0, 1, 2 the table below is computed.

Table 4: Schmidt solution for the Burgers equation for λ= 3.0

	x										
t	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	0.0	0.3090	0.5878	0.8090	0.9511	1.0000	0.9511	0.8090	0.5878	0.3090	0.0
0.002	0.0	0.2892	0.5506	0.7592	0.8942	0.9423	0.8979	0.7651	0.5565	0.2928	0.0

0.004	0.0	0.2710	0.5166	0.7130	0.8415	0.8876	0.8472	0.7228	0.5264	0.2770	0.0
0.006	0.0	0.2544	0.4850	0.6703	0.7913	0.8362	0.7989	0.6824	0.4972	0.2620	0.0

For $\lambda = 10.0$ and j varied from 0, 1, 2 the table below is computed.

Table 5: Schmidt solution for the Burgers equation for λ= 10.0

	x										
t	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	0.0	0.3090	0.5878	0.8090	0.9511	1.0000	0.9511	0.8090	0.5878	0.3090	0.0
0.002	0.0	0.2469	0.4700	0.6485	0.7638	0.8054	0.7694	0.6544	0.4759	0.2505	0.0
0.004	0.0	0.1982	0.3791	0.5206	0.6158	0.6508	0.6131	0.5297	0.3842	0.2016	0.0
0.006	0.0	0.1629	0.2992	0.4270	0.4950	0.5059	0.4927	0.4070	0.3114	0.1644	0.0

From tables 1 to 5 we note that the solutions to the Burgers equation decrease as we move from time level t=0 to t= 0.006

4. Conclusion and Recommendations

The Schmidt method was accurate and could solve the Burgers equation for $\lambda \in (0, \frac{10}{3}]$. The method discussed was unconditionally stable with respect to mesh-ratios. When λ= 10.0 the method gave inconsistent solutions as we would expect since this is a region outside the stability range. We recommend other numerical methods to be explored in solving the Burgers equation especially outside the stability range.

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