A FOCUS ON FIXED POINT THEOREM IN BANACH SPACE

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Abstract

In this paper we present a fixed point theorem with the help of self mapping which satisfy the contractive type of condition in Banach Space. Its purpose is to change the contractive condition by D.P. Shukla & Shivkant Tiwari [4]

Keywords :

Fixed point, self maps, contraction mapping, Banach Space, Cauchy Sequence, Convex Set.

Introduction :

In (1979) Fisher gave the contractive condition for a mapping $S : X \to X$

$$[d(Sx,Ty)]^2 \leq \alpha d(x,Sx)d(y,Sy) + \beta d(x,Sy)d(y,Sx)$$

For all x, y $\in$ X and $0 \leq \alpha < 1$ and $\beta \geq 0$

In 2012 D.P. Shukla [4] established a fixed point theorem satisfying the following condition

$$[d(Sx,Sy]^2 \leq \alpha.\min \begin{bmatrix} \frac{1}{5}\{d(x,Sx)d(x,Sy)+d(x,Sy)d(y,Sx)\}, \\ \frac{1}{5}\{d(x,Sx)d(x,Sy)+d(x,Sx)d(y,Sx)\}, \\ \frac{1}{5}\{d(x,Sy)d(y,Sx)+d(x,Sx)d(y,Sx)\} \end{bmatrix}$$

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Definition and Preliminaries :

- **Banach Space :** A normed linear space which is complete as a metric space is called a Banach Space.
- **Contacting Mapping :** If (X, d) be a complete metric space. A mapping S: X→ X is

  called a contacting mapping if there exists a real number α with

  0 ≤ a <1 such that $d(Sx,Sy) \le \alpha d(x,y) \le d(x,y)$, for every x,y ∈ X

  Thus in a contracting mapping the distance between the images of any two pints is less then the distance between the points.
- **Fixed point :** Let X be a non empty set and Let S : X→ X, for all x ∈ X

  Such that Sx = x, for all x ∈ X

  That is S maps X to itself.

  Then x is called fixed point of the mapping S
- **Convex Set :** A non empty subset X of a Bonach Space is said to be convex

  if (1-α) x+αy ∈ X, for all x,y ∈ X

  where α is any real such that 0 ≤ α<1

Theorem :

Let X be a closed and convex subset of a Banach Space and let S be a self mapping of X into itself which satisfies the following condition :

$$[d(Sx,Sy)]^2 \le \alpha \max \begin{bmatrix} \frac{1}{4} d(x,Sx) d(x,STx), \\ \frac{1}{4} d(x,Sx) d(Tx,Sx), \\ \frac{1}{4} d(x,STx) d(Tx,Sx), \end{bmatrix}, \text{ For all } x \in X$$

And $y \in \{Sx, Tx, STx\}$ & $0 \le \alpha < 1$

Where T is self mapping in X such that

$$Tx = \frac{x+Sx}{2} \text{.......................................}(2)$$

Then S has a fixed point

Proof- By the definition of metric space

$$d(x,Sx) = \|x - Sx\|$$
$$= \|x + x - Sx - x\|$$
$$= 2\left\|\frac{2x-(Sx+x)}{2}\right\|$$
$$= 2\left\|x - \frac{Sx+x}{2}\right\|$$
$$= 2\|x - Tx\|$$
$$= 2\|x - Tx\|, \; by(2)$$
$$= 2d(x,Tx) \text{ ....}(3)$$

Now $d(Sx, Tx) = \|Sx - Tx\|$

$$= \left\|Sx - \frac{x+Sx}{2}\right\|, by(2)$$
$$= \left\|\frac{Sx-x}{2}\right\|$$
$$= \frac{1}{2}\|x - Sx\|$$
$$= \frac{1}{2}d(x,Sx)\text{...................}(4)$$

Again $d(Sx,Tx) = \frac{1}{2}d(x,Sx)$

$$= \frac{1}{2}[2d(x,Tx)], by \; (3)$$

$$d(Sx,Tx) = d(x,Tx)\text{...}(5)$$

Now taking A =2[(Tx-STx)+STx]

$$=2\left[\frac{x+Sx}{2}\text{-STx}\right] + STx$$
$$= x+Sx-STx \text{ ....}(6)$$

Now d (A, STx) = $\|A - STx\|$

$= \|x + Sx - STx - STx\|$, by(6)

$= \|2\,Tx - 2STx\|,\ by 2$

$= \|x + Sx - STx - STx\|$, by(6)

$= \|2\,Tx - 2STx\|,\ by 2$

$= 2\|Tx - STx\|$

$= 2\ d(Tx, STx)$

$= 2\ .2\ d(Tx, TTx)$, by(3)

$= 4\ d(Tx, T^2x)$..(7)

$Now\ \ d(A, STx)\ \leq d(A, Sx) + d(Sx, STx),\ \ \ by\ triangular\ Inequality$

$= \|A - Sx\| + d(Sx, STx)$

$= \|x + Sx - STx - Sx\| + d(Sx, STx),\ by(6)$

$= \|(x - Sx) + (Sx - ST)\| + d(Sx, STx)$

$\leq \|x - Sx\| + \|Sx - STx\| + d(Sx, STx)$

$= d(x, Sx) + d(Sx, STx) + d(Sx, STx)$

$= d(x, Sx) + 2d(Sx, STx)$

$=> d(A, STx) \leq d(x, Sx) + 2d(Sx, STx)$..(8)

By (7) & (8)

$4d(Tx, T^2x) \leq d(x, Sx) + 2d(Sx, STx)$

$= 2d(x, Tx) + 2d(Sx, STx),\ By\ (3)$

$=> 4d(Tx, T^2x) \leq 2d(x, Tx) + 2d(Sx, STx)$

$2d(Tx, T^2x) \leq d(x, Tx) + d(Sx, STx)$..(9)

Now from (1)

$$[d(Sx, Sy)]^2\ \ \leq \alpha.\max\begin{bmatrix} \frac{1}{4}d(x, Sx).d(x, STx), \\ \frac{1}{4}d(x, Sx).d(Tx, Sx), \\ \frac{1}{4}d(x, STx).d(Tx, Sx) \end{bmatrix}$$

$$=> d(Sx,STx) \leq \alpha.\ \max \begin{bmatrix} \frac{1}{4}d(x,Sx)[d(x,Sx)+d(Sx,STx), \\ \frac{1}{4}d(x,Sx)\frac{1}{2}(x,Sx), \\ \frac{1}{4}[d(Sx,Tx)(d(x,Sx)+d(Sx,STx) \end{bmatrix} \text{ ByTriangular Inequality \& By } (4) \text{ \& } y=Tx$$

$$=\alpha.\max \begin{bmatrix} \frac{1}{4}[(d(x,Sx)^2+d(x,Sx)d(Sx,STx)], \\ \frac{1}{8}[d(x,Sx)]^2 \\ \frac{1}{4}[\frac{1}{2}d(x,Sx)[d(x,Sx)+d(Sx,STx)] \end{bmatrix}$$

$$=\alpha.\max \begin{bmatrix} \frac{1}{4}[(d(x,Sx)]^2+d(x,Sx)d(Sx,STx), \\ \frac{1}{2}[\frac{1}{4}(d(x,Sx))^2], \\ \frac{1}{8}\{[d(x,Sx)]^2+d(x,Sx)d(Sx,STx)\} \end{bmatrix}$$

$$=\alpha \max \begin{bmatrix} 2\left[\frac{1}{8}(d(x,Sx)]^2+\frac{1}{8}d(x,Sx)d(Sx,STx)]\right], \\ \frac{1}{8}(d(x,Sx)]^2, \\ \frac{1}{8}(d(x,Sx)^2+\frac{1}{8}d(x,Sx)d(Sx,STx) \end{bmatrix}$$

$$=\alpha.2\left[\frac{1}{8}[d(x,Sx)]^2+\frac{1}{8}d(x,Sx)d(Sx,STx)\right]$$

$$=\frac{\alpha}{4}\left[[d(x,Sx)]^2+d(x,Sx)d(Sx,STx)\right]$$

$$=> 4[d(Sx,STx)]^2 \leq \alpha\left[[d(x,Sx)]^2+d(x,Sx)d(Sx,STx)\right]$$

$$=> 4[d(Sx,STx)]^2-\alpha d(x,Sx)d(x,STx)-\alpha[d(x,Sx)]^2 \leq 0$$

Which is quadratic equation

Then by the solution for equation $ax^2+bx+c=0$ is given by

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Hence $d(S,STx)-\frac{\alpha(x,Sx)\pm\sqrt{\alpha^2[d(x,Sx)]^2+16\alpha(d[(x,Sx)]^2}}{2\times 4}\le 0$

$d(S,STx)-\frac{\alpha(x,Sx)\pm d(x,Sx)\sqrt{\alpha^2+16\alpha}}{8}\le 0$

$d(S,STx)-\frac{2d(x,Tx)[\alpha+\sqrt{\alpha^2+16\alpha}}{8}\le 0$

$d(S,STx)-\frac{d(x,Tx)[\alpha+\sqrt{\alpha^2+16\alpha}}{4}\le 0,\ by(3)$, $on\ taking(+ve)\ sign$

$d(Sx,STx)-\varsigma d(x,Tx)\le 0$

$d(Sx,STx)\le\varsigma d(x,Tx)$..(10)

Where $\varsigma=\frac{\alpha+\sqrt{\alpha^2+16\alpha}}{4}$

Where $0\le\varsigma<1$

Because if $\varsigma<1$ then $\frac{\alpha+\sqrt{\alpha^2+16\alpha}}{4}<1$

=> $\alpha+\sqrt{\alpha^2+16\alpha}<4$

=> $\sqrt{\alpha^2+16\alpha}<4-\alpha$

=> $\alpha^2+16\alpha<(4-\alpha)^2$

=> $\alpha^2+16\alpha<16+\alpha^2-8\alpha$

=> $24\alpha<16$ => $\alpha<\frac{16}{24}$ => $\alpha<0.66<1$ => $\alpha<1$

Hence $\varsigma<1$

And also $0\le\alpha$ => $0\le\varsigma$ Hence $0\le\varsigma<1$

Now by (9) & (10)

$2d(Tx,T^2x)\le d(x,Tx)+\varsigma d(x,Tx)=(1+\varsigma)d(x,Tx)$

$d(Tx,T^2x)\le\frac{(1+\varsigma)}{2}d(x,Tx)$

Similarly

$$d(T^2x,T^3x) \le \frac{(1+\varsigma)}{2} d(Tx,T^2x)$$

$$\le \frac{(1+\varsigma)}{2}\frac{(1+\varsigma)}{2} d(x,Tx) = \left(\frac{1+\varsigma}{2}\right)^2 d(x,Tx)$$

Similarly we can find

$$d(T^3x,T^4x) \le \left(\frac{1+\varsigma}{2}\right)^3 d(x,Tx)$$

$$and \quad d(T^4x,T^5x) \le \left(\frac{1+\varsigma}{2}\right)^4 d(x,Tx)$$

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$$d(T^nx,T^{n+1}x) \le \left(\frac{1+\varsigma}{2}\right)^n d(x,Tx) \quad \text{...(11)}$$

$$\because \varsigma < 1 \quad \Rightarrow 1+\varsigma < 2 \quad \Rightarrow \frac{(1+\varsigma)}{2} < 1$$

$$Then \quad \lim_{n\to\infty}\left(\frac{1+\varsigma}{2}\right)^n = 0$$

$\therefore$ *from euqtion* (11) $\quad d(T^nx,T^{n+1}x) \to 0 \quad as\ n \to \infty$

$d(T^nx,T^{n+1}x) < \varepsilon$ for $\varepsilon > 0$ $\ as\ n \to \infty$

$\therefore \{T^nx\}_{n=1}^{\infty}$ *is* a Cauchy Sequence in X. (By the defination of Cauchy Sequence)

But X is a Banach Space. Then by the property of completeness

$\therefore \{T^nx\}_{n=1}^{\infty}$ *is a convergent sequence in* X, *which* converges to a fixed point.

Let there exists a point ω in X such that $\lim_{n\to\infty} T^nx = \omega$..(12)

Now consider $d(\omega,S\omega) \le d(\omega,T^{n+1}\omega) + d(T^{n+1}\omega,S\omega)$ *by* triangular, *Inequality*

$$= d(\omega,T^{n+1}\omega) + d(TT^n\omega,S\omega)$$

$$= d(\omega,T^{n+1}\omega) + \left\|TT^n\omega - S\omega\right\|$$

$$= d(\omega, T^{n+1}\omega) + \left\| \frac{1}{2}(T^n\omega + ST^n\omega) - S\omega \right\|, \quad \text{by}\,(2)$$

$$= d(\omega, T^{n+1}\omega) + \left\| \frac{1}{2}T^n\omega - \frac{1}{2}S\omega + \frac{1}{2}ST^n\omega - \frac{1}{2}S\omega \right\|$$

$$\leq d(\omega, T^{n+1}\omega) + \frac{1}{2}\left\| ST^n\omega - S\omega \right\| + \frac{1}{2}\left\| T^n\omega - S\omega \right\|$$

$$= d(\omega, T^{n+1}\omega) + \frac{1}{2}d(ST^n\omega, S\omega) + \frac{1}{2}d(T^n\omega, S\omega)$$

$$= d(\omega, T^{n+1}\omega) + \frac{1}{2}\big(d(T^n\omega, S\omega) + d(S\omega, ST^n\omega)\big)$$

$$= d(\omega, T^{n+1}\omega) + \frac{1}{2}\big(d(T^n\omega, ST^n\omega)\big)$$

$$= d(\omega, T^{n+1}\omega) + d(T^n\omega, TT^n\omega), \; by (3)$$

$$= d(\omega, T^{n+1}\omega) + d(T^n\omega, T^{n+1}\omega)$$

$$= d(\omega, T^{n+1}\omega) + d(T^{n+1}\omega, T^n\omega)$$

$$= d(\omega, T^n\omega)$$

$$= d(\omega, T^n\omega) \quad \because\; X \text{ is convex.}$$

$\Rightarrow d(\omega, S\omega) \leq d(\omega, T^n\omega)$

$As \;\; n \to \infty \qquad then \;\; d(\omega, T^n\omega) \to 0, \quad by \;\; (12)$

$\therefore\; d(\omega, S\omega) \leq 0 \qquad \Rightarrow\; S\omega = \omega$

$\therefore\; S \;\; has \;\; a \;\; fixed \;\; \text{point}\; \omega \;\; \text{in X}$

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