DIRECT STEADY STATE THERMOELASTIC PROBLEM OF A THIN ANNULAR DISC

Mrs. Shinde Anjali K.*[*]

ABSTRACT :

In this paper, an attempt has been made to study thermoelastic response of a thin annular disc occupying the space D: $a \le r \le b$, $-h \le z \le h$, with the stated boundary conditions. The temperature, displacement and stress functions of the disc are determined by using the finite Hankel transform techniques.

Keywords: Annular disc, steady-state problem, direct thermoelastic problem, Hankel Transform.

[*] Assistant professor, Dept. of Mathematics, K.T.H.M. College, Nashik, 422002, Maharashtra.

1. Introduction:

During recent years, the theory of thermo elasticity has found considerable applications in the solutions of engineering problems. The thermo elastic behaviour of an annular disc constituting foundations of containers for hard gases or liquids, in the foundations for furnaces, in applications involving turbine motors, flywheels, gears etc. is increasingly important.

In this paper, an attempt has been made to study the direct steady state thermoelastic problem to determine the temperature, displacement and stress functions of a thin annular disc of thickness 2h. The homogeneous boundary conditions of the third kind are maintained on the lower plane surface, while upper plane surface is maintained at f (r), which is known function of r . The finite Hankel transform technique is used to find the solution of the problem.

2. The Transformation and its Essential Property

If f(x) satisfies Dirichlet's conditions in the range $b \le x \le a$, and if its finite Hankel transform in that range is defined to be

$$H[f(x)] = \bar{f}_{\mu}(\xi_i) =$$

$$\int_a^b x f(x)[J_{\mu}(x\xi_i)G_{\mu}(a\xi_i) - J_{\mu}(a\xi_i)G_{\mu}(x\xi_i)]dx \qquad (2.1)$$

Where , J_μ is Bessel function of order μ of first kind, G_μ is Bessel function of order μ of second kind, and ξ_i is a root of the transcendental equation,

$$J_{\mu}(\xi_i b)\, G_{\mu}(\xi_i a) - J_{\mu}(\xi_i a)G_{\mu}(\xi_i b) = 0 \qquad (2.2)$$

Then at each point of the interval (b, a) at which the function f(x) is continuous,

$$f(x) = \sum_i \frac{2\xi_i^2 J_{\mu}^2(\xi_i b)\bar{f}_{\mu}(\xi_i)}{J_{\mu}^2(a\xi_i) - J_{\mu}^2(b\xi_i)} \times [J_{\mu}(x\xi_i)\, G_{\mu}(a\xi_i) - J_{\mu}(a\xi_i)G_{\mu}(x\xi_i)] \qquad (2.3)$$

the summation extending over all the positive roots of (2.2).

Property of Hankel transform :

$$\int_a^b \left[\frac{\partial^2 f}{\partial x^2} + \frac{1}{x}\frac{\partial f}{\partial x}\right][J_{\mu}(x\xi_i)\, G_{\mu}(a\xi_i) - J_{\mu}(a\xi_i)G_{\mu}(x\xi_i)]dx$$

$$= -\xi_i^2 \bar{f}_\mu(\xi_i) + a[J_\mu(x\xi_i)G_\mu(a\xi_i) - J_\mu(a\xi_i)G_\mu(x\xi_i)]_{x=a}$$
$$+ b[J_\mu(x\xi_i)G_\mu(a\xi_i) - J_\mu(a\xi_i)G_\mu(x\xi_i)]_{x=b}$$
$$= -\xi_i^2 \bar{f}_\mu(\xi_i)$$

3. Formulation of the Problem: Governing Equation

Consider a thin annular disc of thickness 2h occupying the space D: $a \le r \le b$, $-h \le z \le h$, the material being Homogeneous and isotropic.

The differential equation governing the displacement function U(r, z) as in Nowacki W. [4] is,

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} = (1+\nu)a_t T \qquad (3.1)$$

With $U_r = 0$ at r = a and r = b. (3.2)

ν and a_t are the Poisson's ratio and the linear coefficient of thermal expansion of the material of disc respectively and T is the temperature of the disc satisfying the differential equation

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = 0 \qquad (3.3)$$

Subject to the boundary conditions

$$\frac{\partial T}{\partial r} = g(r) \qquad at \quad r = a, \quad -h \le z \le h \qquad (3.4)$$

$$\left[T(r,z) + k_1 \frac{\partial T(r,z)}{\partial z}\right]_{z=h} = f(r) \qquad (3.5)$$

$$\left[T(r,z) + k_2 \frac{\partial T(r,z)}{\partial z}\right]_{z=-h} = 0 \qquad (3.6)$$

The stress function σ_{rr} and $\sigma_{\theta\theta}$ are given by

$$\sigma_{rr} = -2\mu \frac{1}{r}\frac{\partial U}{\partial r} \qquad (3.7)$$

$$\sigma_{\theta\theta} = -2\mu \frac{\partial^2 U}{\partial r^2} \qquad (3.8)$$

Where μ is the Lame's constant , while each of the stress functions $\sigma_{rz}, \sigma_{zz}, \sigma_{\theta z}$ are zero within the disc in the plane state of stress.

The equations (3.1) to (3.8) constitute the mathematical formulation of the problem under consideration.

4. Solution of the Problem

4.1 : Determination of the Temperature T (r, z):

Applying finite Hankel transform stated in [7] to (3.3) to (3.6), ,one obtains

$$\frac{d^2T^*}{dz^2} - \mu_n^2 T^* = 0 \tag{4.1}$$

$$\left[T^*(\mu_n, z) + k_1 \frac{dT^*(\mu_n, z)}{dz}\right]_{z=h} = f^*(\mu_n) \tag{4.2}$$

$$\left[T^*(\mu_n, z) + k_2 \frac{dT^*(\mu_n, z)}{dz}\right]_{z=-h} = 0 \tag{4.3}$$

Where T^* denotes the finite Hankel transform of T and μ_n is the Hankel Transform parameter.

The equation (4.1) is a second order differential equation whose solution is given by

$$T^*(\mu_n, z) = A\cosh(\mu_n z) + B\sinh(\mu_n z) \tag{4.4}$$

Where A and B are constants.

Using (4.2) and (4.3) in (4.4) , we obtain the values of A and B.

Substituting these values of A and B in (4.4),

$$T^*(\mu_n, z) = f^*(\mu_n)\left[\frac{[\sinh(\mu_n(z+h)) - k_2\mu_n\cosh(\mu_n(z+h))]}{[(1-k_1k_2\mu_n^{\,2})\sinh(2\mu_n h) + \mu_n(k_1-k_2)\cosh(2\mu_n h)]}\right] \tag{4.5}$$

and then inversion of finite Hankel transform lead to

$$T(r,z) = \sum_{n=1}^{\infty}\frac{2\mu_n^2 J_0^2(\mu_n a)}{J_0^2(b\mu_n) - J_0^2(a\mu_n)} \times f^*(\mu_n)\left[\frac{[\sinh(\mu_n(z+h)) - k_2\mu_n\cosh(\mu_n(z+h))]}{[(1-k_1k_2\mu_n^{\,2})\sinh(2\mu_n h) + \mu_n(k_1-k_2)\cosh(2\mu_n h)]}\right] \times \left[J_0(r\mu_n)G_0(b\mu_n) - J_0(b\mu_n)G_0(r\mu_n)\right] \tag{4.6}$$

And g (r) =

$$-\sum_{n=1}^{\infty}\frac{2\mu_n^3 J_0^2(\mu_n a)}{J_0^2(b\mu_n) - J_0^2(a\mu_n)} \times f^*(\mu_n)\left[\frac{[\sinh(\mu_n(z+h)) - k_2\mu_n\cosh(\mu_n(z+h))]}{[(1-k_1k_2\mu_n^{\,2})\sinh(2\mu_n h) + \mu_n(k_1-k_2)\cosh(2\mu_n h)]}\right]$$

$$\times \left[J_1(a\mu_n)G_0(b\mu_n) - J_0(b\mu_n)G_1(a\mu_n)\right] \qquad (4.7)$$

Where ,

$$f^*(\mu_n) = \int_a^b r f(r)\left[J_0(r\mu_n)G_0(b\mu_n) - J_0(b\mu_n)G_0(r\mu_n)\right]dr$$

And μ_n is a root of the transcendental equation,

$$\left[J_0(\mu_n a)G_0(\mu_n b) - J_0(\mu_n b)G_0(\mu_n a)\right] = 0$$

Equations (4.6) and (4.7) are the desired solution of the given problem.

4.2. Determination of Displacement Function

Substituting this value of T(r , z) from (4.6) in (3.1), one obtains the thermoelastic displacement function U(r , z) as

$$U(r,z) = -(1+\nu)a_t \sum_{n=1}^{\infty} \frac{2\mu_n^2 J_0^2(\mu_n a)}{J_0^2(b\mu_n) - J_0^2(a\mu_n)} \times f^*(\mu_n)\left[\frac{[\sinh(\mu_n(z+h)) - k_2\mu_n\cosh(\mu_n(z+h))]}{[(1-k_1k_2\mu_n{}^2)\sinh(2\mu_n h) + \mu_n(k_1-k_2)\cosh(2\mu_n h)]}\right]$$
$$\times \left[J_0(r\mu_n)G_0(b\mu_n) - J_0(b\mu_n)G_0(r\mu_n)\right] \qquad (4.8)$$

4.3. Determination of Stress Functions:

Using (4.8) in (3.7) and (3.8), the stress functions are obtained as

$$\sigma_{rr} =$$
$$-2\mu\frac{1}{r}(1+\nu)a_t \sum_{n=1}^{\infty} \frac{2\mu_n^3 J_0^2(\mu_n a)}{J_0^2(b\mu_n) - J_0^2(a\mu_n)} \times f^*(\mu_n)\left[\frac{[\sinh(\mu_n(z+h)) - k_2\mu_n\cosh(\mu_n(z+h))]}{[(1-k_1k_2\mu_n{}^2)\sinh(2\mu_n h) + \mu_n(k_1-k_2)\cosh(2\mu_n h)]}\right]$$
$$\times \left[J_1(r\mu_n)G_0(b\mu_n) - J_0(b\mu_n)G_1(r\mu_n)\right] \qquad (4.9)$$

$$\sigma_{\theta\theta} = -2\mu(1+\nu)a_t \sum_{n=1}^{\infty} \frac{2\mu_n^4 J_0^2(\mu_n a)}{J_0^2(b\mu_n) - J_0^2(a\mu_n)} \times f^*(\mu_n)\left[\frac{[\sinh(\mu_n(z+h)) - k_2\mu_n\cosh(\mu_n(z+h))]}{[(1-k_1k_2\mu_n{}^2)\sinh(2\mu_n h) + \mu_n(k_1-k_2)\cosh(2\mu_n h)]}\right]$$
$$\times \left[J_1'(r\mu_n)G_0(b\mu_n) - J_0(b\mu_n)G_1'(r\mu_n)\right] \qquad (4.10)$$

5. Conclusion:

In this paper, the direct steady state problem of thermoelastic deformation of a thin annular disc of thickness 2h, with stated boundary conditions is discussed. The finite Hankel transform technique is used. The temperature, displacement and thermal stresses that are obtained can be applied to the design of useful structures or machines in many engineering applications.

6. References:

[1] Kailash and N.W.Khobragade, "An inverse transient thermoelastic problem of thin annular disc", Applied Mathematics E-Note,6 ,pp17-25,2006

[2] Khobragade N. W. and Wankhede P. C.,: An inverse steady-state thermoelastic problem of thin annular disc, Bull. of pure and appl.sci, vol. 19E(No. 1), pp. 551-555 (2001).

[3] N. M. Ozisik, "Boundary Value Problem of Heat Conduction," International Textbook company Scranton, Pennsylvania, 1968.

[4] Nowaki W.,: The state of stress in a thick circular plate due to temperature field, Bull. Acad. Polo. Sci. Ser. Tech. 5, pp. 227(1957).

[5] N.W.Khobragade and R.T.Walde, "Thermal deflection of a clamped annular disc due to heat generation", Int. J. Latest Trend Math ,Vol. 2,No1, ,pp37-44, March 2012

[6] Payal Hiranwar and N.W Khobragade,"Thermoelastic problem of thin annular disc due to radiation", Int. J. of Pure and Applied mathematics , Vol. 71,No3,2011,pp403-414

[7] Sneddon, I. N.,: The use of integral transforms, McGraw-Hill Book Co., chap. 3, pp. 174-180 (1974)