NEW METHOD TO CALCULATE THE MATRIX EXPONENTIAL

Mohammed Abdullah Salman[1,2]

V. C. Borkar[1]

Abstract

Matrix exponential is widely used in science area especially in matrix analysis. We pay particular attention to the matrix exponential. The matrix exponential is a very important subclass of control theory. In control theory it is needed to evaluate matrix exponential. In classical methods we calculate the eigenvalues of the matrix, but that the problem can be complicated if the eigenvalues are not easy to calculate. In this paper we use same methods and same procedure, but the eigenvalues of A are not needed for the construction of e^{tA}, since most of our results use only the coefficients of the polynomial w, we explain some examples how the procedure works in the method of Dr. Luis Verde Star, in his article, where it develops, he gave in his article only theory without any applied. Finally, we developed the method for evaluate the characteristic polynomial.

Keywords: control theory, exponential of a matrix, exponential of a function, eigenvalues.

[2] Department of Mathematics & Statistics, Yeshwant Mahavidyalaya, Swami Ramamnand Teerth Marthwada University, Nanded, India.

[1] Department of Mathematics, College of Education & languages, University of Amran, Amran, Yemen.

1- Introduction:

In some problems of control theory and other areas of knowing is necessary to resolve the equation of state in continuous time (1) and (2)

$$\dot{x} = Ax + bu \qquad (1)$$

Where x is a vector (n × 1), u is the entry vector (r × 1), A is a constant matrix of (n × n) and B is a constant matrix of (n × r). If equation (1) is written in the form

$$\dot{x} - Ax = bu \qquad (2)$$

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and multiply (2) by e^{tA} becomes

$$\frac{d(e^{-At}x)}{dt} = e^{-At}Bu \qquad (3)$$

By integrating equation (3) between 0 and t gives us:

$$x = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \qquad (4)$$

Therefore to determine x we need to compute e^{At}.

2- Some methods to compute exponential matrices:

There are several methods to compute the exponential matrix$A$. One of these methods is compute the exponential matrix$A$ through series:

$$e^{At} = \sum_{k=0}^{\infty}\frac{(At)^k}{k!} = I + At + \frac{(At)^2}{2!} + \ldots\ldots + \frac{(At)^{n-1}}{(n-1)!} + \ldots\ldots\ldots. \qquad (5)$$

However, calculating the powers of matrix A which is an infinite sum makes this method impractical. Another method we obtain the exponential matrix$A$ by using canonical Jordan form J of the matrix$A$, if P is non-singular matrix such that:

$$A = PJP^{-1} \tag{6}$$

then (3) becomes:

$$e^{At} = Pe^{Jt}P^{-1} \tag{7}$$

To compute e^{Jt} we can assume that J is the direct sum of blocks $A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ or we can write it as $J = \oplus \lambda_k$ where J is a Jordan block corresponding to the eigenvalues λ_k of A such that

$$J_k = J_k(\lambda_k) = \begin{pmatrix} \lambda_k & 1 & 0 & 0 \\ 0 & \ddots & 1 & 0 \\ 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & \lambda_k \end{pmatrix}$$

and evaluation of this matrix is made with the formula

$$f(J_k) = \begin{pmatrix} f(\lambda_k) & f'(\lambda_k) & \cdots & \dfrac{f^{(m_k-1)}(\lambda_k)}{(m_k-1)!} \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & f'(\lambda_k) \\ 0 & 0 & 0 & f(\lambda_k) \end{pmatrix} \tag{8}$$

Where f the exponential is function and m_k is the block of Jordan with size j_k. Then (4)

$$e^{Jt} = \oplus e^{J_k t} \tag{9}$$

The problem with this method is that the Jordan canonical form is not easy to calculate, because we need the eigenvalues of matrix A. In addition we calculate the P matrix and its inverse.

Example (1):

How we apply the procedure above for a matrix A where

$$A = \begin{pmatrix} -1 & 1 & 1 \\ -3 & 3 & 1 \\ -4 & 3 & 2 \end{pmatrix}$$

Solution:

First we need to get the eigenvalues and eigenvectors we get

$$\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \text{And} \begin{pmatrix} \frac{3}{5} & 1 & 0 \\ \frac{4}{5} & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

The first column contains the eigenvalues vector, while the columns of the matrix are the eigenvectors. From this we can conclude that the eigenvalue2 has an eigenvector $(\frac{3}{5}, \frac{4}{5}, 1)$, i.e. the eigenvalue 2 has algebraic multiplicity 1 and geometric dimension 1. Therefore the canonical Jordan form of 2 has a block of form

$$J_1 = (2)$$

Furthermore, the eigenvalue 1 has algebraic multiplicity 2 and geometric dimension 1, so for this eigenvalue there is a single block of Jordan is

$$J_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

now we will go to the next step is to obtain the dimension of space

$$\ker(A-2I)^k, k = 1,2,\ldots \ \ and \ \ker(A-1I)^k, k = 1,2,\ldots.$$

until we get to the geometric dimensions of each eigenvalue. We start with $\ker(A-2I)^k$ when $k = 1$ we obtain the basis $\begin{pmatrix} \frac{3}{5} & \frac{4}{5} & 1 \end{pmatrix}$ already coincides with the geometric dimension eigenvalue 2. Then this vector will be the first column of the matrix P.

now we do the same with $\ker(A-1I)^k$ when $k = 2$ to obtain the basis for $\ker(A-1)^2$ which is $(0\ 0\ 1),(1\ 1\ 0)$, which coincides with the geometric and dimensional eigenvalue 1. We take one of two vectors, example, $v = (0\ 0\ 1)$ and calculate $(A-1I)v$ to obtain w = (1 1 1). Then the last two columns of the matrix P formed by v and w then the matrix P is

$$P = \begin{pmatrix} \frac{3}{5} & 1 & 0 \\ \frac{4}{5} & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \text{ and the inverse is } P^{-1} = \begin{pmatrix} -5 & 5 & 0 \\ 4 & -3 & 0 \\ 1 & -2 & 1 \end{pmatrix}$$

Now we apply the formula PAP^{-1} to get the canonical Jordan form

$$J = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Now we apply equation (7)

$$e^{At} = P(e^{J_1 t \oplus J_2 t})P^{-1} \qquad (10)$$

Calculating the exponential matrix by using (8) as follow

$$e^{J_1 t} = \begin{pmatrix} e^{2t} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } e^{J_2 t} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & e^{t} & e^{t} \\ 0 & 0 & e^{t} \end{pmatrix}$$

then

$$e^{J_1 t \oplus J_2 t} = \begin{pmatrix} e^{2t} & 0 & 0 \\ 0 & e^{t} & e^{t} \\ 0 & 0 & e^{t} \end{pmatrix}$$

where the direct sum of the matrices is obtained by joining them. What we are doing now to compute the final step from e^{At} we use (10)

$$e^{At} = P(e^{J_1 t \oplus J_2 t})P^{-1} = \begin{pmatrix} -3e^{2t} + 5e^{t} & 3e^{2t} - 5e^{t} & e^{t} \\ -4e^{2t} + 5e^{t} & 4e^{2t} - 5e^{t} & e^{t} \\ -5e^{2t} + 6e^{t} & 5e^{2t} - 5e^{t} & e^{2t} \end{pmatrix}$$

Through the example we can see that if the size of the matrix increases the calculation are more complicated for more details see (2, 3, 4, 5, 6, 7,8).

3-Luis Verde's Method and main results:

Luis Verde's method for more details see(1)

3.1- Horner polynomials.

In this section we look how to write the powers $1, z, z^2, ...$ in terms of Horner polynomials, which we define as:

Definition (1): let

$$w(z) = z^{n+1} + b_1 z^n + ... + b_{n+1} \quad (11)$$

the characteristic polynomial of the matrix $A \in (n+1)(n+1)$. We define the sequence $\{w_k\}_{k=0}^{\infty}$ Horner polynomials associated with w as follows:

$$w_k(z) = z^k + b_1 z^{k-1} + ... + b_k \; , k \geq 0 \quad (12)$$

where $b_o = 1$ and $b_j = 0$ for $j > n+1$.

it is clear that $w_{n+1}(z) = w(z)$ and $\mathrm{w}_{n+1+k}(z) = z^k w(z)$, because:

$$\begin{aligned} w_{n+1}(z) &= z^{n+1+k} + b_1 z^{n+k} + ... + b_{n+k+1} \\ &= z^{n+1+k} + b_1 z^{n+k} + ... + b_{n+1} z^k \\ &= z^k (z^{n+1} + b_1 z^n + ... + b_{n+1}) = z^k w(k) \end{aligned}$$

Moreover $\{w_k\}_{k=0}^{n}$ is a basis of subspace of all polynomials of degree at most n.

Also these polynomials have the property:

$$w_{k+1}(z) = z w_k(z) + \mathrm{b}_{k+1} \; , \mathrm{k} \geq 0 \quad (13)$$

because:

$$\begin{aligned} z w_k(z) + b_{k+1} &= z(z^k + b_1 z^{k-1} + ... + b_k) + b_{k+1} \\ &= z^{k+1} + b_1 z^k + ... + b_k z + b_{k+1} = w_{k+1}(z) \end{aligned}$$

Definition (2): We define w^* reverse w as:

$$w^*(t) = 1 + b_1 t + ... + b_{n+1} t^{n+1}.$$

Theorem (1): let

$$w^*(t) = (1-zt)\sum_{k=0}^{\infty} w_k(z)t^k.$$

Proof:

The proof is done by using the definition of formula 11 and 13 as follows:

$$(1-zt)\sum_{k=0}^{\infty} w_k(z)t^k = \sum_{k=0}^{\infty} w_k(z)t^k - \sum_{k=0}^{\infty} zw_k(z)t^{k+1}$$

By removing the first term in the first sum we obtain:

$$1+\sum_{k=1}^{\infty} w_k(z)t^k - \sum_{k=0}^{\infty} zw_k(z)t^{k+1}$$

Now by making the first sum starts at zero we obtain:

$$1+\sum_{k=0}^{\infty} w_{k+1}(z)t^{k+1} - \sum_{k=0}^{\infty} zw_k(z)t^{k+1} = 1+\sum_{k=0}^{\infty}[w_{k+1}(z) - zw_k(z)]t^{k+1}$$

$$= 1+\sum_{k=0}^{\infty} b_{k+1}t^{k+1} = 1+\sum_{k=0}^{n} b_{k+1}t^{k+1} = w^*(t),$$

because $b_j = 0$ for $j > n+1$.

Definition (3): let us to define the series

$$h(t) = 1 + h_1 t + h_2 t + ... \qquad (14)$$

as the power series satisfying $h(t)w^*(t) = 1$.

Now we make a remark about the region of convergence of the series (14). As

$$w^*(t) = 1 + b_1 t + b_2 t + ... + b_{n+1}t^{n+1} = t^{n+1}\left[\frac{1}{t^{n+1}} + \frac{b_1}{t^n} + ... + b_{n+1}\right] = t^{n+1} w\left(\frac{1}{t}\right)$$

then the roots of w^* are the reciprocals of the roots of w and hence the series(14) converges for

$$|t| < \rho \qquad (15)$$

where $\rho = \frac{1}{M}$ and M It is the maximum modules of roots w.

Theorem (2):

$$z^l = w_l(z) + h_1 w_{l-1}(z) + h_2 w_{l-2}(z) + ... + h_l, l \geq 0.$$

Proof:

The proof is done by using the formula in Theorem (1) by writing:

$$\frac{1}{1-zt} = h(t)\sum_{k=0}^{\infty} w_k(z)t^k = \sum_{k=0}^{\infty} h_k t^k \sum_{k=0}^{\infty} w_k(z)t^k$$

using the product of series and using increase series we obtain the geometric series:

$$1 + zt + z^2t^2 + ... = h_o w_o + (h_o w_1 + h_1 w_o)t + (h_o w_2 + h_1 w_1 + h_2 w_o)t^2 + ...$$

Comparing the coefficients of t^l on both sides we obtain the formula of Theorem.

3.2- Functions of Matrices:

We will use what we did in the previous section to define functions of matrices.

Theorem (3): let

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

a power series with positive radius of convergence R. Let A be a matrix of $(n+1)\times(n+1)$ and $w(z)$ its characteristic polynomial of degree $n+1$ where $w(A) = 0$. Then

$$f(tA) = \sum_{k=0}^{n} g_k(t) w_k(A) \tag{16}$$

where

$$g_k(t) = \sum_{i=k}^{\infty} a_i h_{i-k} t^i \ , \ k \geq 0 \tag{17}$$

Proof:

We started writing the series of powers such as:

$$f(zt) = a_o + a_1 zt + a_2 z^2 t^2 + ... \tag{18}$$

Now we substitute the powers of z with the formula of Theorem (2) to obtain:

$$\begin{aligned}
&a_o(h_o w_o)t^o + \\
&a_1(h_1 w_o + h_o w_1)t^1 + \\
&a_2(h_2 w_o + h_1 w_1 + h_o w_2)t^2 + \\
&a_3(h_3 w_o + h_2 w_1 + h_1 w_2 + h_o w_3)t^3 + \\
&\vdots \qquad \vdots \qquad \vdots \qquad \vdots
\end{aligned}$$

By adding columns and collecting the terms containing w_o, and the terms containing w, etc., we get:

$$\sum_{i=0}^{\infty} a_i h_i t^i w_o + \sum_{i=1}^{\infty} a_i h_{i-1} t^i w_1 + \sum_{i=2}^{\infty} a_i h_{i-2} t^i w_2 + \dots\dots\dots$$

Finally, this sum can be written as:

$$f(tz) = \sum_{k=0}^{m} \left(\sum_{i=k}^{\infty} a_i h_{i-k} t^i \right) w_k(z) \qquad (19)$$

as $w_k(A) = 0$ for $k > n$, replacing z by A in equation (19) we get:

$$f(tA) = \sum_{k=0}^{m} \left(\sum_{i=k}^{\infty} a_i h_{i-k} t^i \right) w_k(A)$$

to apply (16) and (17) we need to say how to compute the characteristic polynomial $w(z)$ and the coefficients of the series $h(t)$.

3.3- The characteristic polynomial.

The characteristic polynomial of a matrix $A \in (n+1) \times (n+1)$ it is defined as:

$$w(z) = \det(zI - A) \qquad (20)$$

we obtain this polynomial from (20), It is very difficult in some times to compute the determinant. Instead of using the formula (20) will use Cayley Hamilton theorem to compute the coefficients of the characteristic polynomial. We know that if

$$w(z) = z^{n+1} + b_1 z^n + \dots\dots + b_{n+1}$$

then,

$$w(A) = A^{n+1} + b_1 A^n + \dots\dots + b_{n+1} I = 0 \qquad (21)$$

equation (21) expresses as two matrices (6) of size $(n+1)\times(n+1)$, and where $b_1, b_2, \dots\dots\dots, b_{n+1}$ are unknowns to be determined. To find the unknowns take the first column from (21) to form a system of $(n+1)$ linear equations with $(n+1)$ unknowns:

$$b_1 (A^n)^{(1)} + \dots\dots\dots + b_{n+1} (I)^{(1)} = (-A^{n+1})^{(1)} \qquad (22)$$

where the superscript indicates the one of the column matrix. Note that for the first columns can use the following procedure:

$$\begin{aligned} &(I)^{(1)}, \\ &(A)^{(1)} = A(I)^{(1)} \\ &(A^2)^{(1)} = A(A)^{(1)} \\ &(A^3)^{(1)} = A(A^2)^{(1)} \\ &(A^4)^{(1)} = A(A^3)^{(1)} \\ &\vdots \\ &\vdots \\ &(A^{n+1})^{(1)} = A(A^n)^{(1)} \end{aligned}$$

After finding the values of the unknowns we check if they satisfy the equation (21). If this does not happen we take another column or another line.

3.4- Calculating coefficients of the series h(t):

As we know the series $h(t)$ is defined as:

$$h(t) w^*(t) = 1$$

or

$$\sum_{j=0}^{n+1} b_j t^j \sum_{j=0}^{\infty} h_j t^j = 1$$

by using the product of series we get:

$$\sum_{j=0}^{k} b_j h_{k-j} = \delta_{o,k}, \quad k \geq 0$$

where $b_j = 0$ for $j > n+1$.

then from above equation we get :

$$h_k = -\sum_{j=1}^{k} b_j h_{k-j} \qquad (22)$$

note that $b_j = 0$ for $j > n+1$.

Example (2):

With all that we are have done everything, now we give example to show the procedure for compute exp (At), our example is to calculate the exponential, which use the matrix of Example (1).

Solution: The number of digits which we do operations is 20. We first calculate the characteristic polynomial of the formula (22) which generates the system of equations:

$$-6b_1 - b_2 + b_3 = 17$$
$$-10b_1 - 3b_2 = 25$$
$$-13b_1 - 4b_2 = 32$$

and the solution of this system are:

$$b_1 = -4, b_2 = 5 \text{ and } b_3 = -2$$

Then the characteristic polynomial is $w(z) = z^3 - 4z^2 + 5z - 2.$ with this polynomial we construct Horner polynomials by formula (12) so that we get

$$w_, = 1, \quad w_1 = z - 4, \text{ and } w_1 = z^2 - 4z + 5$$

Now the formula (16) gives:

$$e^{At} = g_o(t)w_o(A) + g_1(t)w_1(A) + g_2(t)w_2(A) \qquad (23)$$

now we need to calculate the functions g_k by using the formula(17), for this purpose we need the coefficients of the Series $h(t)$ which are calculated by using the formula (22). then we get :

$$g_o(t) \cong 1 - 4t - .... + 0.000041425t^{20}$$

$$g_1(t) \cong t - 2t^2 + ... - 0.0000081786t^{20}$$

$$g_2(t) \cong 0.5t^2 - 0.666666668t^3 + ... + 0.00000161409t^{20}$$

The function $g_2(t)$ is called dynamic solution and the following properties are satisfied:

$$g_2'(t) = g_1(t) \qquad\qquad g_1'(t) = g_o(t)$$

That is, from the function $g_2(t)$ we can calculate the functions $g_1(t)$ and $g_o(t)$.

For example, if t = 1, we obtain for (23)

$$B = \begin{pmatrix} -8.5757591544965891037 & 8.5757591544965891070 & 2.7182818284590452347 \\ -15.964815253427194201 & 15.964815253427194203 & 2.7182818284590452347 \\ -20.635589523898754060 & 17.917307695439708825 & 5.436563656918090472 \end{pmatrix}$$

While the formula give us from example (1)

$$C = \begin{pmatrix} -8.575759154496724505 & 8.575759154496724505 & 2.7182818284590452354 \\ -15.964815253427374732 & 15.964815253427374732 & 2.7182818284590452354 \\ -20.635589523898979724 & 17.917307695439934488 & 5.4365636569180904708 \end{pmatrix}$$

Finally, evaluate errors to compare the results obtained in Examples (1) and (2), considering that the result of Example (1) is correct, then

$$\|B - C\|_\infty = 4.513282 * 10^{(-13)}$$

4- Conclusions:

The first conclusion we get from the method that proposed it needs to calculate the eigenvalues of matrix A to compute e^{At}, as the method of canonical Jordan form. It is more complicated when we use a big matrix than in the method of Luis Verde that calculating the coefficients of series g_k also only we need to do multiplication and division of numbers that can be easily obtained to calculate the coefficients. In contrast the method of canonical Jordan form must calculate the eigenvalues, eigenvectors, the kernels of the matrices $(A - \lambda I)^k$ and the inverse matrix P. But also if the matrix is large, say size 20, then the calculations to canonical Jordan form are more complicated than the method of Luis Verde. Although we have applied the formula (16) for the exponential function, it is general and it can calculate matrices and other functions like $\ln(A)$ and others. In order to increase the accuracy we can cut the series g_k after t^{40} or t^{100} and then we can ask whether there will be any way to write the formula (16) such that taking a few terms in g_k good accuracy.

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