A REMARKABLE INTEGER SEQUENCE

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Abstract:

In this paper, we present a new integer sequence is developed from the recurrence relation $J_{n+2} = \beta J_{n+1} - \alpha J_n, \alpha \neq \beta, (\alpha, \beta \succ 0)$ with the initial conditions $J_0 = a, J_1 = b$ where a,b are not zeros simultaneously, is illustrated.

Keywords: Derived k-Fibonacci sequence and derived k-Lucas sequence, Binet's formula.

2010 Mathematics Subject Classification: 11B39, 11B83.

Introduction:

It is well known that the Fibonacci sequence is famous for its wonderful and amazing properties. Fibonacci composed a number text in which he did important work in number theory and the solution of algebraic equations. The equation of rabbit problem posed by Fibonacci is known as the first mathematical model for population growth. From the statement of rabbit problem, the famous Fibonacci numbers can be derived. This sequence of Fibonacci numbers is extremely fruitful and appears in different areas in mathematics and science.

The Fibonacci sequence, Lucas sequence, Pell sequence, Pell-Lucas sequence, Jacobsthal sequence and Jacobsthal –Lucas sequence are most prominent examples of recursive sequences.

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The Fibonacci sequence [7] is defined by the recurrence relation $F_k = F_{k-1} + F_{k-2}, k \geq 2$ with $F_0 = 0, F_1 = 1.$ The Lucas sequence [7] is defined by the recurrence relation $L_k = L_{k-1} + L_{k-2}, k \geq 2$ with $L_0 = 2, L_1 = 1.$

The second order recurrence sequence has been generalized in two ways mainly, first by preserving the initial conditions and second by preserving the recurrence relation. In this context, one may refer [10].

D. Kalman and R.Mena [6] generalized the Fibonacci sequence by $F_n = aF_{n-1} + bF_{n-2}, n \geq 2$ with $F_0 = 0, F_1 = 1.$

A. F. Horadam[5] defined generalized the Fibonacci sequence $\{H_n\}$ by $H_n = H_{n-1} + H_{n-2}, n \geq 3$ with $H_1 = p, H_2 = p + q$ where p and q are arbitrary integers.

B. Singh, O. Sikhwal and S. Bhatnagar [11], defined Fibonacci like sequence by recurrence relation $S_k = S_{k-1} + S_{k-2}, k \geq 3$ with $S_0 = 2, S_1 = 2.$ The associated initial conditions $S_0 and S_1$ are the sum of the Fibonacci and Lucas sequence respectively. i.e, $S_0 = F_0 + L_0 and S_1 = F_1 + L_1.$

L.R. Natividad [8], Deriving a formula in solving Fibonacci like sequence. He found missing terms in Fibonacci like sequence and solved by standard formula.

V.K. Gupta, V.Y. Panwar and O. Sikhwal [3], defined generalized Fibonacci sequences and derivd its identies connection formulae and other results. V.K. Gupta, V.Y. Panwar and N.Gupta [4], stated and derived identies for Fibonacci like sequence. Also, described and derived connection formulae and negation formualae for Fibonacci like sequence. B.Singh, V.K.Gupta and V.Y.Panwar [12], present many combination of higher powers of Fibonacci like sequence.

The k-Fibonacci numbers defined by Falco 'n' Plaza.A [1], depending only an one integer parameter k as follows, for any positive real number k, the Fibonacci sequence is defined recurrently by $F_{n,k} = kF_{k,n-1} + F_{k,n-2}, n \geq 2$ with $F_{k,0} = 0, F_{k,1} = 1$.

In [2], A.D. Godase and M.B. Dhakne have presented some properties of k-Fibonacci and k-Lucas numbers by using matrices.

In [9], Yashwant , K.Panwar, G.P. Rathore and Richa Chawla have established some interesting properties of k-Fibonacci like numbers.

In this communication, a new integer sequence is developed by defining the recurrence releation $J_{n+2} = \beta J_{n+1} - \alpha J_n, \alpha \neq \beta, (\alpha, \beta \succ 0)$ with the initial conditions $J_0 = a, J_1 = b$ where a,b are not zeros simultaneously. Various interesting relations among these numbers are exhibited.

Method of Analysis:

In this section a new integer sequence generated from the recurrence relation $J_{n+2} = \beta J_{n+1} - \alpha J_n, \alpha \neq \beta, (\alpha, \beta \succ 0)$ with the initial conditions $J_0 = a, J_1 = b$ where a,b are not zeros simultaneously, is illustrated

Consider a sequence $\{J_n\}$ defined by

$$J_{n+2} = \beta J_{n+1} - \alpha J_n, \alpha \neq \beta, (\alpha, \beta \succ 0) \quad (1)$$

with the initial conditions

$J_0 = a, J_1 = b$

where a,b are not zeros simultaneously.

The auxiliary equation associated with the recurrence relation (1) is given by

$$m^2 - \beta m - \alpha = 0$$

whose roots are $m_1 = \frac{\beta + \sqrt{\beta^2 + 4\alpha}}{2}, m_2 = \frac{\beta - \sqrt{\beta^2 + 4\alpha}}{2}$

Note that $m_1 + m_2 = \beta, m_1 m_2 = -\alpha$.

Thus, the general solution of (1) is

$$J_n = Am_1^n + Bm_2^n.$$

From the initial conditions, we infer that

$$A + B = a; Am_1 + Bm_2 = b$$

Solving for A and B, we get

$$A = \frac{b - am_2}{m_1 - m_2}; B = \frac{am_1 - b}{m_1 - m_2}.$$

Thus, a notable sequence $\{J_n\}$ whose terms are given below, is obtained.

$$J_n = \frac{(b - am_2)m_1^n + (am_1 - b)m_2^n}{m_1 - m_2} = Am_1^n + Bm_2^n (say) \quad (2)$$

where $A = \frac{(b - am_2)}{m_1 - m_2}$ and $B = \frac{(am_1 - b)}{m_1 - m_2}$ (3)

The new sequence $\{J_n\}$ is found to satisfy the following relations:

Identities:

(i). $6\left(\frac{aJ_{4k} - J_{2k}^2}{AB}\right)$ **is a Nasty Number.**

Proof:

$$J_{2k}{}^2 = (Am_1^{2k} + Bm_2^{2k})^2$$

$$= A^2 m_1^{4k} + B^2 m_2^{4k} + 2ABm_1^{2k} m_2^{2k}$$

$$= A(Am_1^{4k} + Bm_2^{4k} - Bm_2^{4k}) + B(Am_1^{4k} + Bm_2^{4k} - Am_1^{4k}) + 2ABm_1^{2k} m_2^{2k}$$

$$= A[J_{4k} - Bm_2^{4k}] + B[J_{4k} - Am_1^{4k}] + 2ABm_1^{2k} m_2^{2k}$$

$$= [A + B]J_{4k} - AB(m_1^{4k} + m_2^{4k} - 2m_1^{2k} m_2^{2k})$$

$$= [A + B]J_{4k} - AB[m_1^{2k} - m_2^{2k}]^2$$

$$= aJ_{4k} - AB[m_1^{2k} - m_2^{2k}]^2$$

Hence, $6\left(\frac{aJ_{4k} - J_{2k}^2}{AB}\right)$ **is a Nasty Number.**

(ii). $6\left(\frac{aJ_{4k} - J_{2k}^2}{AB} + 4\alpha^{2k}\right)$ **is a Nasty Number.**

Proof:

$$\frac{aJ_{4k}-J_{2k}^2}{AB}=[m_1^{2k}+m_2^{2k}]^2-4m_1^{2k}m_2^{2k}$$

$$\frac{aJ_{4k}-J_{2k}^2}{AB}+4\alpha^{2k}=[m_1^{2k}+m_2^{2k}]^2$$

Hence, $6\left(\frac{aJ_{4k}-J_{2k}^2}{AB}+4\alpha^{2k}\right)$ **is a Nasty Number.**

(iii). $J_{2k}=(\beta^{2s}-\sum_{i=1}^{2s-1}2s_{c_i}m_1^{2s-i}m_2^i)J_{2k-2s}-\alpha^{2s}J_{2k-4s},\forall k\geq 2s\succ 0.$

Proof:

$$J_{2k}=Am_1^{2k}+Bm_2^{2k}$$

$$=(Am_1^{2k-2s}+Bm_2^{2k-2s})(m_1^{2s}+m_2^{2s})-Am_1^{2k-2s}m_2^{2s}-Bm_2^{2k-2s}m_1^{2s}$$

$$=J_{2k-2s}(m_1^{2s}+m_2^{2s})-m_1^{2s}m_2^{2s}[Am_1^{2k-4s}+Bm_2^{2k-4s}](k\geq 2s)$$

$$=J_{2k-2s}(m_1^{2s}+m_2^{2s})-\alpha^{2s}J_{2k-4s} \qquad (4)$$

Since, $(m_1+m_2)^{2s}=m_1^{2s}+2s_{c_1}m_1^{2s-1}m_2+2s_{c_2}m_1^{2s-2}m_2^2+........+2s_{c_{2s-1}}m_1m_2^{2s-1}+m_2^{2s}$

$$\therefore m_1^{2s}+m_2^{2s}=\beta^{2s}-\sum_{i=1}^{2s-1}2s_{c_i}m_1^{2s-i}m_2^i$$

(5)

Using (5) in (4), we get

$$J_{2k}=(\beta^{2s}-\sum_{i=1}^{2s-1}2s_{c_i}m_1^{2s-i}m_2^i)J_{2k-2s}-\alpha^{2s}J_{2k-4s},\forall k\geq 2s\succ 0.$$

(iv). $J_kJ_{k+2s}=J_{k+s}^2+AB\alpha^k[\beta^{2s}-2\alpha^s-\sum_{i=1}^{2s-1}2s_{c_i}m_1^{2s-i}m_2^i]$

Proof:

$$J_kJ_{k+2s}=(Am_1^k+Bm_2^k)(Am_1^{k+2s}+Bm_2^{k+2s})$$

$$=A^2m_1^{2k+2s}+B^2m_2^{2k+2s}+AB(m_1^km_2^{k+2s}+m_2^km_1^{k+2s})$$

$$= (Am_1^{k+s} + Bm_2^{k+s})^2 - 2ABm_1^{k+s}m_2^{k+s} + AB(m_1^k m_2^{k+2s} + m_2^k m_1^{k+2s})$$

$$= J_{k+2}^2 + ABm_1^k m_2^k (m_1^{2s} + m_2^{2s} - 2m_1^s m_2^s) \qquad (6)$$

Using (5) in (6), we get

$$= J_{k+s}^2 + AB\alpha^k[\beta^{2s} - \sum_{i=1}^{2s-1} 2s_{c_i} m_1^{2s-i} m_2^i - 2\alpha^s]$$

Hence, $J_k J_{k+2s} = J_{k+s}^2 + AB\alpha^k[\beta^{2s} - 2\alpha^s - \sum_{i=1}^{2s-1} 2s_{c_i} m_1^{2s-i} m_2^i]$

(v). $6\left[\frac{J_k J_{k+2s} - J_{k+s}^2}{AB\alpha^k}\right]$ **is a Nasty Number.**

Proof:

From the identity (iv) we have

$$\frac{J_k J_{k+2s} - J_{k+s}^2}{AB\alpha^k} = (m_1^s - m_2^s)^2$$

$$\therefore 6\left[\frac{J_k J_{k+2s} - J_{k+s}^2}{AB\alpha^k}\right]$$ **is a Nasty Number.**

(vi). $6\left[\frac{J_k J_{k+2s} - J_{k+s}^2}{AB\alpha^k} + 4\alpha^s\right]$ **is a Nasty Number.**

Proof:

Since, $\frac{J_k J_{k+2s} - J_{k+s}^2}{AB\alpha^k} = (m_1^s - m_2^s)^2$

$$= (m_1^s + m_2^s)^2 - 4\alpha^s$$

$$\therefore 6\left[\frac{J_k J_{k+2s} - J_{k+s}^2}{AB\alpha^k} + 4\alpha^s\right]$$ **is a Nasty Number.**

(vii). $\frac{(J_{k+s}J_{k-s} - aJ_{2k})\alpha^s}{AB} = \alpha^k[\beta^{2s} - \sum_{i=1}^{2s-1} 2s_{c_i} m_1^{2s-i} m_2^i] - \alpha^s[\beta^{2k} - \sum_{i=1}^{2k-1} 2k_{c_i} m_1^{2k-i} m_2^i]$

Proof:

$$J_{k+s}J_{k-s} = (Am_1^{k+s} + Bm_2^{k+s})(Am_1^{k-s} + Bm_2^{k-s})$$

$$= A^2 m_1^{2k} + B^2 m_2^{2k} + AB(m_1^{k+s} m_2^{k-s} + m_2^{k+s} m_1^{k-s})$$

$$= A\{Am_1^{2k}+Bm_2^{2k}-Bm_2^{2k}\}+B\{Am_1^{2k}+Bm_2^{2k}-Am_1^{2k}\}+ABm_1^k m_2^k(m_1^s m_2^{-s}+m_2^s m_1^{-s})$$

$$= (A+B)J_{2k}-AB(m_1^{2k}+m_2^{2k})+ABm_1^k m_2^k\left(\frac{m_1^s}{m_2^s}+\frac{m_2^s}{m_1^s}\right)$$

$$= aJ_{2k}-AB(m_1^{2k}+m_2^{2k})+AB\frac{\alpha^k}{\alpha^s}\left(m_1^{2s}+m_2^{2s}\right)$$

$$\frac{(J_{k+s}J_{k-s}-aJ_{2k})\alpha^s}{AB}=\alpha^k\left(m_1^{2s}+m_2^{2s}\right)-\left(m_1^{2k}+m_2^{2k}\right)\alpha^s \qquad (7)$$

Using (5) in (7), we get

$$\frac{(J_{k+s}J_{k-s}-aJ_{2k})\alpha^s}{AB}=\alpha^k[\beta^{2s}-\sum_{i=1}^{2s-1}2s_{c_i}m_1^{2s-i}m_2^i]-\alpha^s[\beta^{2k}-\sum_{i=1}^{2k-1}2k_{c_i}m_1^{2k-i}m_2^i]$$

(viii). $(m_1-1)(m_2-1)\sum_{k=0}^{N-1}J_k=(1-\beta)(\alpha-J_N)-J_{N+1}+b$

Proof:

Since, $\sum_{k=0}^{N-1}J_k=\sum_{k=0}^{N-1}(Am_1^k+Bm_2^k)$

$$= A\sum_{k=0}^{N-1}m_1^k+B\sum_{k=0}^{N-1}m_2^k$$

$$= A\frac{m_1^N-1}{m_1-1}+B\frac{m_2^N-1}{m_2-1}$$

$$(m_1-1)(m_2-1)\sum_{k=0}^{N-1}J_k=A(m_2-1)(m_1^N-1)+B(m_1-1)(m_2^N-1)$$

$$=A(m_2m_1^N-m_2-m_1^N+1)+B(m_1m_2^N-m_1-m_2^N+1)$$

$$=(A+B)-(Am_1^N+Bm_2^N)-(Am_2+Bm_1)+m_2(Am_1^N-Bm_2^N-Bm_2^N)+m_1(Am_1^N+Bm_2^N-Am_1^N)$$

$$=a-J_N-(Am_2+Bm_1)+(m_2+m_1)J_N-J_{N+1}$$

$$= a+(\beta-1)J_N-J_{N+!}-(\alpha\beta-b)$$

$$= a(1-\beta)-(1-\beta)J_N-J_{N+1}+b$$

$$= (1-\beta)(a-J_N)-J_{N+1}+b$$

Hence, $(m_1-1)(m_2-1)\sum_{k=0}^{N-1}J_k=(1-\beta)(\alpha-J_N)-J_{N+1}+b$

(ix). $\frac{aJ_{2n}-J_n^2+aJ_{2n+2}-J_{n+1}^2}{AB}$ **is written as sum of two squares.**

Proof:

$$J_n^2+J_{n+1}^2=(Am_1^n-Bm_2^n)^2+(Am_1^{n+1}+Bm_2^{n+1})^2$$

$$=A^2m_1^{2n}+B^2m_2^{2n}+2ABm_1^nm_2^n+A^2m_1^{2n+2}+B^2m_2^{2n+2}+2ABm_1^{n+1}m_2^{n+1}$$

$$=A(J_{2n}-Qm_2^{2n})+B(J_{2n}-Pm_1^{2n})+2ABm_1^nm_2^n+A(J_{2n+2}-Qm_2^{2n+2})+B(J_{2n+2}-Pm_1^{2n+2})+2ABm_1^{n+1}m_2^{n+1}$$

$$=aJ_{2n}-AB(m_1^n-m_2^n)^2+aJ_{2n+2}-AB(m_1^{n+1}-m_2^{n+1})^2$$

Hence, $\frac{aJ_{2n}-J_n^2+aJ_{2n+2}-J_{n+1}^2}{AB}$ **is written as the sum of two squares.**

(x). $6\left(\frac{J_{2k}J_{2s}-J_{k+s}^2}{AB}\right)$ **is a Nasty Number.**

Proof:

$$J_{2k}J_{2s}=(Am_1^{2k}+Bm_2^{2k})(Am_1^{2s}+Bm_2^{2s})$$

$$=A^2m_1^{2(k+s)}+B^2m_2^{2(k+s)}+AB(m_1^{2k}m_2^{2s}+m_2^{2k}m_1^{2s})$$

$$=(Am_1^{k+s}+Bm_2^{k+s})^2-2ABm_1^{k+s}m_2^{k+s}+AB(m_1^{2k}m_2^{2s}+m_1^{2s}m_2^{2k})$$

$$=J_{k+s}^2+AB(m_1^km_2^s-m_2^km_1^s)^2$$

Hence, $6\left(\frac{J_{2k}J_{2s}-J_{k+s}^2}{AB}\right)$ **is a Nasty Number.**

Conclusion:

In this paper, we have presented a remarkable integer sequence developed by the recurrence relation $J_{n+2}=\beta J_{n+1}-\alpha J_n, \alpha\neq\beta, (\alpha,\beta\succ 0)$ with the initial conditions $J_0=a, J_1=b$ where a,b are not zeros simultaneously. One may search for other choices of integer sequences with suitable initial conditions.

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