INVARIANCE OF SEPARATION AXIOMS IN ISOTONIC SPACES WITH RESPECT TO PERFECT MAPPINGS

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Abstract

The behavior of separation axioms under perfect mappings has been studied in the realm of topological spaces. In this paper, we extend the characterization of perfect mappings to isotonic spaces and then use this class of continuous functions to investigate the behavior of separation axioms.The hierarchy of separation axioms that is familiar from topological spaces generalizes to spaces with an isotone and expansive closure functions. Neither additivity nor idempotence of the closure function need to be assumed.

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1. Introduction

The concept of a topological space is generally introduced and studied in terms of the axioms of open sets. However, alternate methods of describing a topology are often used; neighborhood systems, family of closed sets, closure operator and interior operator. Of these, the closure operator, records Kelley (1955), was axiomated by Kuratowski. He associated a topology in a closure space by taking closed sets as the sets which are fixed with respect to closure operation, that is, A is closed if $A = cl(A)$. It is also found that $cl(A)$ is the smallest closed set containing A.

The class of continuous functions forms a very broad spectrum of mappings comprising of different subclasses with varying properties. Perhaps of greater interest in topological spaces is the class of homeomorphisms, which is classically used to study equivalent classes of topological spaces. In the spectrum of continuous functions is the class of perfect mappings which, although weaker than homeomorphisms, provides a general yet satisfactory means of investigating topological invariants and hence the equivalence of topological spaces. It can therefore be used in substitution of homeomorphisms.

In an attempt to extend the boundaries of topology, Mashhour and Ghanim (1983), have shown that topological spaces do not constitute a natural boundary for the validity of theorems and results in topology. Moreover, Joshi (1983) remarks that when a branch of mathematics gets rich in terms of depth and applicability, people begin to investigate whether some of the basic axioms can either be dropped totally or at least be replaced by some weaker ones. Many results therefore, can be extended to closure spaces where some of the basic axioms in this space can be dropped. On the other hand, almost all approaches to extend the framework of topology, including Hammer (1964), at least use the closure function with the assumption that it is isotonic. Consequently, many properties which hold in basic topological spaces hold in spaces possessing the isotonic property.

2. Literature review

2.1 Closure Operator and Generalized Closure Space

A closure operator is an arbitrary set-valued, set-function $cl: \mathcal{P}(X) \to \mathcal{P}(X)$, where $\mathcal{P}(X)$ is the power set of a non-void set X (Thron, 1981). Consequently, various combinations of the following axioms have been used in the past in an attempt to define closure operators (Sunitha, 1994). Let $A, B \subset \mathcal{P}(X)$.

i. Grounded: $cl(\emptyset) = \emptyset$
ii. Expansive: $A \subset cl(A)$
iii. Sub-additive: $cl(A \cup B) \subset cl(A) \cup cl(B)$. This axiom implies the Isotony axiom: $A \subset B$ implies $cl(A) \subset cl(B)$
iv. Idempotent: $cl\big(cl(A)\big) = cl(A)$

The structure (X, cl), where cl satisfies the first three axioms is called a closure space. If in addition the idempotent axiom is satisfied, then the structure is a topological space.

2.2 Isotonic Space

A closure space (X, cl) satisfying only the grounded and the isotony closure axioms is called an isotonic space (Elzenati and Habil, 2006). This is the space of interest in this study and clearly, it is more general than a closure space.

In the dual formulation, a space (X, cl) is isotonic if and only if the interior function $int: \mathcal{P}(X) \to \mathcal{P}(X)$ satisfies;

i. $int(X) = X$.
ii. $A \subseteq B \subseteq X$ implies $int(A) \subseteq int(B)$.

2.3 Interior Function

From Elzenati and Habil (2008), the dual of the closure function, called the interior function, $int: \mathcal{P}(X) \to \mathcal{P}(X)$, is defined by

$$int(A) = X - cl(X - A)\ \forall A \in \mathcal{P}(X)$$

Given the interior function, $int: \mathcal{P}(X) \to \mathcal{P}(X)$ the closure function is recovered by

$$cl(A) = X - int(X - A)$$

2.4 The Open-Set Topology and the Closure Operator Topology

According to Mashhour and Ghanim (1983), if Γ is the set of all Kuratowski closure operators k that can be defined on a set X and Ψ is the set of all topologies τ which can be defined on X, then;

a) Denote by $\lambda: \Gamma \to \Psi$, the function defined by $\lambda(k) = \tau(k)$, where

$$\tau(k) = \{X - A: k(A) = A\}$$

b) Denote by $\mu: \Psi \to \Gamma$ the function defined by $\mu(\tau) = h_r$, where $h_r(A)$is the τ-closure of A.

Clearly, λ maps the closure operator in Γ to open sets and hence open topology in Ψ. Similarly, μ maps the topologies (open sets) in Ψ to the closures of these open sets in Γ. Therefore, the topological structure obtained by the Kuratowski closure operator and that obtained by a family of open sets are completely equivalent, provided the transition from one to the other is by means of functions λ and μ.

2.5 Continuity in Closure Spaces

Let (X, cl_X) and (Y, cl_Y) be two closure spaces with arbitrary closure functions. Let $f: X \to Y$ be a function from X to Y. According to Stadler and Stadler (2003), f is said to be continuous if the following equivalent conditions are satisfied;

i. $cl(f^{-1}(B)) \subseteq f^{-1}(cl(B)) \forall B \in \mathcal{P}(Y)$.

ii. $f^{-1}(int(B)) \subseteq int(f^{-1}(B)) \forall B \in \mathcal{P}(Y)$.

iii. $B \subseteq \mathcal{N}(f(x)) implies\ f^{-1}(B) \in \mathcal{N}(x) \forall B \in \mathcal{P}(Y) and\ \forall x \in X$.

Theorem: if (X, cl_X) and (Y, cl_Y) are isotonic spaces and $f: X \to Y$ is a bijection, then f is a homeomorphism if and only if $f(cl_X(A)) = cl_Y(f((A)) \ \forall A \in \mathcal{P}(X)$.

2.6 Perfect Mappings

The class of perfect mappings was introduced in metric spaces by Vainstein in 1947 (Engelking, 1989). More studies by Frolik (1960), showed the central role that perfect mappings play among all continuous functions; a role similar to that of compactness among topological spaces. He did this by proving that the Cartesian product of perfect mappings is perfect. This theorem is analogous to the Tychonoff theorem of compact spaces. Throughout the 20th century, many theorems on perfect mapping were formulated and proven. This include a theorem by Henriksen and Isbell (1958) showing that complete regularity is not an inverse invariant of perfect mappings, although it had earlier been shown that regularity is an inverse invariant of perfect mappings.

2.7 Compactness

The notion of compactness has been characterized,in closure spaces, using nets and filters as well as by means of covers by Cech (1966).

A closure space (X, cl) is compact if each net in X has an accumulation point. This notion of compactness has also been reinforced by Mresvic (2001). Equivalently, Thron (1981), provided the following definition. Let (X, cl) be a closure space. A family $\{A_i: i \in I\}$ of subsets of X is called a c-cover of X if $\{int\ A_i: i \in I\}$covers X. A closure space is c-compact if every c-cover of X has a finite subcover. We will adopt this latter definition.

3. Main Results

In this section, the results of the study have been given.

3.1 Perfect mappings on isotonic spaces

Perfect mappings are always assumed to be defined on Hausdorff spaces. In this section, this class of continuous functions is defined beginning with characterizations of closed mappings and compactness in isotonic spaces.

3.1.1 Closed mapping

A closed function maybe defined in different equivalent ways. The choice of which characterization one will use in a given instance depends on which definition best describes the notion of a closed mapping in that particular setting. In this research, an equivalent characterization that adopts the dual of the closure operator in closure spaces is established.

Definition

Let $f: X \to Y$ be a continuous mapping from an isotonic space X to an isotonic space Y. $f$is said to be closed if for every set V open in X the set $V^* = Y - f\big(X - int(V)\big)$ is open in Y.

Classically, if a function is closed, then it guarantees that a closed set in the domain will always be carried to a closed set in the co-domain by that function. In the above definition, the set $int(V) = X - cl(X - V)$ is open in X hence the set $X - int(V)$ is closed in X since the complement of an open set is closed.

If $V^* = Y - f\big(X - int(V)\big)$ is open in Y then $f\big(X - int(V)\big)$ is closed in Y. Therefore, f takes the closed set $X - int(V)$ to the closed set $f\big(X - int(V)\big)$. Clearly, if $V^* = Y - f\big(X - int(V)\big)$

is open in Y for every V open in X, then f is closed. Therefore, this is an alternative way of defining a closed mapping in isotonic spaces and more broadly, in closure spaces.

3.1.2 Compactness

Having established a definition for a closed mapping in an isotonic space, compactness becomes the next notion of interest. However, compactness in an isotonic space has previously been defined. That definition can be found in section 2.7 of this thesis. Nevertheless, this topological notion may as well be characterized via open refinements. In order to do this, it is necessary to extend the definition of open refinement from topological spaces in a similar way as Thron (1981) did for open covers. This extension is provided in the definition below.

Definition

Let $\mathcal{A} = \{A_\alpha : \alpha \in I\}$ be a c-cover for an isotonic space (X, cl). Then another c-cover, $\mathfrak{B} = \{B_\beta : \beta \in J\}$is an open refinement of $\mathcal{A}$ if for every $\beta \in J$, there exists $\alpha \in I$ such that $intB_\beta \subseteq intA_\alpha$.

It is clear from the definition of open refinement and from section 2.7 that aT_2-isotonic space X is c-compact if and only if every c-cover of X has a finite refinement. This definition can be used in place of the one employed by Thron (1981) in instances where it is easier to talk of open refinements than it is for open covers.

Attention now turns to defining the class of perfect mappings in isotonic spaces. However, it is important to define a c-compact mapping before giving a definition for a perfect map. The definitions below are those of a c-compact mapping as well as of a perfect mapping.

Definition

Let X and Y be isotonic spaces and $f: X \to Y$ be a continuous surjective mapping. $f$is called a c-compact mapping if $f^{-1}(y)$ is a c-compact subset of $X \forall y \in Y$.

A continuous surjective mapping between two isotonic spaces is said to be perfect if it is closed and $\forall y \in Y, f^{-1}(y)$ is a c-compact subset of X.

These two definitions closely correlate with their definitions in general topological spaces, except for the fact that a different approach is employed while defining the foregoing topological notions.

3.1.3 Perfect mappings and homeomorphisms

From general topological spaces, it is known that the class of homeomorphisms is used to investigate topological invariants. In this study, this class is abandoned and in its place, a weaker class of perfect mappings is adopted. However, it may be prudent to probe the extent to which these two classes of mappings are different. To begin with, it is known from literature that these two classes fall under the broad spectrum of continuous functions and their difference only sets in while including additional topological notions. The theorem below gives the conditions under which the two classes are equivalent.

Theorem

Let $f: (X, cl_X) \to (Y, cl_Y)$ be a continuous bijection from a c-compact isotonic space X to an T_2-isotonic space Y, then the following two conditions are equivalent:

i. f is a perfect mapping and hence a homeomorphism.

ii. $f\big(cl_X(A)\big) = cl_Y\big(f(A)\big) \forall A \in \mathcal{P}(X)$.

Proof:

To show that f is perfect, we only need to prove that f is closed. Let V be closed in X. This implies that $V = cl(V)$ and $V^* = X - V = int(X - V)$, is open in X.

$f(V) = f\big(cl(V)\big)$ but $cl(V) = X - int(X - V)$

Hence, $f(V) = f(X - int(X - V)) = f(X) - f\big(int(X - V)\big)$, where $f(X) = Y$ and $int(X - V) = X - V = V^*$.

Therefore $f(V) = Y - f(V^*)$ which is closed in Y. f is closed and hence perfect.

To show that f is a homeomorphism, we only need to prove that f is bicontinuous by showing that f^{-1} is continuous which is equivalent to showing that for every B closed in X, $f(B)$ is closed in Y. Let $B \subset X$ be closed, then by c-compactness of X, B is c-compact. Since f is continuous, then $f(B)$ is c-compact. Moreover, Y is Hausdorff, hence $f(B)$ is closed. Therefore f is bicontinuous and thus a homeomorphism. The theorem under section 2.5, gives the necessary and sufficient conditions for the existence of a homeomorphism in isotonic spaces. Therefore, since f is a homeomorphism, then condition (ii), in the theorem above must hold, that is, $f\big(cl_X(A)\big) = cl_Y\big(f(A)\big) \forall A \in \mathcal{P}(X)$.

3.1.4 Local c-compactness.

In general topological spaces, topologists tend to look at the local formulation of topological properties. This is because some important spaces in analysis may not possess the property of interest but may instead possess its local version (Dungdji, 1966). When a particular topological property is used to define another topological notion, then it's interesting to explore the kind of notion that will result from replacing the property with its local version. In the definition of perfect mappings, c-compactness is used to describe the one-point fibers. It would be interesting to find out what kind of mapping is obtained by requiring the one-point fibers to be locally c-compact instead of c-compact. Below is a formulation of local compactness in closure spaces.

Definition

An isotonic space (X, cl) is said to be locally c-compact if $\forall x \in X$ there exists $N \in \mathcal{N}(x)$ such that $x \in int(N)$ and $cl(N)$ is a c-compact subspace of X

This characterization of local compactness follows directly from the definition of local compactness in general topological spaces. In this context, local compactness is formulated in terms of a relatively compact neighborhood of each point of a Hausdorff space.

Proposition

Let $f: X \to Y$ be a one-to-one mapping of an isotonic space X to an isotonic space Y such that for every $y \in Y$, there exists $N \in \mathcal{N}_X(x)$ such that $f^{-1}(y) \in int(N)$ with $cl(N)$ a c-compact subspace of X. Then f is a locally c-compact mapping.

3.1.5 k-spaces

An interesting class of spaces that follows from local compactness and existence of a quotient mapping between two spaces is the class of k-spaces. This property of spaces is extended to closure spaces and specifically isotonic spaces in the definition below. The theorem that follows shows the invariance of local c-compactness and the property of being a k-space. It should be remarked here that according to section 2.5, every perfect mapping is a quotient mapping.

Definition

An isotonic space (X, cl) is called a k-space if it is both T_2 and an image of a locally c-compact space under a quotient mapping.

Having defined the class of k-spaces in isotonic spaces as well as remarked the fact that every perfect mapping is a quotient mapping, and then the following theorem can be formulated to show the invariance of these properties.

Theorem

Let $f: X \rightarrow Y$ be a perfect mapping of a locally c-compact isotonic space X to a T_2-isotonic space Y. Then Y is a locally c-compact k-space.

Proof:

Let $y \in Y$, and let $x \in f^{-1}(y)$. This is possible since a perfect mapping is surjective. Since X is locally c-compact, then there exists $N \in \mathcal{N}(x)$ such that $x \in int(N)$ and $cl(N)$ is a compact subset of X; $f(x) = f\big(int(N)\big)$ is a neighbourhood of $y \in Y$, that is $f(x) = y \in f\big(int(N)\big)$. Further, $f\big(cl(N)\big)$ is c-compact and closed since compactness is invariant of continuous onto maps. Therefore, $cl\left(f\big(cl(N)\big)\right) \subset f\big(cl(N)\big)$ is compact; hence Y is locally compact. Moreover, Y is a k-space since it is an image of a locally c-compact space under the quotient map f.

Local c-compactness is an invariant of perfect mappings.

Let X be a k-space. Then X is T_2 and is an image of a locally c-compact. Moreover, X is locally c-compact since local c-compactness is an invariant of a continuous mapping. Thus, if $f: X \rightarrow Y$is perfect, hence a quotient mapping, then Y is also a k-space. These facts give the following theorem.

Theorem

The property of being a k-space is invariant with respect to perfect mappings whenever the codomain of the map is a T_2-isotonic space.

3.2Invariance of topological properties

The motivation of topology has always been the study of properties that are fixed with respect to some continuous function. This enables the classification of equivalence classes of spaces and hence the transfer of problems from one space to another. In basic formulation, topological invariants work to simplify the solutions to problems that would otherwise be impossible or difficult to solve in a given space. The aim in this section is to characterize the properties that have not been defined in isotonic spaces and then investigate their behavior under perfect mappings.

3.2.1T_1-isotonic space

Since the assumption in this study is that perfect mappings are defined on T_2 spaces, then the investigation of the behavior of T_1 spaces can only be done under a bijective perfect mapping which according to theorem 3.1.3 is equivalent to a homeomorphism. This behavior of the T_1 condition is described in the following theorem:

Theorem

The property of being a T_1-isotonic space is invariant under a bijective perfect mapping.

Proof:

From theorem 4.2.2, if $f: X \to Y$ is a bijective perfect mapping, then;

$f\big(cl_X(A)\big) = cl_Y\big(f(A)\big) \forall A \in \mathcal{P}(X)$.

Let X be T_1 and $y \in Y$. This implies that $f^{-1}(y) \in X$. Therefore, $cl_X(\{f^{-1}(y)\}) \subseteq \{f^{-1}(y)\}$ since $X$is T_1. Hence, $cl_Y(\{y\}) = cl_Y\big(f(\{f^{-1}(y)\})\big) = f\big(cl_X(\{f^{-1}(y)\})\big) \subseteq f\{f^{-1}(y)\} = \{y\}$

Clearly $cl_Y(\{y\}) = \{y\}$, that is, every singleton set is closed. In conclusion, whenever X is T_1 and $f: X \to Y$ is a bijective perfect mapping, then $Y$is T_1.

3.2.2 T_2-isotonic spaces

Hausdorff topologies have the weakest kind of separation that will be considered in this work. All spaces will be assumed to be Hausdorff and hence alsoT_1. This is a somewhat convenient assumption since the class of perfect mappings is also defined at least on Hausdorff topology. The Hausdorff topology has several equivalents in general topological spaces. Nevertheless, these equivalent definitions have not been extended to closure. The theorem below shows the behavior of the T_2 condition under the effect of perfect mapping.

Theorem

The property of being a T_2- isotonic space is invariant under perfect mapping.

Proof:

Let X be $T_2, x, y \in Y$ such that $x \neq y$. This implies $f^{-1}(x) \neq f^{-1}(y)$ are two disjoint compact subsets of X. Therefore there exists $N' \in \mathcal{N}(x)\ \forall x \in f^{-1}(x), N'' \in \mathcal{N}(y)\ \forall y \in f^{-1}(y)$ with

$N^{'} \cap N^{''} = \emptyset$. The sets $Y - f(X - N^{'})$ and $Y - f(X - N'')$ are open in Y since f is closed, such that $x \in Y - f(X - N')$and $\in Y - f(X - N^{''})$. Moreover,

$[Y - f(X - N^{'})] \cap [Y - f(X - N^{''})] = Y - [f(X - N^{'}) \cup f(X - N^{''})] = Y - f[(X - N^{'}) \cup (X - N")] = Y - f[X - (N' \cap N")] = Y - f[X] = \emptyset.$

The Hausdorff property is neither preserved by continuous functions nor by continuous open functions. In general topological spaces, the invariance property of Hausdorff topologies hold only under closed bijections and under perfect mappings. The theorem above shows the consistency of perfect mappings both in general topological spaces and in isotonic spaces.

3.2.3 Regularity

The regularity axiom was formulated in a bid to resolve certain problems that required stronger separation axioms; for instance, the problem regarding the extension of a continuous function. The regular condition takes different formulations, both in general topological spaces and in isotonic spaces. Since the assumption in this study is that every space that will be considered has at least the Hausdorff topology, then the regularity condition coincides with the T_3 axiom. The following theorem shows that regularity (resp. T_3-axiom) is an invariant of perfect mappings.

Theorem

The property of being a regular (resp. T_3) isotonic spaces is invariant under perfect mapping.

Proof:

Let X be regular, $y \in Y$ and $A \subseteq Y$ such that $y \notin cl_Y(A)$. Therefore, $\forall x \in f^{-1}(y), x \notin f^{-1}\big(cl_Y(A)\big)$. Since $f^{-1}(y)$ is a c-compact subset of X then there exists a family $\{U_i : i \in I\}$ of sets in X such that $f^{-1}(y) \subseteq \bigcup_{i \in I} int(U_i) = U$. Similarly, since $f^{-1}\big(cl_Y(A)\big)$ is c-compact, then there exists a family $\{V_i : i \in I\}$ of sets in X such that $f^{-1}\big(cl_Y(A)\big) \subseteq \bigcup_{i \in I} int(V_i) = V$. Clearly the sets U and V are such that $U = int(U), V = int(V)$ and $int(U) \cap int(V) = \emptyset$. Since f is closed, then $U^* = Y - f\big(X - int(V)\big) \in \mathcal{N}(y)$ and $V^* = Y - f\big(X - int(U)\big) \in \mathcal{N}(A)$ are disjoint sets in Y containing y and A respectively. Therefore Y is a regular (resp. T_3) isotonic space.

3.2.4 Complete regularity

Complete regularity in isotonic spaces, and generally in closure spaces, takes a slightly different formulation from its definition in general topological spaces. In closure spaces, the concept of separation of sets is strengthened via the 'completely within' notion and hence instead of talking of existence of an Urysohn function, one talks of the existence of a neighborhood that is completely within another. However, both formulations are equivalent in the sense that besides the set-theoretic notion, both require the notion of a continuous real-valued function.

In the following theorem, the behavior of complete regularity under perfect mappings is explored.

Theorem

The property of being a completely regular isotonic space is invariant of perfect mappings.

Proof:

Let X be completely regular and $f: X \to Y$ be a perfect mapping. Let for every $y \in Y$, $N \in \mathcal{N}_Y(y)$. Since N is open in Y and f is a closed mapping, then $X - f^{-1}(Y - N)$ is open in X. Further,

$X - f^{-1}(Y - N) \in \mathcal{N}_X\big(f^{-1}(y)\big)$. Since X is completely regular then there exists $U \in \mathcal{N}_X\big(f^{-1}(y)\big)$ such that $U \Subset X - f^{-1}(Y - N)$. Moreover,

$f(U) \Subset f\big(X - f^{-1}(Y - N)\big) = f(X) - (Y - N) = N$. Clearly $f(U) \in \mathcal{N}_Y(y)$ such that $f(U) \Subset N$ hence Y is completely regular.

3.2.5 Normality

In the realm of closure spaces, various kinds of normal spaces exist. This has been made possible by the fact that characterization of normal closure spaces has been done both via separated sets as well as via an Urysohn separating function. In this section, the effect of perfect mapping is restricted to normal spaces and completely normal spaces since the other notions of normality are either trivial, for example *t-normality*, or implied from normal and completely normal spaces.

The following theorem shows the behavior of the normality axiom with respect to perfect mapping

Theorem

The property of being a normal space is invariant under perfect mapping

Proof:

Let $f: X \to Y$ be perfect. Let Y be such that $\forall\, A, B\ \ cl_Y(A) \cap cl_Y(B) = \emptyset$. Thus $f^{-1}\big(cl_Y(A)\big) \cap f^{-1}\big(cl_Y(B)\big) = f^{-1}\big(cl_Y(A) \cap cl_Y(B)\big) = \emptyset$. Since X is normal then there exists $U \in \mathcal{N}\left(f^{-1}\big(cl_Y(A)\big)\right)$ and $V \in \mathcal{N}\left(f^{-1}\big(cl_Y(B)\big)\right)$ such that $U \cap V = \emptyset$. Since U and V are open in X and f is closed, then $U^* = Y - f(X - U)$ and $V^* = Y - f(X - V)$ are open in Y such that $U^* \in \mathcal{N}\big(cl(A)\big)$ and $V^* \in \mathcal{N}\big(cl(B)\big)$ with $U^* \cap V^* = \emptyset$. Therefore, Y is normal.

Since the Hausdorff condition is assumed, then the T_4 axiom is invariant of perfect mappings as well.

3.2.6 Completely Normal Space

Completely normal spaces are classified under higher separation axioms. In this section, special interest is given to the behavior of the relativization operation (subspace operation) of the closure function cl on a subset of the underlying set of an isotonic space. Complete normality is always guaranteed for an isotonic space X whenever any of its subspace is normal in the subspace topology. The theorem below shows the relationship between a normal isotonic space and a completely normal isotonic space when the closure operator is relativized.

Theorem

Let (X, cl) be an isotonic space and $Y \subset X$. If Y is normal in the relativization of cl to Y, then X is completely normal.

Proof:

Since the property of being a normal space is not hereditary in isotonic spaces, then there is no guarantee that the subspace of a normal space X in the relativization of cl is itself normal. Let (X, cl) be normal and $C_Y: \mathcal{P}(Y) \to \mathcal{P}(Y)$ be the relativization of cl to Y. We show that if Y is normal, then every pair of semi-separated sets is separated.

Let $A, B \subseteq X$ be such that $cl(A) \cap B = A \cap cl(B) = \emptyset$ and (Y, C_Y) be normal. Setting $A' = Y \cap A$ and $B' = Y \cap B$, then we have $C_Y(A') = Y \cap cl(A)$ and $C_Y(B') = Y \cap cl(B)$. Since Y is normal then if $C_Y(A') \cap C_Y(B') = \emptyset$, there exists $U_Y \in \mathcal{N}\big(C_Y(A')\big)$ and $V_Y \in \mathcal{N}\big(C_Y(B')\big)$ such that $U_Y \cap V_Y = \emptyset$. But $U_Y = Y \cap U$ and $V_Y = Y \cap V$ where $U, V \subseteq X$ such that $U \in \mathcal{N}(A)$

and $V \in \mathcal{N}(B)$. Moreover $U_Y \cap V_Y = (Y \cap U) \cap (Y \cap V) = Y \cap (U \cap V) = \emptyset$. Therefore $U \cap V = \emptyset$ hence the semi-separated sets $A, B \subseteq X$ are separated. X is completely normal.

Theorem

Let $f: X \to Y$ be a perfect mapping and X be a completely normal isotonic space; then Y is also a completely normal space.

Proof

Let (X, cl) be a completely normal isotonic space and $f: X \to Y$ be a perfect mapping of X to an isotonic space Y. Let $A, B \subseteq Y$ such that $cl(A) \cap B = A \cap cl(B) = \emptyset$. Let $U \in \mathcal{N}_Y(A)$ and $V \in \mathcal{N}_Y(B)$. Now, $f^{-1}(cl(A)), f^{-1}(B) \in X$ such that;

$f^{-1}(cl(A)) \cap f^{-1}(B) = f^{-1}(cl(A) \cap B) = \emptyset$. Similarly, $f^{-1}(U) \in \mathcal{N}_X(f^{-1}(A))$ and $f^{-1}(V) \in \mathcal{N}_X(f^{-1}(B))$. Since X is completely normal then $f^{-1}(U) \cap f^{-1}(V) = \emptyset$. On the other hand, f is continuous, thus $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = \emptyset$. This implies that, $ff^{-1}(U \cap V) = U \cap V = \emptyset$. Therefore for $A, B \subseteq Y$ such that $cl(A) \cap B = A \cap cl(B) = \emptyset$ with $U \in \mathcal{N}_Y(A)$ and $V \in \mathcal{N}_Y(B)$, we have $U \cap V = \emptyset$. This means that Y is completely normal.

3.2.7 Perfect Normality.

Perfect normality has not been defined in closure spaces. Therefore, different characterizations are given under this section as modifications from topological spaces. A few basic concepts have to be carried over from general topological spaces before any meaningful definition of perfectly normal spaces can be given.

Preliminary definitions

A set G is called a G_σ-set if and only if $G = \bigcap_{i=1}^{\infty} A_i$, where $A_i = int(A_i)\ \forall i$. A set F is called an F_∂-set if $F = \bigcup_{i=1}^{\infty} B_i$ where $B_i = cl(B_i)\ \forall i$.

In topological spaces, the class of G_σ-set and F_∂-set constitute the most commonly studied classes of Borel sets.

Let $f: X \to I$ be a continuous function in the closure space (X, cl). A subset A of X is said to be a functionally closed subset if $A = f^{-1}(0)$. The complement of A is called a functionally open set subset.

The above definitions have been modified from topological spaces as defined by Engelking (1989). The aim of carrying over these topological concepts is so as to facilitate the definition of perfectly normal isotonic spaces.

An isotonic space (X, cl) is perfectly normal if X is normal and for every subset $A = cl(A)$ of X, A is a G_σ-set. That is for every closed set in X, $A = \bigcap_{i=1}^{\infty} M_i$, $M_i = int(M_i)\ \forall i$.
Equivalently, a normal isotonic space (X, cl) is perfectly normal if every open subset of X is an F_∂-set. That is, for every set $B = int(B)$, then $B = \bigcup_{i=1}^{\infty} N_i$, $N_i = cl(N_i)\ \forall i$.

Theorem
The isotonic space (X, cl) is perfectly normal if for every $A, B \subseteq X$ such that $A = cl(A)$, $B = cl(B)$ and $cl(A) \cap cl(B) = \emptyset$, then $\exists$ a function $f: X \to [0,1]$ that precisely separates A and B. That is $f^{-1}(0) = A$ and $f^{-1}(1) = B$.

Another definition of perfect mappings is provided via functionally closed (resp. open) sets. In this case, X is said to be perfectly normal if for every pair of disjoint closed subsets A and B, A is functionally closed while B is functionally open.
An isotonic space (X, cl) is T_6 if it is T_2 and perfectly normal.

Having defined perfectly normal spaces, investigation of their invariance under perfect mappings can follow.

Theorem
Perfect normality is invariant of perfect mapping

Proof
Let (X, cl) be a perfectly normal isotonic space and $f: X \to Y$ be a perfect mapping. Let $A, B \subseteq Y$ with $cl_Y(A) \cap cl_Y(B) = \emptyset$. Now, $f^{-1}(A), f^{-1}(B) \subseteq X$ such that,
$f^{-1}(cl_Y(A)) \cap f^{-1}(cl_Y(B)) = f^{-1}(cl_Y(A) \cap cl_Y(B)) = \emptyset$. Since X is perfectly normal, then there exists a continuous function $g: X \to [0,1]$ such that $g^{-1}(0) = f^{-1}(A)$ and $g^{-1}(1) = f^{-1}(B)$. Let $h = gof^{-1}$. Clearly, $h: Y \to [0,1]$ is continuous. Further $0 = gof^{-1}(A) = h(A)$ and $1 = gof^{-1}(B) = h(B)$. Thus, $h^{-1}(0) = h^{-1}(h(A)) = A$ and $h^{-1}(1) = h^{-1}(h(B)) = B$. $A, B \subseteq Y$ are precisely separated in Y and hence Y is perfectly normal.

Alternatively, let $f: X \to Y$ be a perfect mapping of a perfectly normal space X onto an isotonic space Y. Since normality is preserved by perfect mappings, then Y is normal. For every closed set $U \subset Y$, then $f^{-1}(U)$ is closed in X since f is a closed function and $f^{-1}(U) = \bigcap_{i=1}^{\infty} F_i$, where $F_{i's}$ are open in X. This is possible since X is perfectly normal and thus every open subset of X is a G_δ-set. Hence $f^{-1}(U) = U = f(\bigcap_{i=1}^{\infty} F_i) = \bigcap_{i=1}^{\infty} f(F_i)$. U is a G_δ-set in Y and therefore Y is perfectly normal.

Clearly, the T_6 condition is invariant of perfect mappings.

4.0 Conclusion

The results obtained in this thesis show that the invariance characteristic of separation axioms in closure spaces under homeomorphism extends to perfect mappings.

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