LIE SYMMETRY ANALYSIS: FOURTH ORDER NONLINEAR ORDINARY DIFFERENTIAL EQUATION AS A CASE STUDY

Rodgers K. Adenyah*[*]

Titus J.O. Aminer [**]

Oduor, Okoya M[***]

Abstract

Lie group theory is a very useful mathematical tool and it can be used in solving a variety of problems in applied mathematics. It's rich, very interesting and in broad sense a topic of active mathematical research. Efforts to obtain solutions to a variety of differential equations by use of Lie symmetry have been in existence for quite a long period of time [1]. The numerical methods applied in obtaining solutions provide approximations which depend on certain initial and boundary conditions. This in itself is a limitation which can be dealt with by obtaining an analytical solution. In this study we seek to obtain an equation which is an identity in x, y, y', y'' and y''' of the fourth order Non-linear equation $y^{(4)} = (y)^{-1} y'y'''$ which is harmonic in nature. It's from this equation that we obtain partial differential equations known as determining equations [4] which are used to get the generator G of infinitesimal transformation.

Mathematics Subject Classification: Primary 65N30; Secondary 65N22, 65N12, 65F10

***Keywords*—: Lie Symmetry, infinitesimal transformation, ordinary differential equations,**

[*] Department of Mathematics and PhysicsPwani University

[**]School of Mathematics and Actuarial Science, Jaramogi Oginga Odinga University of Science and Technology1

[***] School of Mathematics and Actuarial Science,Jaramogi Oginga Odinga University of Science and Technology

1. Introduction

Many differential equations of practical interest evolve on Lie groups or on manifolds acted upon by Lie groups [5]. A Lie group or symmetry group is a group of transformations which maps any solution of the system to another solution of the same system [7]. The retention of Lie-group structure under discretization is very vital in the minimization of numerical error which arises when numerical methods are applied. This work seeks to get a differential equation which enables us to obtain the Generator G of infinitesimal transformation from which one can obtain the solution to the differential equation.

2. Obtaining the differential equation which is an identity in $x, y, y', y'' and\ y'''$

This is a differential equation which holds for any arbitrary choice of

$x, y, y', y'' and\ y'''$

Let's consider our equation $y^{(4)} = (y)^{-1} y' y'''$ (1)

Our equation is of fourth order and so we will use

$$G^{(4)} = \alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y} + \left(\beta' - \alpha' y'\right)\frac{\partial}{\partial y'} + \left(\beta'' - 2y''\alpha' - y'\alpha''\right)\frac{\partial}{\partial y''}$$

$$+(\beta''' - 3y'''\alpha' - 3y''\alpha'' - y'\alpha''')\frac{\partial}{\partial y'''}$$

$$+(\beta^{(4)} - 4y^{(4)}\alpha' - 6y'''\alpha'' - 4y''\alpha''' - y'\alpha^{(4)})\frac{\partial}{\partial y^{(4)}}$$

Which is the fourth extension of G .[6]

Now $G^{[4]}$ acting on the differential equation (1) will yield

$G^{[4]}\left[y^{(4)} - (y)^{-1} y' y'''\right]$.Which on expanding gives us

$$\Big[\ \alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y} + \left(\beta' - \alpha' y'\right)\frac{\partial}{\partial y'} + \left(\beta'' - 2y''\alpha' - y'\alpha''\right)\frac{\partial}{\partial y''} +$$

$$\left(\beta''' - 3y'''\alpha' - 3y''\alpha'' - y'\alpha'''\right)\frac{\partial}{\partial y'''} + \left(\beta^{(4)} - 4y^{(4)}\alpha' - 6y'''\alpha'' - 4y''\alpha''' - y'\alpha^{(4)}\right)\frac{\partial}{\partial y^{(4)}}\ \Big]$$

$$\left[\ y^{(4)} - (y)^{-1} y' y'''\ \right] = 0 \qquad (2)$$

This produces the equation

$$\alpha\left[y^{(5)}-y^{-1}y'y^{(4)}-y^{-1}y''y'''+y^{-2}(y')^{2}y'''\right]+\beta\left[y^{-2}y'y'''\right]+\left[\beta'-\alpha'y'\right]\left[-y^{-1}y'''\right]$$

$$+\left[\beta''-2y''\alpha'-y'\alpha''\right](0)+\left[\beta'''-3y'''\alpha'-3y''\alpha''-y'\alpha'''\right]\left[-y^{-1}y'\right]$$

$$+\left[\beta^{(4)}-4y^{(4)}\alpha'-6y'''\alpha''-4y''\alpha'''-y'\alpha^{(4)}\right]\ (1)=0 \qquad (3)$$

Remember

$$y^{(5)}=\left(y^{(4)}\right)'$$

$$y^{(5)}=\left(y^{-1}y'y'''\right)'$$

$$y^{(5)}=y^{-1}y'y^{(4)}+y^{-1}y''y'''-y^{-2}(y')^{2}y''' \qquad (4)$$

Substituting values of (4) into (3) we obtain

$$\alpha\left[y^{-1}y'y^{(4)}+y^{-1}y''y'''-y^{-2}(y')^{2}y'''-y^{-1}y'y^{(4)}-y^{-1}y''y'''+y^{-2}(y')^{2}y'''\right]$$

$$+\beta\left[y^{-2}y'y'''\right]$$

$$+(\beta'-\alpha'y')\left[-y^{-1}y'''\right]$$

$$+\ (\beta''-2y''\alpha'-y'\alpha'')(0)$$

$$+(\beta'''-3y'''\alpha'-3y''\alpha''-y'\alpha''')\left[-y^{-1}y'\right]$$

$$+(\beta^{(4)}-4y^{(4)}\alpha'-6y'''\alpha''-4y''\alpha'''-y'\alpha^{(4)})(1) \qquad (5)$$

Or equivalently

$$\alpha y^{-1}y'y^{(4)}+\alpha y^{-1}y''y'''-\alpha y^{-2}y'^{2}y'''-\alpha y^{-1}y'y^{(4)}-\alpha y^{-1}y''y'''+\alpha y^{-2}y'^{2}y'''$$

$$+\beta y^{-2}y'y'''-\beta'y^{-1}y'''+\alpha'y^{-1}y'y'''-\beta'''y^{-1}y'+3\alpha'y^{-1}y'y'''$$

$$+3\alpha''y^{-1}y'y''+\alpha'''y^{-1}(y')^{2}+\beta^{(4)}-4y^{(4)}\alpha'-6\alpha''y'''-4\alpha'''y''-\alpha^{(4)}y'=0 \qquad (6)$$

Substituting $y^{(4)}=(y)^{-1}y'y'''$ into (6) we get

$$\alpha y^{-1}y'(y^{-1}y'y''')+\alpha y^{-1}y''y'''-\alpha y^{-2}(y')^{2}y'''-\alpha y^{-1}y'(y^{-1}y'y''')-\alpha y^{-1}y''y'''+\alpha y^{-2}(y')^{2}y'''$$

$$+\beta y^{-2}y'y'''-\beta'y^{-1}y'''+\alpha'y^{-1}y'y'''-\beta'''y^{-1}y'+3\alpha'y^{-1}y'y'''$$

$$+3\alpha''y^{-1}y'y''+\alpha'''y^{-1}(y')^{2}+\beta^{(4)}-4(y^{-1}y'y''')\alpha'-6\alpha''y'''-4\alpha'''y''-\alpha^{(4)}y'=0 \qquad (7)$$

On opening the brackets we have

$$\alpha(y^{-1})^2(y')^2y''' + \alpha y^{-1}y''y''' - \alpha y^{-2}(y')^2y''' - \alpha(y^{-1})^2(y')^2y'''$$

$$-\alpha y^{-1}y''y''' + \alpha y^{-2}(y')^2y''' + \beta y^{-2}y'y''' - \beta' y^{-1}y'''$$

$$+\alpha' y^{-1}y'y''' - \beta''' y^{-1}y' + 3\alpha' y^{-1}y'y''' + 3\alpha'' y^{-1}y'y''$$

$$+\alpha''' y^{-1}(y')^2 + \beta^{(4)} - 4\alpha' y^{-1}y'y''' - 6\alpha'' y''' - 4\alpha''' y'' - \alpha^{(4)}y' = 0 \qquad (8)$$

Which we simplify to have

$$\beta y^{-2}y'y''' - \beta' y^{-1}y''' + \alpha' y^{-1}y'y''' - \beta''' y^{-1}y' + 3\alpha' y^{-1}y'y''' + 3\alpha'' y^{-1}y'y''$$

$$+\alpha''' y^{-1}(y')^2 + \beta^{(4)} - 4\alpha' y^{-1}y'y''' - 6\alpha'' y''' - 4\alpha''' y'' - \alpha^{(4)}y' = 0 \qquad (9)$$

Remember that the primes in (9) represent the total derivatives [3]. We express the first, second, third and fourth total derivatives of α and β in terms of partial derivatives as follows

$$\alpha' = \frac{\partial \alpha}{\partial x} + y'\frac{\partial \alpha}{\partial y} \qquad \{\text{From } d(\alpha) = \left(\frac{\partial \alpha}{\partial x}\right)dx + \left(\frac{\partial \alpha}{\partial y}\right)dy\}$$

$$\alpha'' = \frac{\partial}{\partial x}\left(\frac{\partial \alpha}{\partial x} + y'\frac{\partial \alpha}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \alpha}{\partial x} + y'\frac{\partial \alpha}{\partial y}\right)y'$$

$$\alpha'' = \frac{\partial^2 \alpha}{\partial x^2} + y'\frac{\partial^2 \alpha}{\partial x \partial y} + y''\frac{\partial \alpha}{\partial y} + y'\frac{\partial^2 \alpha}{\partial x \partial y} + y'^2\frac{\partial^2 \alpha}{\partial y^2} + 0$$

$$\alpha'' = \frac{\partial^2 \alpha}{\partial x^2} + 2y'\frac{\partial^2 \alpha}{\partial x \partial y} + y'^2\frac{\partial^2 \alpha}{\partial y^2} + y''\frac{\partial \alpha}{\partial y}$$

$$\alpha''' = \frac{\partial}{\partial x}\left(\frac{\partial^2 \alpha}{\partial x^2} + 2y'\frac{\partial^2 \alpha}{\partial x \partial y} + y''\frac{\partial \alpha}{\partial y} + y'^2\frac{\partial^2 \alpha}{\partial y^2}\right)$$

$$+y'\frac{\partial}{\partial y}\left(\frac{\partial^2 \alpha}{\partial x^2} + 2y'\frac{\partial^2 \alpha}{\partial x \partial y} + y''\frac{\partial \alpha}{\partial y} + y'^2\frac{\partial^2 \alpha}{\partial y^2}\right)$$

$$\alpha''' = \frac{\partial^3 \alpha}{\partial x^3} + 2y'\frac{\partial^2 \alpha}{\partial x \partial x \partial y} + 2y''\frac{\partial^2 \alpha}{\partial x \partial y} + y''\frac{\partial^2 \alpha}{\partial x \partial y} + y'''\frac{\partial \alpha}{\partial y} + y'^2\frac{\partial^3 \alpha}{\partial x \partial y^2}$$

$$+2y'y''\frac{\partial^2 \alpha}{\partial y^2} + y'\frac{\partial^3 \alpha}{\partial y \partial x^2} + 2y'^2\frac{\partial^3 \alpha}{\partial y \partial x \partial y} + 0 + y'y''\frac{\partial^2 \alpha}{\partial y^2} + 0 + y'^3\frac{\partial^3 \alpha}{\partial y^3} + 0$$

$$\alpha''' = \frac{\partial^3 \alpha}{\partial x^3} + 3y'\frac{\partial^3 \alpha}{\partial x^2 \partial y} + 3y''\frac{\partial^2 \alpha}{\partial x \partial y} + y'''\frac{\partial \alpha}{\partial y} + 3y'^2\frac{\partial^3 \alpha}{\partial x \partial y^2} + 3y'y''\frac{\partial^2 \alpha}{\partial y^2} + y'^3\frac{\partial^3 \alpha}{\partial y^3}$$

$$\alpha^{(4)}=\frac{\partial}{\partial x}\left(\frac{\partial^3\alpha}{\partial x^3}+3y'\frac{\partial^3\alpha}{\partial x^2\partial y}+3y''\frac{\partial^2\alpha}{\partial x\partial y}+y'''\frac{\partial\alpha}{\partial y}+3y'^2\frac{\partial^3\alpha}{\partial x\partial y^2}+3y'y''\frac{\partial^2\alpha}{\partial y^2}+y'^3\frac{\partial^3\alpha}{\partial y^3}\right)$$

$$\alpha^{(4)}=\frac{\partial^4\alpha}{\partial x^4}+3y'\frac{\partial^4\alpha}{\partial x^3\partial y}+3y''\frac{\partial^3\alpha}{\partial x^2\partial y}+3y''\frac{\partial^3\alpha}{\partial x^2\partial y}+3y'''\frac{\partial^2\alpha}{\partial x\partial y}$$

$$+y'''\frac{\partial^2\alpha}{\partial x\partial y}+y^{(4)}\frac{\partial\alpha}{\partial y}+3y'^2\frac{\partial^4\alpha}{\partial x^2\partial y^2}+6y'y''\frac{\partial^3\alpha}{\partial x\partial y^2}$$

$$\alpha^{(4)}=\frac{\partial^4\alpha}{\partial x^4}+4y'\frac{\partial^4\alpha}{\partial x^3\partial y}+6y''\frac{\partial^3\alpha}{\partial x^2\partial y}++4y'''\frac{\partial^2\alpha}{\partial x\partial y}$$

$$+y^{(4)}\frac{\partial\alpha}{\partial y}+3y'^2\frac{\partial^4\alpha}{\partial x^2\partial y^2}+9y'y''\frac{\partial^3\alpha}{\partial x\partial y^2}$$

$$+4y'y'''\frac{\partial^2\alpha}{\partial y^2}+3y''^2\frac{\partial^2\alpha}{\partial y^2}+4y'^3\frac{\partial^4\alpha}{\partial x\partial y^3}+6y'^2y''\frac{\partial^3\alpha}{\partial y^3}$$

$$+3y'^2\frac{\partial^4\alpha}{\partial x^2\partial y^2}+3y'y''\frac{\partial^3\alpha}{\partial x\partial y^2}+y'^4\frac{\partial^4\alpha}{\partial y^4}. \quad [2]$$

The same is applied to β',β'',β''' and $\beta^{(4)}$ so that we have the first, second, third and fourth derivatives of both α and β.Then we substitute α, β and their derivatives into

$$\beta y^{-2}y'y'''-\beta'y^{-1}y'''+\alpha'y^{-1}y'y'''-\beta'''y^{-1}y'+3\alpha'y^{-1}y'y'''+3\alpha''y^{-1}y'y''$$
$$+\alpha'''y^{-1}(y')^2+\beta^{(4)}-4\alpha'y^{-1}y'y'''-6\alpha''y'''-4\alpha'''y''-\alpha^{(4)}y'=0 \tag{9}$$

So as to obtain

$$\beta y^{-2}y'y'''-\left(\frac{\partial\beta}{\partial x}+y'\frac{\partial\beta}{\partial y}\right)y^{-1}y'''+\left(\frac{\partial\alpha}{\partial x}+y'\frac{\partial\alpha}{\partial y}\right)y^{-1}y'y'''$$

$$-\left(\frac{\partial^3\beta}{\partial x^3}+3y'\frac{\partial^3\beta}{\partial x^2\partial y}+3y''\frac{\partial^2\beta}{\partial x\partial y}+y'''\frac{\partial\beta}{\partial y}+3y'^2\frac{\partial^3\beta}{\partial x\partial y^2}+3y'y''\frac{\partial^2\beta}{\partial y^2}+y'^3\frac{\partial^3\beta}{\partial y^3}\right)y^{-1}y'$$

$$+3\left(\frac{\partial\alpha}{\partial x}+y'\frac{\partial\alpha}{\partial y}\right)y^{-1}y'y'''+3\left(\frac{\partial^2\alpha}{\partial x^2}+2y'\frac{\partial^2\alpha}{\partial x\partial y}+y'^2\frac{\partial^2\alpha}{\partial y^2}+y''\frac{\partial\alpha}{\partial y}\right)y^{-1}y'y''$$

$$+\left(\frac{\partial^3\alpha}{\partial x^3}+3y'\frac{\partial^3\alpha}{\partial x^2\partial y}+3y''\frac{\partial^2\alpha}{\partial x\partial y}+y'''\frac{\partial\alpha}{\partial y}+3y'^2\frac{\partial^3\alpha}{\partial x\partial y^2}+3y'y''\frac{\partial^2\alpha}{\partial y^2}+y'^3\frac{\partial^3\alpha}{\partial y^3}\right)y^{-1}y'^2$$

$$+\frac{\partial^4\beta}{\partial x^4}+4y'\frac{\partial^4\beta}{\partial x^3\partial y}+6y''\frac{\partial^3\beta}{\partial x^2\partial y}+4y'''\frac{\partial^2\beta}{\partial x\partial y}+y^{(4)}\frac{\partial\beta}{\partial y}+3y'^2\frac{\partial^4\beta}{\partial x^2\partial y^2}+9y'y''\frac{\partial^3\beta}{\partial x\partial y^2}+4y'y'''\frac{\partial^2\beta}{\partial y^2}$$

$$+3y''^2\frac{\partial^2\beta}{\partial y^2}+4y'^3\frac{\partial^4\beta}{\partial x\partial y^3}+6y'^2y''\frac{\partial^3\beta}{\partial y^3}+3y'^2\frac{\partial^4\beta}{\partial x^2\partial y^2}+3y'y''\frac{\partial^3\beta}{\partial x\partial y^2}+y'^4\frac{\partial^4\beta}{\partial y^4}$$

$$-4\left(\frac{\partial\alpha}{\partial x}+y'\frac{\partial\alpha}{\partial y}\right)y^{-1}y'y'''-6\left(\frac{\partial^2\alpha}{\partial x^2}+2y'\frac{\partial^2\alpha}{\partial x\partial y}+y'^2\frac{\partial^2\alpha}{\partial y^2}+y''\frac{\partial\alpha}{\partial y}\right)y'''$$

$$-4\left(\frac{\partial^3\alpha}{\partial x^3}+3y'\frac{\partial^3\alpha}{\partial x^2\partial y}+3y''\frac{\partial^2\alpha}{\partial x\partial y}+y'''\frac{\partial\alpha}{\partial y}+3y'^2\frac{\partial^3\alpha}{\partial x\partial y^2}+3y'y''\frac{\partial^2\alpha}{\partial y^2}+y'^3\frac{\partial^3\alpha}{\partial y^3}\right)y''$$

$$-\Big(\frac{\partial^4\alpha}{\partial x^4}+4y'\frac{\partial^4\alpha}{\partial x^3\partial y}+6y''\frac{\partial^3\alpha}{\partial x^2\partial y}+4y'''\frac{\partial^2\alpha}{\partial x\partial y}+y^{(4)}\frac{\partial\alpha}{\partial y}+3y'^2\frac{\partial^4\alpha}{\partial x^2\partial y^2}$$

$$+9y'y''\frac{\partial^3\alpha}{\partial x\partial y^2}+4y'y'''\frac{\partial^2\alpha}{\partial y^2}+3y''^2\frac{\partial^2\alpha}{\partial y^2}+4y'^3\frac{\partial^4\alpha}{\partial x\partial y^3}+6y'^2y''\frac{\partial^3\alpha}{\partial y^3}$$

$$+3y'^2\frac{\partial^4\alpha}{\partial x^2\partial y^2}+3y'y''\frac{\partial^3\alpha}{\partial x\partial y^2}+y'^4\frac{\partial^4\alpha}{\partial y^4}\Big)y'=0 \qquad (10)$$

From which we obtain

$$\beta y^{-2}y'y'''-y^{-1}y'''\frac{\partial\beta}{\partial x}-y^{-1}y'y'''\frac{\partial\beta}{\partial y}+y^{-1}y'y'''\frac{\partial\alpha}{\partial x}+y^{-1}y'^2y'''\frac{\partial\alpha}{\partial y}-y^{-1}y'\frac{\partial^3\beta}{\partial x^3}$$

$$-3y^{-1}y'^2\frac{\partial^3\beta}{\partial x^2\partial y}-3y^{-1}y'y''\frac{\partial^2\beta}{\partial x\partial y}-y^{-1}y'y'''\frac{\partial\beta}{\partial y}-3y^{-1}y'^3\frac{\partial^3\beta}{\partial x\partial y^2}-3y^{-1}y'^2y''\frac{\partial^2\beta}{\partial y^2}$$

$$-y^{-1}y'^4\frac{\partial^3\beta}{\partial y^3}+3y^{-1}y'y'''\frac{\partial\alpha}{\partial x}+3y^{-1}y'^2y'''\frac{\partial\alpha}{\partial y}+3y^{-1}y'y''\frac{\partial^2\alpha}{\partial x^2}+6y^{-1}y'^2y''\frac{\partial^2\alpha}{\partial x\partial y}$$

$$+3y^{-1}y'^3y''\frac{\partial^2\alpha}{\partial y^2}+3y^{-1}y'y''^2\frac{\partial\alpha}{\partial y}+y^{-1}y'^2\frac{\partial^3\alpha}{\partial x^3}+3y^{-1}y'^3\frac{\partial^3\alpha}{\partial x^2\partial y}+3y^{-1}y'^2y''\frac{\partial^2\alpha}{\partial x\partial y}$$

$$+y^{-1}y'^2y'''\frac{\partial\alpha}{\partial y}+3y^{-1}y'^4\frac{\partial^3\alpha}{\partial x\partial y^2}+3y^{-1}y'^3y''\frac{\partial^2\alpha}{\partial y^2}+y^{-1}y'^5\frac{\partial^3\alpha}{\partial y^3}+\frac{\partial^4\beta}{\partial x^4}+4y'\frac{\partial^4\beta}{\partial x^3\partial y}$$

$$+6y''\frac{\partial^3\beta}{\partial x^2\partial y}+4y'''\frac{\partial^2\beta}{\partial x\partial y}+y^{(4)}\frac{\partial\beta}{\partial y}+3y'^2\frac{\partial^4\beta}{\partial x^2\partial y^2}+9y'y''\frac{\partial^3\beta}{\partial x\partial y^2}+4y'y'''\frac{\partial^2\beta}{\partial y^2}$$

$$+3y''^2\frac{\partial^2\beta}{\partial y^2}+4y'^3\frac{\partial^4\beta}{\partial x\partial y^3}+6y'^2y''\frac{\partial^3\beta}{\partial y^3}+3y'^2\frac{\partial^4\beta}{\partial x^2\partial y^2}+3y'y''\frac{\partial^3\beta}{\partial x\partial y^2}+y'^4\frac{\partial^4\beta}{\partial y^4}$$

$$-4y^{-1}y'y'''\frac{\partial\alpha}{\partial x}-4y^{-1}y'^2y'''\frac{\partial\alpha}{\partial y}-6y'''\frac{\partial^2\alpha}{\partial x^2}-12y'y'''\frac{\partial^2\alpha}{\partial x\partial y}-6y'^2y'''\frac{\partial^2\alpha}{\partial y^2}$$

$$-6y''y'''\frac{\partial\alpha}{\partial y}-4y''\frac{\partial^3\alpha}{\partial x^3}-12y'y''\frac{\partial^3\alpha}{\partial x^2\partial y}-12y''^2\frac{\partial^2\alpha}{\partial x\partial y}-4y''y'''\frac{\partial\alpha}{\partial y}$$

$$-12y'^2y''\frac{\partial^3\alpha}{\partial x\partial y^2}-12y'y''^2\frac{\partial^2\alpha}{\partial y^2}-4y'^3y''\frac{\partial^3\alpha}{\partial y^3}-y'\frac{\partial^4\alpha}{\partial x^4}-4y'^2\frac{\partial^4\alpha}{\partial x^3\partial y}-6y'y''\frac{\partial^3\alpha}{\partial x^2\partial y}$$

$$-4y'y'''\frac{\partial^2\alpha}{\partial x\partial y}-y'y^{(4)}\frac{\partial\alpha}{\partial y}-3y'^3\frac{\partial^4\alpha}{\partial x^2\partial y^2}-9y'^2y''\frac{\partial^3\alpha}{\partial x\partial y^2}-4y'^2y'''\frac{\partial^2\alpha}{\partial y^2}$$

$$-3y'y''^2\frac{\partial^2\alpha}{\partial y^2}-4y'^4\frac{\partial^4\alpha}{\partial x\partial y^3}-6y'^3y''\frac{\partial^3\alpha}{\partial y^3}-3y'^3\frac{\partial^4\alpha}{\partial x^2\partial y^2}-3y'^2y''\frac{\partial^3\alpha}{\partial x\partial y^2}-y'^5\frac{\partial^4\alpha}{\partial y^4} \qquad (11)$$

Now replacing $y^{(4)}=y^{-1}y'y'''$ into equation (11) above we obtain

$$\beta y^{-2}y'y'''-y^{-1}y''\frac{\partial\beta}{\partial x}-y^{-1}y'y'''\frac{\partial\beta}{\partial y}+y^{-1}y'y'''\frac{\partial\alpha}{\partial x}+y^{-1}y'^2y'''\frac{\partial\alpha}{\partial y}-y^{-1}y'\frac{\partial^3\beta}{\partial x^3}$$

$$-3y^{-1}y'^2\frac{\partial^3\beta}{\partial x^2\partial y}-3y^{-1}y'y''\frac{\partial^2\beta}{\partial x\partial y}-y^{-1}y'y'''\frac{\partial\beta}{\partial y}-3y^{-1}y'^3\frac{\partial^3\beta}{\partial x\partial y^2}-3y^{-1}y'^2y''\frac{\partial^2\beta}{\partial y^2}$$

$$-y^{-1}y'^4\frac{\partial^3\beta}{\partial y^3}+3y^{-1}y'y'''\frac{\partial\alpha}{\partial x}+3y^{-1}y'^2y'''\frac{\partial\alpha}{\partial y}+3y^{-1}y'y''\frac{\partial^2\alpha}{\partial x^2}+6y^{-1}y'^2y''\frac{\partial^2\alpha}{\partial x\partial y}$$

$$+3y^{-1}y'^3y''\frac{\partial^2\alpha}{\partial y^2}+3y^{-1}y'y''^2\frac{\partial\alpha}{\partial y}+y^{-1}y'^2\frac{\partial^3\alpha}{\partial x^3}+3y^{-1}y'^3\frac{\partial^3\alpha}{\partial x^2\partial y}+3y^{-1}y'^2y''\frac{\partial^2\alpha}{\partial x\partial y}$$

$$+y^{-1}y'^2y'''\frac{\partial\alpha}{\partial y}+3y^{-1}y'^4\frac{\partial^3\alpha}{\partial x\partial y^2}+3y^{-1}y'^3y''\frac{\partial^2\alpha}{\partial y^2}+y^{-1}y'^5\frac{\partial^3\alpha}{\partial y^3}+\frac{\partial^4\beta}{\partial x^4}+4y'\frac{\partial^4\beta}{\partial x^3\partial y}$$

$$+6y''\frac{\partial^3\beta}{\partial x^2\partial y}+4y'''\frac{\partial^2\beta}{\partial x\partial y}+y^{-1}y'y'''\frac{\partial\beta}{\partial y}+3y'^2\frac{\partial^4\beta}{\partial x^2\partial y^2}+9y'y''\frac{\partial^3\beta}{\partial x\partial y^2}+4y'y'''\frac{\partial^2\beta}{\partial y^2}$$

$$+3y''^2\frac{\partial^2\beta}{\partial y^2}+4y'^3\frac{\partial^4\beta}{\partial x\partial y^3}+6y'^2y''\frac{\partial^3\beta}{\partial y^3}+3y'^2\frac{\partial^4\beta}{\partial x^2\partial y^2}+3y'y''\frac{\partial^3\beta}{\partial x\partial y^2}+y'^4\frac{\partial^4\beta}{\partial y^4}$$

$$-4y^{-1}y'y'''\frac{\partial\alpha}{\partial x}-4y^{-1}y'^2y'''\frac{\partial\alpha}{\partial y}-6y'''\frac{\partial^2\alpha}{\partial x^2}-12y'y'''\frac{\partial^2\alpha}{\partial x\partial y}-6y'^2y'''\frac{\partial^2\alpha}{\partial y^2}$$

$$-6y''y'''\frac{\partial\alpha}{\partial y}-4y''\frac{\partial^3\alpha}{\partial x^3}-12y'y''\frac{\partial^3\alpha}{\partial x^2\partial y}-12y''^2\frac{\partial^2\alpha}{\partial x\partial y}-4y''y'''\frac{\partial\alpha}{\partial y}$$

$$-12y'^2y''\frac{\partial^3\alpha}{\partial x\partial y^2}-12y'y''^2\frac{\partial^2\alpha}{\partial y^2}-4y'^3y''\frac{\partial^3\alpha}{\partial y^3}-y'\frac{\partial^4\alpha}{\partial x^4}-4y'^2\frac{\partial^4\alpha}{\partial x^3\partial y}-6y'y''\frac{\partial^3\alpha}{\partial x^2\partial y}$$

$$-4y'y'''\frac{\partial^2\alpha}{\partial x\partial y}-y^{-1}y'^2y'''\frac{\partial\alpha}{\partial y}-3y'^3\frac{\partial^4\alpha}{\partial x^2\partial y^2}-9y'^2y''\frac{\partial^3\alpha}{\partial x\partial y^2}-4y'^2y'''\frac{\partial^2\alpha}{\partial y^2}$$

$$-3y'y''^2\frac{\partial^2\alpha}{\partial y^2}-4y'^4\frac{\partial^4\alpha}{\partial x\partial y^3}-6y'^3y''\frac{\partial^3\alpha}{\partial y^3}-3y'^3\frac{\partial^4\alpha}{\partial x^2\partial y^2}-3y'^2y''\frac{\partial^3\alpha}{\partial x\partial y^2}-y'^5\frac{\partial^4\alpha}{\partial y^4}$$

(12)

Conclusion

Clearly equation (12) is an identity in x, y, y', y'' and y''', i.e.it holds for any arbitrary choice of x, y, y', y'' and y'''[4]. From equation (12) we can easily obtain our determining equations that would be used in getting the generator G of infinitesimal transformation for our fourth order non-linear equation. The transformation helps in obtaining the solution to our nonlinear equation. This is of great significance in mechanics.

References

1. **Aminer, T.J.O,(2014).** *Lie symmetry Solution of Fourth Order Non- linearOrdinary Differential Equations, PhD thesis,* Jaramogi Oginga Odinga University of Science and Technology, Bondo, Kenya.
2. **Bluman, G.W., Anco, S.C. (2002).** *Symmetry and Integration Methods for Differential Equations;* Springer: New York, NY, USA.
3. **Bluman, G.W., Kumei, S. (1989).** *Symmetries and differential Equations;* Springer-Verlag: New York, NY, USA.
4. **Dresner, L. (1999).** *Applications of Lie's Theory of Ordinary and Partial Differential Equations;* London, Institute of Physics.
5. **H.Stephani (1989)**.*Differential Equations:Their solutions using symmetries,* Cambridge University Press, Cambridge 1989.
6. **Mohamed F M and Leach P G L. (1990).***Symmetry Lie algebras of nth order ordinary differential equations***.** *J. Math. Anal. Appl.* 151 80-107
7. **Olver, P.J. (1986).** *Applications of Lie Groups to Differential Equations;* Springer: New York, N Y, USA.