STABILITY ANALYSIS OF MATHEMATICAL SYN-ECOLOGICAL MODEL COMPRISING OF PREY-PREDATOR, HOST-COMMENSAL, MUTUALISM AND NEUTRAL PAIRS-III

(TWO OF THE FOUR SPECIES ARE WASHED OUT STATES)

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Abstract

This investigation deals with a mathematical model of a four species (S_1, S_2, S_3 and S_4) Syn-Ecological system (Two of the four species are washed out states). S_2 is a predator surviving on the prey S_1. The predator S_2 is a commensal to the host S_3. The pairs S_2 and S_4, S_1 and S_3 are neutral. The mathematical model equations characterizing the syn-ecosystem constitute a set of four first order non-linear coupled differential equations. There are in all sixteen equilibrium points. Criteria for the asymptotic stability of six of the sixteen equilibrium points: Two of the four species are washed out states only are established in this paper. The linearized equations for the perturbations over the equilibrium points are analyzed to establish the criteria for stability and the trajectories illustrated.

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1. INTRODUCTION

Mathematical modeling is an important interdisciplinary activity which involves the study of some aspects of diverse disciplines. Biology, Epidemiodology, Physiology, Ecology, Immunology, Bio-economics, Genetics, Pharmocokinetics are some of those disciplines. This mathematical modeling has raised to the zenith in recent years and spread to all branches of life and drew the attention of every one. Mathematical modeling of ecosystems was initiated by Lotka [9] and by Volterra [18]. The general concept of modeling has been presented in the treatises of Meyer [11], Cushing [4], Paul Colinvaux [11], Freedman [5], Kapur [6, 7]. The ecological interactions can be broadly classified as Prey-Predation, Competition, Mutualism and so on. N.C. Srinivas [17] studied the competitive eco-systems of two species and three species with regard to limited and unlimited resources. Later, Lakshmi Narayan [8] has investigated the two species prey-predator models. Stability analysis of competitive species was carried out by Archana Reddy [3] while Acharyulu [1, 2] investigated Ammensalism between two species. Recently local stability analysis for a two-species ecological mutualism model has been investigated by present author et al [12, 13, 14, 15, 16]. Example for S_1, S_2, S_3 and S_4 are Insects, Insectivorous Plants (nephantis, drosera etc.), VAM associated with the plant roots, Soil bacteria respectively.

2. BASIC EQUATIONS

The model equations for a four species multi-system are given by a set of four non-linear ordinary differential equations as

(i) For S_1**:** The Prey of S_1 and Neutral to S_3

$$\frac{dN_1}{dt} = a_1N_1 - a_{11}N_1^2 - a_{12}N_1N_2 \qquad \text{…. (2.1)}$$

(ii) For S_2**:** The Predator surviving on S_1 and Commensal to S_3

$$\frac{dN_2}{dt} = a_2N_2 - a_{22}N_2^2 + a_{21}N_2N_1 + a_{23}N_2N_3 \qquad \text{…. (2.2)}$$

(iii) For S_3**:** The Host of S_2 and Mutual to S_4

$$\frac{dN_3}{dt} = a_3N_3 - a_{33}N_3^2 + a_{34}N_3N_4 \qquad \text{…. (2.3)}$$

(iv) For S_4**:** Mutual to S_3 and Neutral to S_2

$$\frac{dN_4}{dt} = a_4N_4 - a_{44}N_4^2 + a_{43}N_4N_3 \qquad \text{…. (2.4)}$$

with the following notation.

N_i (t): Population strengths of the species S_i at time t, i=1, 2, 3, 4.
a_i : The natural growth rates of S_i, i = 1,2,3,4
a_{12},a_{21} : Interaction (Prey-Predator) coefficients of S_1 due to S_2 and S_2 due to S_1
a_{13} : Coefficient for commensal for S_1 due to the Host S_3
a_{34}, a_{43} : Mutually interaction between S_3 and S_4
K_i: $\frac{a_i}{a_{ii}}$: Carrying capacities of S_i, i=1, 2, 3, 4.

Further the variables N_1, N_2, N_3, N_4 are non-negative and the model parameters a_1, a_2, a_3, a_4; a_{11}, a_{22}, a_{33}, a_{44}; a_{12}, a_{21}, a_{13}, a_{24} are assumed to be non-negative constants.

3. EQUILIBRIUM STATES:

The system under investigation has sixteen equilibrium states defined by

$$\frac{dN_i}{dt}=0,\ i=1,2,3,4 \qquad \ldots\ldots \quad (3.1)$$

are given in the following table.

I. Fully washed out state:

E1: $\overline{N_1}=0, \overline{N_2}=0, \overline{N_3}=0, \overline{N_4}=0$

II. States in which three of the four species are washed out and fourth is surviving

E2: $\overline{N_1}=0, \overline{N_2}=0, \overline{N_3}=0, \overline{N_4}=\frac{a_4}{a_{44}}$

E3: $\overline{N_1}=0, \overline{N_2}=0, \overline{N_3}=\frac{a_3}{a_{33}}, \overline{N_4}=0$

E4: $\overline{N_1}=0, \overline{N_2}=\frac{a_2}{a_{22}}, \overline{N_3}=0, \overline{N_4}=0$

E5: $\overline{N_1}=\frac{a_1}{a_{11}}, \overline{N_2}=0, \overline{N_3}=0, \overline{N_4}=0$

III. States in which two of the four species are washed out while the other two are surviving

E6: $\overline{N_1}=0, \overline{N_2}=0, \overline{N_3}=\frac{a_4a_{34}+a_3a_{44}}{a_{33}a_{44}-a_{34}a_{43}}, \overline{N_4}=\frac{a_3a_{43}+a_4a_{33}}{a_{33}a_{44}-a_{34}a_{43}}$

This state exists only when $a_{33}a_{44}-a_{34}a_{43}>0$

E7: $\overline{N_1}=0, \overline{N_2}=\frac{a_2}{a_{22}}, \overline{N_3}=0, \overline{N_4}=\frac{a_4}{a_{44}}$

E8: $\overline{N_1}=0, \overline{N_2}=\frac{a_3}{a_{22}}\frac{a_{23}}{a_{33}}+\frac{a_2}{a_{22}}, \overline{N_3}=\frac{a_3}{a_{33}}, \overline{N_4}=0$

E9: $\overline{N_1}=\frac{a_1}{a_{11}}, \overline{N_2}=0, \overline{N_3}=0, \overline{N_4}=\frac{a_4}{a_{44}}$

E10: $\overline{N_1}=\frac{a_1}{a_{11}}, \overline{N_2}=0, \overline{N_3}=\frac{a_3}{a_{33}}, \overline{N_4}=0$

E11: $\overline{N_1} = \frac{a_1 a_{22} - a_2 a_{12}}{a_{11} a_{22} + a_{12} a_{21}}, \overline{N_2} = \frac{a_1 a_{21} + a_2 a_{11}}{a_{11} a_{22} + a_{12} a_{21}}, \overline{N_3} = 0, \overline{N_4} = 0$

This state exists only when $a_1 a_{22} - a_2 a_{12} > 0$

IV. States in which one of the four species is washed out while the other three are surviving

E12: $\overline{N_1} = 0, \overline{N_2} = \frac{a_{23}(a_4 a_{34} + a_3 a_{44})}{a_{22}(a_{33} a_{44} - a_{34} a_{43})} + \frac{a_2}{a_{22}}, \overline{N_3} = \frac{a_4 a_{34} + a_3 a_{44}}{a_{33} a_{44} - a_{34} a_{43}},$

$\overline{N_4} = \frac{a_4 a_{33} + a_3 a_{43}}{a_{33} a_{44} - a_{34} a_{43}}$

This state exists only when $a_{33} a_{44} - a_{34} a_{43} > 0$

E13: $\overline{N_1} = \frac{a_1}{a_{11}}, \overline{N_2} = 0, \overline{N_3} = \frac{a_4 a_{34} + a_3 a_{44}}{a_{33} a_{44} - a_{34} a_{43}}, \overline{N_4} = \frac{a_4 a_{33} + a_3 a_{43}}{a_{33} a_{44} - a_{34} a_{43}}$

This state exists only when $(a_{33} a_{44} - a_{34} a_{43}) > 0$

E14: $\overline{N_1} = \frac{a_1 a_{22} - a_2 a_{12}}{a_{11} a_{22} + a_{12} a_{21}}, \overline{N_2} = \frac{a_1 a_{21} + a_2 a_{11}}{a_{11} a_{22} + a_{12} a_{21}}, \overline{N_3} = 0, \overline{N_4} = \frac{a_4}{a_{44}}$

This state exists only when $a_1 a_{22} - a_2 a_{12} > 0$

E15: $\overline{N_1} = \frac{\beta_4}{\beta_1}, \overline{N_2} = \frac{\beta_5}{\beta_1}, \overline{N_3} = \frac{a_3}{a_{33}}, \overline{N_4} = 0$

Where

$\beta_1 = a_{33}(a_{11} a_{22} + a_{12} a_{21}), \beta_4 = a_{33}(a_1 a_{22} - a_2 a_{12}) - a_3 a_{23} a_{12}$

$\beta_5 = a_{33}(a_1 a_{21} + a_2 a_{11}) + a_3 a_{23} a_{11}$

This state exists only when $\beta_4 > 0$

V. The co-existent state (or) Normal steady state

E16: $\overline{N_1} = \frac{\gamma_1 + a_{12} a_{23} \gamma_2}{\gamma_3 (a_{33} a_{44} - a_{34} a_{43})}, \overline{N_2} = \frac{\gamma_4 + a_{11} a_{23} \gamma_2}{\gamma_3 (a_{33} a_{44} - a_{34} a_{43})},$

$\overline{N_3} = \frac{a_4 a_{34} + a_3 a_{44}}{a_{33} a_{44} - a_{34} a_{43}}, \overline{N_4} = \frac{a_4 a_{33} + a_3 a_{43}}{a_{33} a_{44} - a_{34} a_{43}}$

where

$\gamma_1 = (a_1 a_{22} + a_2 a_{12})(a_{33} a_{44} - a_{34} a_{43}), \gamma_2 = a_3 a_{44} + a_4 a_{34}$

$\gamma_3 = a_{11} a_{22} + a_{12} a_{21}, \gamma_4 = (a_1 a_{21} - a_2 a_{11})(a_{33} a_{44} - a_{34} a_{43})$

This state exists only when $(a_1 a_{21} - a_2 a_{11}) > 0$ *and* $(a_{33} a_{44} - a_{34} a_{43}) > 0$.

The present paper deals with two of the four species are washed out states only. The stability of the other equilibrium states will be presented in the forth coming communications.

4. STABILITY OF TWO OF THE FOUR SPECIES WASHED OUT EQUILIBRIUM STATES

(Sl. Nos 6, 7, 8, 9, 10, 11 in the above Equilibrium states)

4.1 Stability of the Equilibrium State E6:

Let us consider small deviations $u_1(t), u_2(t), u_3(t), u_4(t)$ from the steady state i.e.

$$N_i(t) = \bar{N}_i + u_i(t), \; i = 1,2,3,4 \qquad \text{--- (4.1.1)}$$

Substituting (4.1.1) in (2.1), (2.2), (2.3), (2.4) and neglecting products and higher powers of u_1, u_2, u_3, u_4, we get

$$\frac{du_1}{dt} = a_1 u_1 \qquad \text{..... (4.1.2)} \qquad \frac{du_2}{dt} = l_2 u_2 \qquad \text{... (4.1.3)}$$

$$\frac{du_3}{dt} = -a_{33}\overline{N_3}u_3 + a_{34}\overline{N_3}u_4 \qquad \text{..... (4.1.4)} \qquad \frac{du_4}{dt} = a_{43}\overline{N_4}u_3 - a_{44}\overline{N_4}u_4 \qquad \text{.... (4.1.5)}$$

Here $l_2 = a_2 + a_{23}\overline{N_3}$ ... (4.1.6)

The characteristic equation of which is

$$(\lambda - a_1)(\lambda - l_2)\left[\lambda^2 + (a_{33}\overline{N_3} + a_{44}\overline{N_4})\lambda + (a_{33}a_{44} - a_{34}a_{43})\overline{N_3}\,\overline{N_4}\right] = 0 \qquad \text{... (4.1.7)}$$

The characteristic roots of (4.6.7) are

$$\lambda = a_1, \lambda = l_2, \lambda = \frac{-(a_{33}\bar{N}_3 + a_{44}\bar{N}_4) \pm \sqrt{(a_{33}\bar{N}_3 - a_{44}\bar{N}_4)^2 + 4a_{34}a_{43}\bar{N}_3\bar{N}_4}}{2}$$

Two roots of the equation (4.1.7) are positive and the other two roots are negative.
Hence the equilibrium state is **unstable**.

The solutions of the equations (4.1.2), (4.1.3), (4.1.4), (4.1.5) are

$$u_1 = u_{10}e^{a_1 t} \qquad \text{... (4.1.8)} \qquad u_2 = u_{20}e^{l_2 t} \qquad \text{... (4.1.9)}$$

$$u_3 = \left[\frac{u_{30}(\lambda_3 + a_{44}\bar{N}_4) + u_{40}a_{34}\bar{N}_3}{\lambda_3 - \lambda_4}\right]e^{\lambda_3 t} + \left[\frac{u_{30}(\lambda_4 + a_{44}\bar{N}_4) + u_{40}a_{34}\bar{N}_3}{\lambda_4 - \lambda_3}\right]e^{\lambda_4 t} \qquad \text{... (4.1.10)}$$

$$u_4 = \left[\frac{u_{40}(\lambda_3 + a_{33}\bar{N}_3) + u_{30}a_{43}\bar{N}_4}{\lambda_3 - \lambda_4}\right]e^{\lambda_3 t} + \left[\frac{u_{40}(\lambda_4 + a_{33}\bar{N}_3) + u_{30}a_{43}\bar{N}_4}{\lambda_4 - \lambda_3}\right]e^{\lambda_4 t} \qquad \text{...(4.1.11)}$$

where $u_{10}, u_{20}, u_{30}, u_{40}$ are the initial values of u_1, u_2, u_3, u_4 respectively.

There would arise in all 576 cases depending upon the ordering of the magnitudes of the growth rates a_1, a_2, a_3, a_4 and the initial values of the perturbations $u_{10}(t), u_{20}(t), u_{30}(t), u_{40}(t)$ of the species S_1, S_2, S_3, S_4. Of these 576 situations some typical variations are illustrated through respective solution curves that would facilitate to make some reasonable observations. The solutions are illustrated in figures.

Case (i): If $u_{40} < u_{20} < u_{10} < u_{30}$ and $l_2 < a_1 < a_3 < a_4$

In this case initially the host (S_3) of S_2 dominates the Prey (S_1), the Predator (S_2) and S_4 till the time instant $t^*_{13}, t^*_{23}, t^*_{43}$ respectively and thereafter the dominance is reversed.

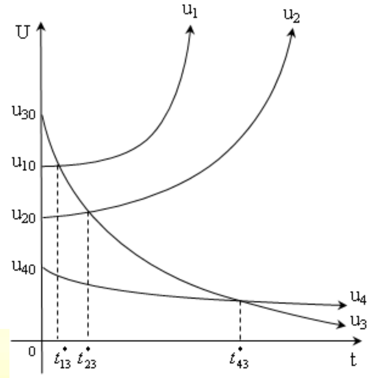


Fig. 1

Case (ii): If $u_{20} < u_{30} < u_{40} < u_{10}$ and $a_1 < a_3 < l_2 < a_4$

In this case initially the Prey (S_1) dominates the Predator (S_2) till the time instant t^*_{21} and thereafter the dominance is reversed. Also S_4 dominates the Predator (S_2) till the time instant t^*_{24} and the dominance gets reversed thereafter. Similarly the host (S_3) of S_2 dominates the Predator (S_2) till the time instant t^*_{23} and thereafter the dominance is reversed.

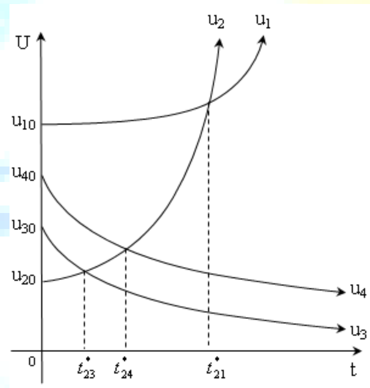


Fig. 2

4.2 Stability of the Equilibrium State E_7:

Substituting (4.1.1) in (2.1), (2.2), (2.3), (2.4) and neglecting products and higher powers of u_1, u_2, u_3, u_4, we get

$$\frac{du_1}{dt} = r_1 u_1 \quad \ldots (4.2.1)$$

$$\frac{du_2}{dt} = -a_2 u_2 + \frac{a_{21} a_2}{a_{22}} u_1 + \frac{a_{23} a_2}{a_{22}} u_3 \quad \ldots (4.2.2)$$

$$\frac{du_3}{dt} = l_3 u_3 \quad \ldots (4.2.3) \qquad \frac{du_4}{dt} = -a_4 u_4 + \frac{a_{43} a_4}{a_{44}} u_3 \quad \ldots (4.2.4)$$

Here $r_1 = a_1 - \frac{a_{12} a_2}{a_{22}}$ ... (4.2.5)

$$l_3 = a_3 + \frac{a_{34} a_4}{a_{44}} \quad \ldots (4.2.6)$$

The characteristic equation of which is

$$(\lambda - r_1)(\lambda + a_2)(\lambda - l_3)(\lambda + a_4) = 0 \quad \text{... (4.2.7)}$$

Case (A): When $r_1 < 0$ (i.e., when $a_1 < \frac{a_{12}a_2}{a_{22}}$)

The roots $r_1, -a_2, -a_4$ are negative and l_3 is positive.

Hence the equilibrium state is **unstable.**

The solutions of the equations (4.2.1) (4.2.2), (4.2.3), (4.2.4) are

$$u_1 = u_{10}e^{r_1 t} \quad \text{... (4.2.8)}$$

$$u_2 = [u_{20} - \frac{a_{21}a_2u_{10}}{a_{22}(r_1 + a_2)} - \frac{a_{23}a_2u_{30}}{a_{22}(l_3 + a_2)}]e^{-a_2 t} + \frac{a_{21}a_2u_{10}}{a_{22}(r_1 + a_2)}e^{r_1 t} + \frac{a_{23}a_2u_{30}}{a_{22}(l_3 + a_2)}e^{l_3 t} \quad \text{... (4.2.9)}$$

$$u_3 = u_{30}e^{l_3 t} \quad \text{... (4.2.10)}$$

$$u_4 = [u_{40} - \frac{a_{43}a_4u_{30}}{a_{44}(l_3 + a_4)}]e^{-a_4 t} + \frac{a_{43}a_4u_{30}}{a_{44}(l_3 + a_4)}e^{l_3 t} \quad \text{... (4.2.11)}$$

The solutions are illustrated in figures.

Case (i): If $u_{30} < u_{40} < u_{20} < u_{10}$ and $a_2 < l_3 < a_4 < r_1$

In this case initially the Prey (S_1) dominates the Predator (S_2), S_4 and the host (S_3) of S_2 till the time instant t^*_{21}, t^*_{41}, t^*_{31} respectively and thereafter the dominance is reversed. Also the Predator (S_2) dominates the host (S_3) of S_2 till the time instant t^*_{32} and the dominance gets reversed thereafter. Similarly S_4 dominates the host (S_3) of S_2 till the time instant t^*_{34} and thereafter the dominance is reversed.

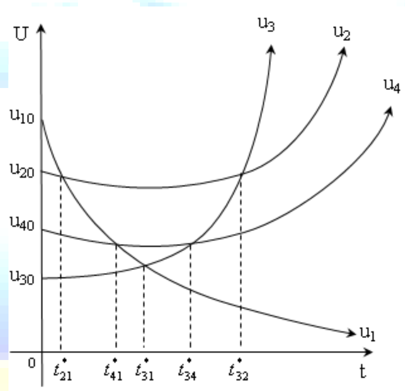


Fig. 3

Case (ii): If $u_{10} < u_{30} < u_{20} < u_{40}$ and $r_1 < a_2 < l_3 < a_4$

In this case initially S_4 dominates the Predator (S_2) and the host (S_3) of S_2 till the time instant t^*_{24}, t^*_{34} respectively and thereafter the dominance is reversed. Also the Predator (S_2) dominates the host (S_3) of S_2 till the time instant t^*_{32} and the dominance gets reversed thereafter.

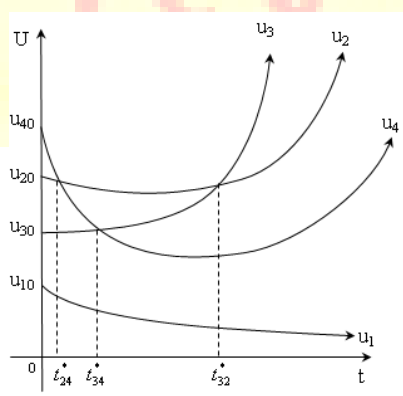


Fig. 4

Case (B): When $r_1 > 0$ (i.e., when $a_1 > \frac{a_{12}a_2}{a_{22}}$)

The roots $-a_2$, $-a_4$ are negative and r_1, l_3 are positive. Hence the equilibrium state is **unstable.**

In this case the solutions are same as in case (A). The solutions are illustrated in figures.

Case (ii): If $u_{20} < u_{30} < u_{40} < u_{10}$ and $l_3 < a_4 < a_2 < r_1$

In this case initially S_4 dominates the host (S_3) of S_2 and the Predator (S_2) till the time instant t^*_{34}, t^*_{24} respectively and thereafter the dominance is reversed.

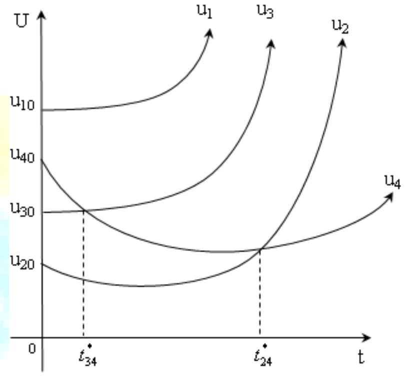


Fig. 5

Case (ii): If $u_{40} < u_{30} < u_{20} < u_{10}$ and $a_2 < r_1 < l_3 < a_4$

In this case initially the Prey (S_1) dominates the Predator (S_2) and the host (S_3) of S_2 till the time instant t^*_{21}, t^*_{31} respectively and thereafter the dominance is reversed. Also the Predator (S_2) dominates the host (S_3) of S_2 till the time instant t^*_{32} and the dominance gets reversed thereafter.

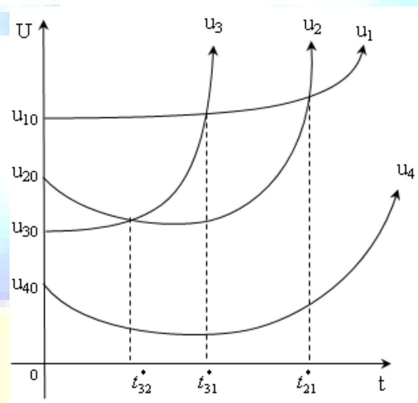


Fig. 6

4.3 Stability of the Equilibrium State E_8:

Substituting (4.1.1) in (2.1), (2.2), (2.3), (2.4) and neglecting products and higher powers of u_1, u_2, u_3, u_4, we get

$$\frac{du_1}{dt} = w_1 u_1 \quad \ldots (4.3.1)$$

$$\frac{du_2}{dt} = -M_2 u_2 + a_{21}\overline{N_2}u_1 + a_{23}\overline{N_2}u_3 \quad \ldots (4.3.2)$$

$$\frac{du_3}{dt} = -a_3 u_3 + \frac{a_{34}a_3}{a_{33}}u_4 \quad \ldots (4.3.3) \qquad \frac{du_4}{dt} = n_4 u_4 \quad \ldots (4.3.4)$$

Here $w_1 = a_1 - a_{12}\overline{N_2}$ ... (4.3.5)

$$n_4 = a_4 + \frac{a_{43}a_3}{a_{33}}, M_2 = a_2 + \frac{2a_3a_{23}}{a_{33}} \quad \ldots (4.3.6)$$

The characteristic equation of which is

$$(\lambda - w_1)(\lambda + M_2)(\lambda + a_3)(\lambda - n_4) = 0 \quad \ldots (4.3.7)$$

Case (A): When $w_1 < 0$ (i.e., when $a_1 < a_{12}\overline{N_2}$)

The roots $w_1, -M_2, -a_3$ are negative and n_4 is positive.

Hence the equilibrium state is **unstable.**

The solutions of the equations (4.3.1) (4.3.2), (4.3.3), (4.3.4) are

$$u_1 = u_{10}e^{w_1t} \quad \ldots (4.3.8)$$

$$u_2 = [u_{20} - \frac{a_{21}\overline{N_2}u_{10}}{(w_1 + M_2)} - \frac{a_{23}\overline{N_2}u_{30}}{(-a_3 + M_2)}]e^{-M_2t} + \frac{a_{21}\overline{N_2}u_{10}}{(w_1 + M_2)}e^{w_1t} + a_{23}\overline{N_2}[\frac{(u_{30} - \eta_7)e^{-a_3t} + \eta_7e^{n_4t} +}{(-a_3 + M_2)}] \quad \ldots (4.3.9)$$

$$u_3 = [u_{30} - \frac{a_3a_{34}u_{40}}{a_{33}(n_4 + a_3)}]e^{-a_3t} + \frac{a_3a_{34}u_{40}}{a_{33}(n_4 + a_3)}e^{n_4t} \quad \ldots (4.3.10)$$

$$u_4 = u_{40}e^{n_4t} \quad \ldots (4.3.11)$$

Where $\eta_7 = \frac{a_3a_{34}u_{40}}{(n_4 + a_3)a_{33}}$

The solutions are illustrated in figures.

Case (i): If $u_{30} < u_{20} < u_{10} < u_{40}$ and $a_3 < M_2 < w_1 < n_4$

In this case initially the Prey (S_1) dominates the Predator (S_2) and the host (S_3) of S_2 till the time instant t^*_{21}, t^*_{31} respectively and thereafter the dominance is reversed. Also the Predator (S_2) dominates the host (S_3) of S_2 till the time instant t^*_{32} and the dominance gets reversed thereafter.

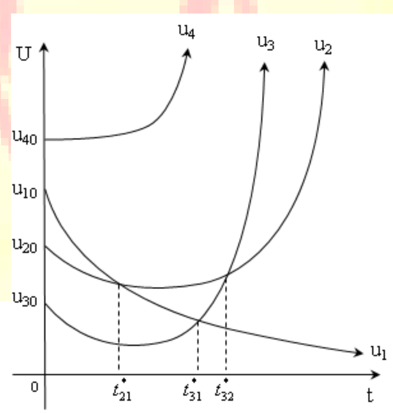


Fig. 7

Case (ii): If $u_{40} < u_{30} < u_{20} < u_{10}$ and $w_1 < M_2 < a_3 < n_4$

In this case initially the Prey (S_1) dominates the Predator (S_2), the host (S_3) of S_2 and S_4 till the time instant t^*_{21}, t^*_{31}, t^*_{41} respectively and thereafter the dominance is reversed. Also the Predator (S_2) dominates the host (S_3) of S_2 and S_4 till the time instant t^*_{32}, t^*_{42} respectively and the dominance gets reversed thereafter. Similarly the host (S_3) of S_2 dominates S_4 till the time instant t^*_{43} and thereafter the dominance is reversed.

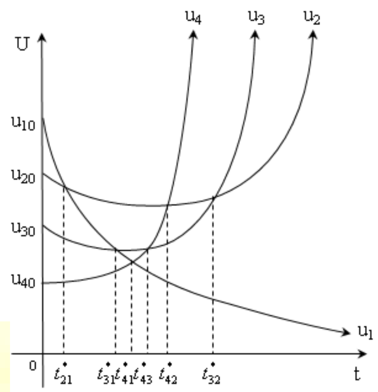


Fig. 8

Case (B): When $w_1 > 0$ (i.e., when $a_1 > a_{12}\overline{N_2}$)

The roots $-M_2$, $-a_3$ are negative and w_1, n_4 are positive.

Hence the equilibrium state is **unstable.**

In this case the solutions are same as in case (A) and the solutions are illustrated in figures.

Case (i): If $u_{30} < u_{40} < u_{10} < u_{20}$ and $M_2 < a_3 < n_4 < w_1$

In this case initially the Predator (S_2) dominates the Prey (S_1), S_4 and the host (S_3) of S_2 till the time instant t^*_{12}, t^*_{42}, t^*_{32} respectively and thereafter the dominance is reversed. Also the Prey (S_1) dominates S_4 and the host (S_3) of S_2 till the time instant t^*_{41}, t^*_{31} respectively and the dominance gets reversed thereafter. Similarly S_4 dominates the host (S_3) of S_2 till the time instant t^*_{34} and thereafter the dominance is reversed.

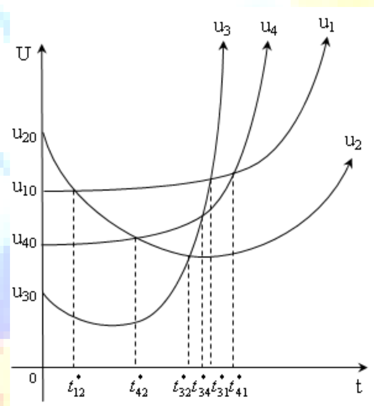


Fig. 9

Case (ii): If $u_{40} < u_{30} < u_{10} < u_{20}$ and $a_3 < n_4 < M_2 < w_1$

In this case initially the Predator (S_2) dominates the Prey (S_1), the host (S_3) of S_2 and S_4 till the time instant t^*_{12}, t^*_{32}, t^*_{42} respectively and thereafter the dominance is reversed. Also the Prey (S_1) dominates the host (S_3) of S_2 and S_4 till the time instant t^*_{31}, t^*_{41} respectively and thereafter the dominance is reversed.

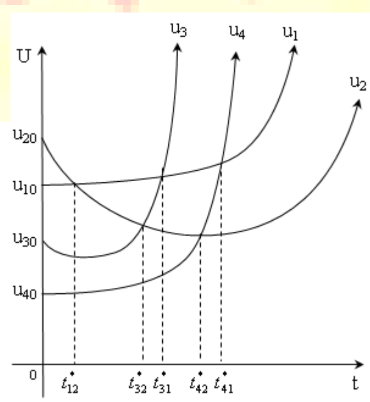


Fig. 10

4.4 Stability of the Equilibrium State E_9:

Substituting (4.1.1) in (2.1), (2.2), (2.3), (2.4) and neglecting products and higher powers of u_1, u_2, u_3, u_4, we get

$$\frac{du_1}{dt} = -a_1 u_1 - \frac{a_{12} a_1}{a_{11}} u_2 \quad \ldots (4.4.1)$$

$$\frac{du_2}{dt} = q_2 u_2 \quad \ldots (4.4.2)$$

$$\frac{du_3}{dt} = l_3 u_3 \quad \ldots (4.4.3) \qquad \frac{du_4}{dt} = -a_4 u_4 + \frac{a_{43} a_4}{a_{44}} u_3 \quad \ldots (4.4.4)$$

Here $q_2 = a_2 + \frac{a_{21} a_1}{a_{11}}$ ... (4.4.5)

$$l_3 = a_3 + \frac{a_{34} a_4}{a_{44}} \quad \ldots (4.4.6)$$

The characteristic equation of which is

$$(\lambda + a_1)(\lambda - q_2)(\lambda - l_3)(\lambda + a_4) = 0 \quad \ldots (4.4.7)$$

The roots q_2, l_3 are positive and $-a_1, -a_4$ are negative.

Hence the equilibrium state is **unstable.**

The solutions of the equations (4.4.1) (4.4.2), (4.4.3), (4.4.4) are

$$u_1 = [u_{10} + \frac{a_{12} a_1 u_{20}}{a_{11}(q_2 + a_1)}] e^{-a_1 t} - \frac{a_{12} a_1 u_{20}}{a_{11}(q_2 + a_1)} e^{q_2 t} \quad \ldots (4.4.8)$$

$$u_2 = u_{20} e^{q_2 t} \quad \ldots (4.4.9)$$

$$u_3 = u_{30} e^{l_3 t} \quad \ldots (4.4.10)$$

$$u_4 = [u_{40} - \frac{a_{43} a_4 u_{30}}{a_{44}(l_3 + a_4)}] e^{-a_4 t} + \frac{a_{43} a_4 u_{30}}{a_{44}(l_3 + a_4)} e^{l_3 t} \quad \ldots (4.4.11)$$

The solutions are illustrated in figures.

Case (i): If $u_{20} < u_{10} < u_{30} < u_{40}$ and $l_3 < a_4 < q_2 < a_1$

In this case initially S_4 dominates the host (S_3) of S_2 and the Predator (S_2) till the time instant t^*_{34}, t^*_{24} respectively and thereafter the dominance is reversed. Also the host (S_3) of S_2 dominates the Predator (S_2) till the time instant t^*_{23} and the dominance gets reversed thereafter. Similarly the Prey (S_1) dominates the Predator (S_2) till the time instant t^*_{21} and thereafter the dominance is reversed.

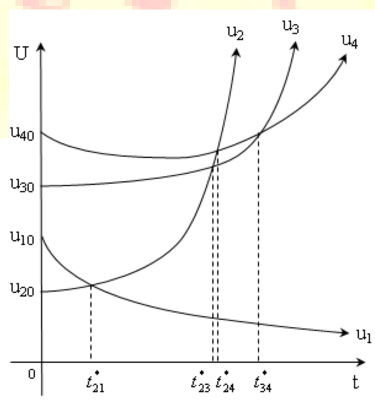


Fig. 11

Case (ii): If $u_{30} < u_{20} < u_{10} < u_{40}$ and $q_2 < a_1 < l_3 < a_4$

In this case initially S_4 dominates the host (S_3) of S_2 till the time instant t^*_{34} and thereafter the dominance is reversed. Also the Prey (S_1) dominates the Predator (S_2) and the host (S_3) of S_2 till the time instant t^*_{21}, t^*_{31} respectively and the dominance gets reversed thereafter. Similarly the Predator (S_2) dominates the host (S_3) of S_2 till the time instant t^*_{32} and thereafter the dominance is reversed.

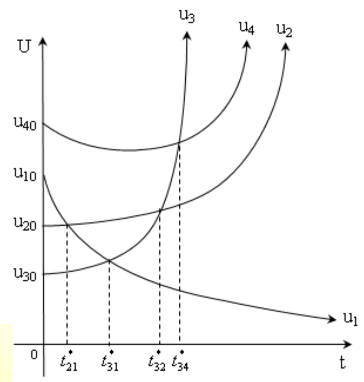


Fig. 12

4.5 Stability of the Equilibrium State E_{10}:

Substituting (4.1.1) in (2.1), (2.2), (2.3), (2.4) and neglecting products and higher powers of u_1, u_2, u_3, u_4, we get

$$\frac{du_1}{dt} = a_1 u_1 - a_{12}\overline{N_1} u_2 \qquad \ldots (4.5.1)$$

$$\frac{du_2}{dt} = w_2 u_2 \qquad \ldots (4.5.2)$$

$$\frac{du_3}{dt} = -a_3 u_3 + \frac{a_{34} a_3}{a_{33}} u_4 \quad \ldots (4.5.3) \qquad \frac{du_4}{dt} = n_4 u_4 \quad \ldots (4.5.4)$$

Here $w_2 = a_2 + a_{21}\overline{N_1} + a_{23}\overline{N_3}$ … (4.5.5)

$n_4 = a_4 + a_{43}\overline{N_3}$ … (4.5.6)

The characteristic equation of which is

$$(\lambda + a_1)(\lambda - w_2)(\lambda + a_3)(\lambda - n_4) = 0 \qquad \ldots (4.5.7)$$

The roots w_2, n_4 are positive and $-a_1, -a_3$ are negative.

Hence the equilibrium state is **unstable.**

The solutions of the equations (4.5.1) (4.5.2), (4.5.3), (4.5.4) are

$$u_1 = \left\{ u_{10} + \frac{a_{12} a_1 u_{20}}{a_{11}(w_2 + a_1)} \right\} e^{-a_1 t} - \frac{a_{12} a_1 u_{20}}{a_{11}(w_2 + a_1)} e^{w_2 t} \qquad \ldots (4.5.8)$$

$$u_2 = u_{20} e^{w_2 t} \qquad \ldots (4.5.9)$$

$$u_3 = [u_{30} - \frac{a_{34} a_3 u_{40}}{a_{33}(n_4 + a_3)}] e^{-a_3 t} + \frac{a_{34} a_3 u_{40}}{a_{33}(n_4 + a_3)} e^{n_4 t} \qquad \ldots (4.5.10)$$

$$u_4 = u_{40} e^{n_4 t} \qquad \ldots (4.5.11)$$

Where $\eta_7 = \frac{a_{34}a_3u_{40}}{a_{33}(n_4 + a_3)}$

The solutions are illustrated in figures.

Case (i): If $u_{10} < u_{20} < u_{40} < u_{30}$ and $n_4 < a_3 < w_2 < a_1$

In this case initially the host (S_3) of S_2 dominates the Predator (S_2) till the time instant t^*_{23} and thereafter the dominance is reversed. Also S_4 dominates the Predator (S_2) till the time instant t^*_{24} and the dominance gets reversed thereafter.

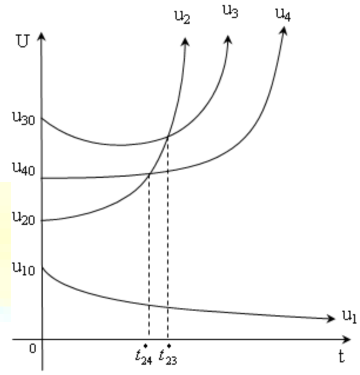


Fig. 13

Case (ii): If $u_{40} < u_{20} < u_{10} < u_{30}$ and $a_3 < a_1 < n_4 < w_2$

In this case initially the host (S_3) of S_2 dominates the Predator (S_2) and S_4 till the time instant t^*_{23}, t^*_{43} respectively and thereafter the dominance is reversed. Also the Prey (S_1) dominates the Predator (S_2) and S_4 till the time instant t^*_{21}, t^*_{41} and thereafter the dominance is reversed.

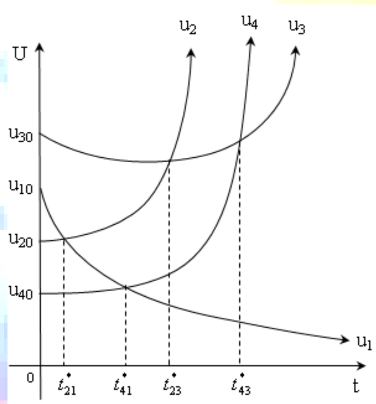


Fig. 14

4.6 Stability of the Equilibrium State E_{11}:

Substituting (4.1.1) in (2.1), (2.2), (2.3), (2.4) and neglecting products and higher powers of u_1, u_2, u_3, u_4, we get

$$\frac{du_1}{dt} = -a_{11}\bar{N}_1u_1 - a_{12}\bar{N}_1u_2 \quad \text{----- (4.6.1)}$$

$$\frac{du_2}{dt} = a_{21}\bar{N}_2u_1 - a_{22}\bar{N}_2u_2 + a_{23}\bar{N}_2u_3 \quad \text{----- (4.6.2)}$$

$$\frac{du_3}{dt} = a_3u_3 \quad \text{------ (4.6.3)}$$

$$\frac{du_4}{dt} = a_4u_4 \quad \text{----- (4.6.4)}$$

The characteristic equation of which is

$$\left[\lambda^2 + (a_{11}\bar{N}_1 + a_{22}\bar{N}_2)\lambda + (a_{11}a_{22} + a_{12}a_{21})\bar{N}_1\bar{N}_2\right](\lambda - a_3)(\lambda - a_4) = 0 \qquad \text{----- (4.6.5)}$$

The characteristic roots of (4.6.5) are

$$\lambda = \frac{-(a_{11}\bar{N}_1 + a_{22}\bar{N}_2) \pm \sqrt{(a_{11}\bar{N}_1 + a_{22}\bar{N}_2)^2 - 4(a_{11}a_{22} + a_{12}a_{21})\bar{N}_1\bar{N}_2}}{2}\ , \ \lambda = a_3, \lambda = a_4 \qquad \text{--- (4.6.6)}$$

Two roots of the equation (4.6.5) are positive and the other two roots are negative. Hence the equilibrium state is unstable.

The trajectories are given by

$$u_1 = \left[\frac{a_{12}\bar{N}_1\ \ u_{10} + u_{20}\ \ - H_1(\lambda_2 - a_3)}{\lambda_2 - \lambda_1}\right]e^{\lambda_1 t}$$

$$+ \left[\frac{(u_{10} - H_1)(\lambda_2 - \lambda_1) - a_{12}\bar{N}_1\ \ u_{10} + u_{20}\ \ + H_1(\lambda_2 - a_3)}{\lambda_2 - \lambda_1}\right]e^{\lambda_2 t} + H_1 e^{a_3 t} \qquad \text{… (4.6.7)}$$

$$u_2 = \left[\frac{a_{12}\bar{N}_1\ \ u_{10} + u_{20}\ \ - H_1(\lambda_2 - a_3)}{\lambda_2 - \lambda_1}\right]\xi_1 e^{\lambda_1 t}$$

$$+ \left[\frac{(u_{10} - H_1)(\lambda_2 - \lambda_1) - a_{12}\bar{N}_1\ \ u_{10} + u_{20}\ \ + H_1(\lambda_2 - a_3)}{\lambda_1 - \lambda_2}\right]\xi_2 e^{\lambda_2 t} + H_2 e^{a_3 t} \qquad \text{…. (4.6.8)}$$

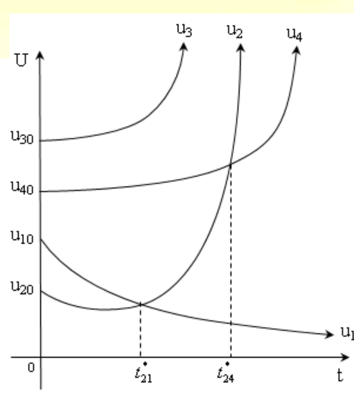
$$u_3 = u_{30}e^{a_3 t} \qquad \text{---- (4.6.9)} \qquad\qquad u_4 = u_{40}e^{a_4 t} \qquad \text{--- (4.6.10)}$$

Here

$$H_1 = \frac{G_1}{a_3{}^2 + \psi_1 a_3 + \beta_1},\quad H_2 = \frac{-H_1(a_3 + P_3)}{a_{12}\bar{N}_1},\quad P_3 = a_{11}\bar{N}_1$$

$$\beta_1 = (a_{11}a_{22} + a_{12}a_{21})\bar{N}_1\bar{N}_2\ ,\ G_1 = -a_{12}a_{23}\bar{N}_1\bar{N}_2\ ,\ \xi_1 = \frac{-(\lambda_1 + P_3)}{a_{12}\bar{N}_1},\quad \xi_2 = \frac{-(\lambda_2 + P_3)}{a_{12}\bar{N}_1}$$

The solutions are illustrated in figures.



Case (i): If $u_{20} < u_{10} < u_{40} < u_{30}$ and $a_2 < a_4 < a_3 < a_1$

In this case initially S_4 dominates the Predator (S_2) till the time instant t^*_{24} and thereafter the dominance is reversed. Also the Prey (S_1) dominates the Predator (S_2) till the time instant t^*_{21} and the dominance gets reversed thereafter.

Fig. 15

Case (ii): If $u_{40} < u_{30} < u_{10} < u_{20}$ and $a_3 < a_2 < a_1 < a_4$

In this case initially the Predator (S_2) dominates the host (S_3) of S_2 and S_4 till the time instant t^*_{32}, t^*_{42} respectively and thereafter the dominance is reversed. Also the Prey (S_1) dominates the host (S_3) of S_2 and S_4 till the time instant t^*_{31}, t^*_{41} respectively and the dominance gets reversed thereafter. Similarly the host (S_3) of S_2 dominates S_4 till the time instant t^*_{43} and thereafter the dominance is reversed.

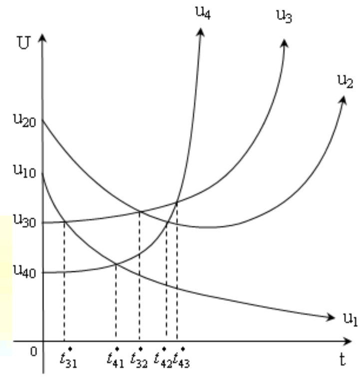


Fig. 16

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