FURTHER CHARACTERIZATIONS AND SOME APPLICATIONS OF UPPER AND LOWER WEAKLY QUASI CONTINUOUS FUZZY MULTIFUNCTIONS

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Abstract

In his paper we characterize upper and lower weakly quasi continuous fuzzy multifunction's [3] be a new type of convergence of a net in a topological space and also characterize lower weakly quasi continuous fuzzy multifunction by a newly defined convergence of a fuzzy net. Again a new concept of regularity in a topological space has been introduced and characterized and using this regularity several applications of upper weakly quasi continuous fuzzy multifunctions have been shown.

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INTRODUCTION

This paper is a continuation of [3]. A fuzzy multifunction is a function which carries a point of an ordinary topological space X to a fuzzy set in a fuzzy topological space Y according to Papageorgiou [13] in 1985. After introducing fuzzy multifunction by Papageorgiou, a good many researchers have been inspired to study it and a number of fuzzy multifunctions have been introduced and studied. Papageorgiou defined fuzzy upper and lower inverses of a fuzzy multifunction. But the definition of lower inverse was not suitable for further study so that Mukherjee and Malakar[11] redefined it suitably in1991 and this new definition of lower inverse has been accepted to many of the researchers. In this paper we take the definition of upper inverse by Papageorgiou and the definition of lower inverse by Mukherjee and Malakar.

Throughout this paper, by (X, τ) or simply by X we shall mean an ordinary topological space, while (Y, τ_Y) or simply Y stands for a fuzzy topological space (fts, for short) in the sense of Chang [5]. The support of a fuzzy set A in Y will be denoted as $suppA$ [16] and is defined by $suppA = \{y \in Y : A(y) \neq 0\}$. A fuzzy point [14] with the singleton support $y \in Y$ and the value $\alpha\ (0 < \alpha \leq 1)$ at y will be denoted by y_α. 0_Y and 1_Y are the constant fuzzy sets taking respectively the constant values 0 and 1 on Y. The complement of a fuzzy set A in Y will be denoted by $1_Y \setminus A$ [16]. For two fuzzy sets A and B in Y, we write $A \leq B$ iff $A(y) \leq B(y)$, for each $y \in Y$, while we write AqB to mean A is quasi-coincident (q-coincident, for short) with B [14] if there exists $y \in Y$ such that $A(y) + B(y) > 1$; the negation of AqB is written as $A\bar{q}B$. clA and $intA$ of a set A in X (respectively, a fuzzy set A [16] in Y) respectively stand for the closure and interior of A in X (respectively, in Y). A fuzzy set B is called a quasi neighbourhood (q-nbd, for short) of a fuzzy set A in an fts Y if there is a fuzzy open set U in Y such that $AqU \leq B$. If, in addition, B is fuzzy open then B is called a fuzzy open q-nbd of A. In particular, a fuzzy (open) set B in Y is a fuzzy open q-nbd of a fuzzy point y_α in Y if $y_\alpha qU \leq B$, for some fuzzy open set U in Y. A set A (or a fuzzy set A) in a topological space X (in an fts Y) is said to be semiopen [10] (fuzzy semiopen [1]) if there exists an open set (respectively a fuzzy open set) U in X (in Y) such that $U \subseteq A \subseteq clU$ (resp. $U \leq A \leq clU$), or equivalently, if $A \subseteq cl\,intA$ (resp. $A \leq cl\,intA$). By $SO(X)$ (resp. $FSO(Y)$) we mean the set of all semiopen (resp. fuzzy semiopen) sets in X (in Y). The complement of a semiopen set (resp. fuzzy semiopen set) in X (resp. in Y) is called a semiclosed (fuzzy semiclosed) set. We mean semiclosure (resp. fuzzy semiclosure) of a set A in X (resp. of a fuzzy set A in Y), to be written as $sclA$, which is the union of all points (resp. fuzzy points) x in X (resp. y_α in Y) such that for every semiopen set (resp. fuzzy semiopen set) U in X (in Y) with $x \in U$ (resp. $y_\alpha qU$), it follows that $U \cap A \neq \emptyset$ [6] (resp. UqA [9]). A set A in X (resp. a fuzzy set A in Y) is semiclosed (fuzzy semiclosed) iff $A = scl\ A$.

1. SOME WELL KNOWN DEFINITIONS AND THEOREMS

In this section, we recall some definitions and theorems for ready references.

DEFINITION 1.1 [13]. Let (X, τ) and (Y, τ_Y) be respectively an ordinary topological space and an fts. We say that $F : X \to Y$ is a fuzzy multifunction if corresponding to each $x \in X$, $F(x)$ is a unique fuzzy set in Y.

Henceforth by $F : X \to Y$ we shall mean a fuzzy multifunction in the above sense.

DEFINITION 1.2 [13, 11]. For a fuzzy multifunction $F : X \to Y$, the upper inverse F^+ and lower inverse F^- are defined as follows:

For any fuzzy set A in Y, $F^+(A) = \{x \in X : F(x) \leq A\}$ and $F^-(A) = \{x \in X : F(x)qA\}$.

There is a following relationship between the upper and the lower inverses of a fuzzy multifunction.

THEOREM 1.3 [11]. For a fuzzy multifunction $F : X \to Y$, we have $F^-(1_Y \backslash A) = X \backslash F^+(A)$, for any fuzzy set A in Y.

We now recall the following two definitions for ready references.

DEFINITION 1.4 [3]. A fuzzy multifunction $F : X \to Y$ is said to be

(a) Fuzzy upper weakly quasi continuous (f.u.w.q.c., for short) (resp. fuzzy upper quasi continuous, f.u.q.c., for short) at a point $x \in X$ if for each open set U in X containing x and each fuzzy open set V in Y containing $F(x)$, there exists a non-empty open set G in X such that $G \subseteq U$ and $F(G) \leq clV$ (resp., $F(G) \leq V$),

(b) Fuzzy lower weakly quasi continuous (f.l.w.q.c., for short) (resp. fuzzy lower quasi continuous, f.l.q.c., for short) at a point $x \in X$ if for each open set U in X containing x and each fuzzy open set V in Y with $F(x)qV$, there exists a non-empty open set G in X such that $G \subseteq U$ and $F(g)qcl\, V$ (resp. $F(g)qV$), for each $g \in G$,

(c) f.u.w.q.c. (f.l.w.q.c.) on X if F has the corresponding property at each point x of X.

DEFINITION 1.5 [2]. A fuzzy multifunction $F : X \to Y$ is said to be

(a) fuzzy upper almost quasi continuous (f.u.a.q.c., for short) at a point $x \in X$, if for each open set U in X containing x and each fuzzy open set V in Y containing $F(x)$, there exists a non-empty open set G in X such that $G \subseteq U$ and $F(G) \leq scl\, V$,

(b) fuzzy lower almost quasi continuous (f.l.a.q.c., for short) at a point $x \in X$, if for each open set U in X containing x and each fuzzy open set V in Y with $F(x)qV$, there exists a non-empty open set G in X such that $G \subseteq U$ and $F(g)qscl\, V$, for all $g \in G$,

(c) f.u.a.q.c. (f.l.a.q.c.) on X if F has the corresponding property at each point x of X.

We now recall the following theorem from [3].

THEOREM 1.6. A fuzzy multifunction $F : X \to Y$ is f.l.w.q.c. at a point $x \in X$ iff for any fuzzy open set
V in Y with $F(x)qV$, there exists $U \in SO(X)$ with $x \in U$ such that $F(u)q\, cl\, V$, for each $u \in U$.

THEOREM 1.7. A fuzzy multifunction $F : X \to Y$ is f.u.w.q.c. at a point $x \in X$ iff for any fuzzy open set V in Y containing $F(x)$, there exists $U \in SO(X)$ with $x \in U$ such that $F(U) \leq cl\, V$.

We recall the following theorem from [2] for ready reference.

THEOREM 1.8. A fuzzy multifunction $F : X \to Y$ is f.u.a.q.c. at a point $x \in X$ iff for any fuzzy open set V in Y containing $F(x)$, there exists $U \in SO(X)$ with $x \in U$ such that $F(U) \leq scl\, V$.

2. CHARACTERIZATIONS OF UPPER AND LOWER WEAKLY QUASI CONTINUOUS FUZZY MULTIFUNCTIONS VIA A NET (FUZZY NET) IN X(INY)

In this section a new type of convergence of a net in an ordinary topological space has been introduced and we characterize upper and lower weakly quasi continuous fuzzy multifunctions via this newly defined convergence of a net. Also we define fuzzy inferior limit points of a fuzzy net and also a new type of convergence of a fuzzy net and characterize lower weakly quasi continuous fuzzy multifunctions via these new concepts.

DEFINITION 2.1. A net $\{S_n : n \in (D, \geq)\}$ in a topological space X with the directed set $(D, \geq)$ as the domain, is said to s-converge to a point $x \in X$ if for each semiopen set $U \subseteq X$ containing x, there exists $m \in D$ such that $S_n \in U$, for all $n \geq m$ $(n \in D)$.

THEOREM 2.2. A fuzzy multifunction $F : X \to Y$ is f.l.w.q.c. on X, iff for each point $x \in X$, if $\{S_n : n \in (D,\geq)\}$ is a net in X, s-converging to x, then for each fuzzy open set V in Y with $F(x)qV$, there exists $m \in D$ such that $F(S_n)qclV$, for all $n \geq m$ $(n \in D)$.

Proof. Suppose that$F$ is f.l.w.q.c. Let $\{S_n : n \in (D,\geq)\}$ be a net in X, s-converging to $x \in X$ and V be a fuzzy open set in Y with $F(x)qV$. Then $x \in F^-(V)$. By Theorem 1.6, there exists $U \in SO(X)$ containing x such that $U \subseteq F^-(clV)$. Then by Definition 2.1, there exists $m \in D$ such that $S_n \in U$, for all $n \geq m$ $(n \in D) \Rightarrow S_n \in F^-(clV)$, for all $n \geq m \Rightarrow F(S_n)qclV$, for all $n \geq m$.

Conversely, suppose that F is not f.l.w.q.c. at some point $x \in X$. Then there exists a fuzzy open set V in Y with $F(x)qV$ such that for each $U \in SO(X)$ containing x such that $F(x_U)\bar{q}clV$, for some $x_U \in U$. Let $(D,\geq)$ be the directed set consisting of all pairs (x_U,U) with $(x_U,U) \geq (x_W,W)$ iff $U \subseteq W$ (U,W being semiopen sets in X containing x and $F(x_U)\bar{q}clU$, $F(x_W)\bar{q}clW$) and consider the net $S(x_U,U) = x_U$ in X. Then evidently, the net $\{S_n : n \in (D,\geq)\}$ is s-convergent to x, but $F(S_n)\bar{q}clV$, for each $n \in D$, contradiction.

THEOREM 2.3. A fuzzy multifunction $F : X \to Y$ is f.u.w.q.c. iff for each point $x \in X$ if $\{S_n : n \in (D,\geq)\}$ is a net in X which is s-convergent to x, then for each fuzzy open set V in Y with $x \in F^+(V)$, there exists $m \in D$ such that $F(S_n) \leq clV$, for all $n \geq m$ ($n \in D$).

Proof. Suppose that F is f.u.w.q.c. Let $\{S_n : n \in (D,\geq)\}$ be a net in X, s-converge to $x \in X$ and V be a fuzzy open set in Y with $x \in F^+(V)$. By Theorem 1.7, there exists $U \in SO(X)$ containing x such that $U \subseteq F^+(clV)$. So by Definition 2.1, there exists $m \in D$ such that $S_n \in U$, for all $n \geq m$ ($n \in D$) $\Rightarrow S_n \in F^+(clV)$, for all $n \geq m$. Hence $F(S_n) \leq cl\,V$, for all $n \geq m$.

Conversely suppose that F is not f.u.w.q.c. at some point $x \in X$. Then there exists a fuzzy open set V in Y with $F(x) \in V$ such that for each $U \in SO(X)$ containing x such that $F(x_U) \not\leq clV$, for some $x_U \in U$. Let $(D,\geq)$ be the directed set consisting of all pairs (x_U, U) with $(x_U, U) \geq (x_V, V)$ iff $U \subseteq V$ (U,V being any semiopen sets in X containing x) and consider the net $S(x_U, U) = x_U$ in X. Then evidently, the net $\{S_n : n \in (D,\geq)\}$ is s-convergent to x, but $F(S_n) \not\leq clV$, i.e., $S_n \notin F^+(clV)$ for each $n \in D$, a contradiction.

DEFINITION 2.4. Let (Y, τ_Y) be an fts and $\{S_n : n \in (D,\geq)\}$ be a fuzzy net in Y. A fuzzy point y_α in Y is said to be a fuzzy θ-inferior limit point of the net if for each fuzzy open q-nbd V of y_α, there exists $m \in D$ such that $S_n q clV$, for all $n \geq m$ ($n \in D$). The union of all fuzzy θ-inferior limit points of the net will be denoted by $f - \theta - \inf - LiS_n$.

THEOREM 2.5. A fuzzy multifunction $F : X \to Y$ is f.l.w.q.c. at a point $x \in X$ iff for any net $\{S_n : n \in (D,\geq)\}$ in X, which s-converges to x, the net of fuzzy sets $\{F(S_n) : n \in (D,\geq)\}$ satisfies the relation $clF(x) \leq f - \theta - \inf - LiF(S_n)$.

Proof. Let F be f.l.w.q.c. at $x \in X$ and $\{S_n : n \in (D,\geq)\}$ be a net in X, which s-converge to x. If $y_\alpha \leq clF(x)$, then for all fuzzy open q-nbd V of y_α, $VqF(x) \Rightarrow x \in F^-(V)$. Since F is f.l.w.q.c., by Theorem 1.6, $U \subseteq F^{-}(clV)$, for some $U \in SO(X)$ containing x. Then by Definition 2.1, there exists $m \in D$ such that $S_n \in U \subseteq F^{-}(clV)$, for all $n \geq m$ ($n \in D$), i.e., $F(S_n)qclV$, for all $n \geq m$. Consequently, $y_\alpha \in f-\theta-\inf-LiF(S_n)$.

Conversely, let F be not f.l.w.q.c. at x. Then by Theorem 2.2, there exists a net $\{S_n : n \in (D,\geq)\}$ in X, which s-converges to x and there exists a fuzzy open set V in Y with $F(x)qV$, such that for any $n \in D$, we have $F(S_m)\bar{q}clV$, for some $m \geq n$ ($m \in D$). Now there exists $y \in supp(F(x) \cap V)$ such that $y_\alpha qV$ where $\alpha = [F(x)](y)$. Thus $y_\alpha \not\leq f-\theta-\inf-LiF(S_n)$. But $y_\alpha \leq F(x)$ and hence $y_\alpha \leq clF(x)$. Thus $clF(x) \not\leq f-\theta-\inf-LiF(S_n)$.

DEFINITION 2.6. Let (Y,τ_Y) be an fts and y_α be a fuzzy point in Y. Then a fuzzy net $\{S_n : n \in (D,\geq)\}$ is said to θ-converge to y_α, written as $S_n \xrightarrow{\theta} y_\alpha$, if for any fuzzy open q-nbd U of y_α, there exists $m \in D$ such that $S_n qclU$, for all $n \geq m$ ($n \in D$).

THEOREM 2.7. A fuzzy multifunction $F : X \to Y$ is f.l.w.q.c. at a point $x \in X$ iff for every fuzzy point $y_t \leq F(x)$ and for every net $\{x_\alpha : \alpha \in D\}$ in X, θ-converging to x, there exists a subnet $\{z_\beta : \beta \in E\}$ of $\{x_\alpha : \alpha \in D\}$ and there exists a fuzzy point $A^\beta \leq F(z_\beta)$, corresponding to each $\beta \in E$ such that the fuzzy net $\{A^\beta : \beta \in E\}$ is fuzzy θ-convergent to y_t.

Proof. Let F be f.l.w.q.c. at $x \in X$ and $\{x_\alpha : \alpha \in D\}$ be a net in X θ-converging to x. Also, let y_t be a fuzzy point such that $y_t \leq F(x)$. For each fuzzy open q-nbd V of y_t, by f.l.w.q.c. of F, there exists $U_V \in SO(X)$ containing x such that $F(x)qclV$, for all $x \in U_V$. Since $\{x_\alpha : \alpha \in D\}$ is θ-convergent to x, there exists $\alpha_V \in D$ such that $\beta \geq \alpha_V$ and $\beta \in D \Rightarrow x_\beta \in U_V \Rightarrow F(x_\beta)qclV$. Let $D_V = \{\beta \in D : \beta \geq \alpha_V\}$ and put $E = \bigcup_{V \in \Gamma}[D_V \times \{clV\}]$ where Γ denotes the system of all fuzzy open q-nbds of y_t. Clearly E is a directed set under "$\succeq$" given by $(\alpha, clV) \succeq (\alpha', clV')$ iff $\alpha \geq \alpha'$ in D and $clV \leq clV'$. For any β (=(α, clV))$\in E$, set $z_\beta = x_\alpha$. Then $\{z_\beta : \beta \in E\}$ is a subnet of the net

$\{x_\alpha : \alpha \in D\}$. Infact, for any $\alpha \in D$, consider any $(\alpha_V, clV) \in E$ and find $\alpha' \in D$ such that $\alpha' \geq \alpha, \alpha_V$. Then $(\alpha', clV) \in E$ such that whenever $(\alpha'', clW) \in E$ (where $W \in \Gamma$) with $(\alpha'', clW) \succeq (\alpha', clV)$, one has $\alpha'' \geq \alpha'$. For any β $(=(\alpha, clV)) \in E$, we have $F(x_\alpha)qclV$ so that $F(z_\beta)qclV$. Choose a fuzzy point $A^\beta \leq F(z_\beta)$ such that $A^\beta qclV$. Let $W \in \Gamma$ be arbitrary. Now $\beta(=(\alpha_W, clW)) \in E$ be such that $F(z_\beta)qclW$. If $\gamma(=(\alpha, clV')) \in E$ with $(\alpha, clV') \succeq (\alpha_W, clW)$, then $\alpha \geq \alpha_W$ and $clV' \leq clW$. Also $A^\gamma qclV' \leq clW \Rightarrow A^\gamma qclW$. Thus $\{A^\beta : \beta \in E\}$ is fuzzy θ-convergent to y_t.

Conversely, let F be not f.l.w.q.c. at x. Then there exists a fuzzy open set G in Y such that $x \in F^-(G)$, and for every $U \in SO(X)$ containing x, there exists $x_U \in U$ for which $F(x_U)\bar{q}clG$. Then $\{x_U : U \in \Gamma\}$ where Γ is the system of all semiopen sets in X containing x (directed by inclusion relation) is a net in X, which θ-converges to x. Let $y_t \leq F(x)$ be such that $y_t q\, G$ (such y_t exists as $F(x) q\, G$). By hypothesis, there is a subnet $\{z_W : W \in (\Sigma, \geqslant)\}$ of the net $\{x_U : U \in \Gamma\}$ and corresponding to each $W \in \Sigma$, there exists a fuzzy point $A^W \leq F(z_W)$ such that the fuzzy net $\{A^W : W \in \Sigma\}$ is fuzzy θ-convergent to y_t. Since G is a fuzzy open q-nbd of y_t, there exists $W_0' \in \Sigma$ such that $A^W cl\, G$, for all $W \geqslant W_0'$ $(W \in \Sigma)$ … (1). Now, since $\{z_W : W \in \Sigma\}$ is a subnet of $\{x_U : U \in \Gamma\}$, there exists a function $\varphi : \Sigma \to \Gamma$ which is cofinal and $z_W = x_{\varphi(W)}$, for each $W \in \Sigma$. Consider any $U \in \Gamma$. Then there exists $W_0'' \in \Sigma$ such that $\varphi(W) \subseteq U$ for each $W \geqslant W_0''$ in Σ $(W \in \Sigma)$. Let $W \in \Sigma$ be such that $W \geqslant W_0', W_0''$. Then $W \geqslant W_0'$ and hence $A^W q\, clG$. Also, $z_W = x_{\varphi(W)}$ and hence $F(z_W)\bar{q}clG$. But $A^W \leq F(z_W) \Rightarrow A^W \bar{q}\, clG$, contradicting (1).

3. SOME APPLICATIONS

Let us now recall some definitions for ready references.

DEFINITION 3.1 [5]. Let A be a fuzzy set in an fts Y. A collection $\mathcal{U}$ of fuzzy sets in Y is called a fuzzy cover of A if $\sup\{U(x) : U \in \mathcal{U}\} = 1$, for each $x \in suppA$. If, in addition, the members of $\mathcal{U}$ are fuzzy open, then $\mathcal{U}$ is called a fuzzy open cover of A. In particular, if $A = 1_Y$, we get the definition of fuzzy cover (open cover) of the fts Y.

DEFINITION 3.2 [8]. A fuzzy cover $\mathcal{U}$ of a fuzzy set A in an fts Y is said to have a finite subcover $\mathcal{U}_0$ if $\mathcal{U}_0$ is a finite subcollection of $\mathcal{U}$ such that $\cup \mathcal{U}_0 \geq A$. Clearly, if $A = 1_Y$ in particular, then the requirements on $\mathcal{U}_0$ is $\cup \mathcal{U}_0 = 1_Y$.

DEFINITION 3.3. An fts Y is said to be fuzzy compact [8] (resp., fuzzy almost compact [12], fuzzy s-closed [15]) if every open (resp., semiopen) cover of Y has a finite subcover (resp., finite proximate, finite semi-proximate) subcover, i.e., there exists a finite subcollection $\mathcal{U}_0$ of the fuzzy open (resp., semiopen) cover $\mathcal{U}$ such that $\mathcal{U}_0$ (resp., $\{clU : U \in \mathcal{U}_0\}$, $\{scl\, U : U \in \mathcal{U}_0\}$) is again a cover of Y.

DEFINITION 3.4 [7]. A topological space X is said to be semicompact if every semiopen cover of X has a finite subcover.

Using above definitions, now we get the following theorem.

THEOREM 3.5. Let $F : X \to Y$ be a surjective fuzzy multifunction and $F(x)$ be a compact set in Y for each $x \in X$. If F is f.u.w.q.c., and X is semicompact, then Y is fuzzy almost compact.

Proof. Let $\{A_\alpha : \alpha \in \Lambda\}$ be a fuzzy open cover of Y. Now for each $x \in X$, $F(x)$ is a fuzzy compact set in Y, and so there is a finite subset Λ_x of Λ such that $F(x) \leq \cup \{A_\alpha : \alpha \in \Lambda_x\}$. Let $A_x = \cup \{A_\alpha : \alpha \in \Lambda_x\}$. Then $F(x) \leq A_x$ and A_x is fuzzy open in Y. Since F is f.u.w.q.c., there exists $U_x \in SO(X)$ with $x \in U_x$ such that $F(U_x) \leq clA_x$. The family $\{U_x : x \in X\}$ is then a semiopen cover of X. As X is semicompact, there are finitely many points $x_1, x_2, \dots, x_n$ in X such that $X = \cup_{i=1}^{n} U_{x_i}$. As F is surjective, $1_Y = F(X) = F(\cup_{i=1}^{n} U_{x_i}) = \cup_{i=1}^{n} F(U_{x_i}) \leq \cup_{i=1}^{n} clA_{x_i} \leq \cup_{i=1}^{n} \cup_{\alpha \in \Lambda_i} clA_\alpha$. Hence Y is fuzzy almost compact.

Now we introduce a new concept of regularity in a topological space.

DEFINITION 3.6. A topological space (X, τ) is said to be s-regular if for each semiclosed set F in X and each $x \in X$ with $x \notin F$, there exist an open set U and a semiopen set V in X such that $x \in U$, $F \subseteq V$ and $U \cap V = \emptyset$.

THEOREM 3.7. For a topological space (X, τ), the following statements are equivalent :

(a) X is s-regular.

(b) For each $x \in X$ and each semiopen set U in X with $x \in U$, there exists an open set V in X such that $x \in V \subseteq scl\, V \subseteq U$.

(c) For each semiclosed set F in X, $\cap\{cl\, V : F \subseteq V$ and $V \in SO(X)\} = F$.

(d) For each set G in X and each semiopen set U in X with $G \cap U \neq \emptyset$, there exists $V \in \tau$ such that $G \cap V \neq \emptyset$ and $scl\, V \subseteq U$.

Proof. (a)⇒(b) : Let $x \in X$ and U, a semiopen set in X with $x \in U$. Then $x \notin X\backslash U$ which is semiclosed in X. Then by (a), there exist $V \in \tau$ and $W \in SO(X)$ such that $x \in V$, $X\backslash U \subseteq W$ and $V \cap W = \emptyset$. Then $V \subseteq X\backslash W \subseteq U$. Therefore, $x \in V \subseteq scl\, V \subseteq scl\,(X\backslash W) = X\backslash W \subseteq U$.

(b)⇒(a) : Let F be semiclosed set in X and $x(\in X) \notin F$. Then $x \in X\backslash F \in SO(X)$. By (b), there exists an open set V in X such that $x \in V \subseteq scl\, V \subseteq X\backslash F$. We take $U = X\backslash scl\, V$; so that $U \in SO(X)$ such that $F \subseteq U$ and $U \cap V = \emptyset$ and so (a) follows.

(b)⇒(c) : Let F be a semiclosed set in X. It is clear that $F \subseteq \cap\{cl\, V : F \subseteq V \in SO(X)\}$. Conversely, let $x \notin F$. Then $x \in X\backslash F \in SO(X)$. By (b), there exists $U \in \tau$ such that $x \in U \subseteq scl\, U \subseteq X\backslash F$. Put $V = X\backslash scl\, U$. Then $F \subseteq V$ and $V \cap U = \emptyset \implies x \notin cl\, V \implies \cap\{cl\, V : F \subseteq V \in SO(X)\} \subseteq F$.

(c) ⇒(b) : Let $x \in X$ and V be a semiopen set in X with $x \in V$. Then $x \notin X\backslash V$ which is semiclosed in X. By (c), there exists $G \in SO(X)$ such that $X\backslash V \subseteq G$ and $x \notin cl\, G$. Hence there exists $U \in \tau$ with $x \in U$ such that $U \cap G = \emptyset \implies U \subseteq X\backslash G \subseteq V$. Therefore, $x \in U \subseteq scl\, U \subseteq scl\,(X\backslash G) = X\backslash G \subseteq V$.

(c)⇒(d) : Let $G \subseteq X$ and $U \in SO(X)$ with $G \cap U \neq \emptyset$. Let $x \in G \cap U$. Then$x \in G$ and $x \in U$ and so $x \notin X\backslash U$ which is semiclosed in X. By (c), there exists $W \in SO(X)$ such that $X\backslash U \subseteq W$ and $x \notin cl\, W$. Then $x \in X\backslash cl\, W$. Put $V = X\backslash cl\, W$. Then $V \in \tau$ and $x \in V$ and so $x \in G \cap V \implies G \cap V \neq \emptyset$. Now, $V = X\backslash cl\, W \subseteq X\backslash W$. Then $scl\, V \subseteq scl\,(X\backslash W) = X\backslash W \subseteq U$.

(d)⇒(b) : Obvious.

THEOREM 3.8. Let $F : X \to Y$ be a surjective fuzzy multifunction and $F(x)$ be a fuzzy compact set in Y for each $x \in X$. If F is f.u.w.q.c. and X is s-regular and compact, then Y is fuzzy almost compact.

Proof. Let $\{A_\alpha : \alpha \in \Lambda\}$ be a fuzzy open cover of Y. Now for each $x \in X$, $F(x)$ is a fuzzy compact set in Y, and so there is a finite subset Λ_x of Λ such that $F(x) \leq \cup\{A_\alpha : \alpha \in \Lambda_x\}$. Let $A_x = \cup\{A_\alpha : \alpha \in \Lambda_x\}$. Then

$F(x) \leq A_x$ and A_x is fuzzy open in Y. Since F is f.u.w.q.c., there exists $U_x \in SO(X)$ with $x \in U_x$ such that $F(U_x) \leq clA_x$. As X is s-regular and $x \in U_x \in SO(X)$, by Theorem 3.7, there exists an open set V_x in X such that $x \in V_x \subseteq scl\, V_x \subseteq U_x$. Then $\{V_x : x \in X\}$ is an open cover of $X$and so by compactness of X, there are finitely many points $x_1, x_2, \ldots, x_n$ in X such that $X = \bigcup_{i=1}^{n} V_{x_i}$. As F is surjective, $1_Y = F(X) = F(\bigcup_{i=1}^{n} V_{x_i}) = \bigcup_{i=1}^{n} F(V_{x_i}) \leq \bigcup_{i=1}^{n} clA_{x_i} \leq \bigcup_{i=1}^{n} \bigcup_{\alpha \in \Lambda_i} clA_\alpha$. Hence Y is fuzzy almost compact.

DEFINITION 3.9 [15]. An fts Y is said to be fuzzy s-closed space if every semiopen cover of Y has a finite semi-proximate subcover for Y.

THEOREM 3.10. Let $F : X \to Y$ be a surjective fuzzy multifunction and $F(x)$ be a fuzzy compact set in Y for each $x \in X$. If F is f.u.a.q.c. and X is semicompact, then Y is fuzzy s-closed space.

Proof. Let $\{A_\alpha : \alpha \in \Lambda\}$ be a fuzzy semiopen cover of Y. Now for each $x \in X$, $F(x)$ is a fuzzy compact set in Y, and so there is a finite subset Λ_x of Λ such that $F(x) \leq \cup \{A_\alpha : \alpha \in \Lambda_x\}$. Let $A_x = \cup \{A_\alpha : \alpha \in \Lambda_x\}$. Then $F(x) \leq A_x$ and A_x is fuzzy open in Y. Since F is f.u.a.q.c., there exists $U_x \in SO(X)$ with $x \in U_x$ such that $F(U_x) \leq sclA_x$. The family $\{U_x : x \in X\}$ is then a semiopen cover of X. As X is semicompact, there are finitely many points $x_1, x_2, \ldots, x_n$ in X such that $X = \bigcup_{i=1}^{n} U_{x_i}$. As F is surjective, $1_Y = F(X) = F(\bigcup_{i=1}^{n} U_{x_i}) = \bigcup_{i=1}^{n} F(U_{x_i}) \leq \bigcup_{i=1}^{n} sclA_{x_i} \leq \bigcup_{i=1}^{n} \bigcup_{\alpha \in \Lambda_i} sclA_\alpha$. Hence Y is fuzzy s-closed space.

In a similar manner, we can set the following theorems.

THEOREM 3.11. Let $F : X \to Y$ be a surjective fuzzy multifunction and $F(x)$ be a fuzzy compact set in Y for each $x \in X$. If F is f.u.q.c. and X is semicompact, then Y is fuzzy compact.

THEOREM 3.12. Let $F : X \to Y$ be a surjective fuzzy multifunction and $F(x)$ be a fuzzy compact set in Y for each $x \in X$. If F is f.u.a.q.c. (resp., f.u.q.c.) and X is s-regular and compact, then Y is fuzzy s-closed space (resp., fuzzy compact space).

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