ON THE CATEGORY OF ORDERED TOPOLOGICAL MODULES OPTIMIZATION AND LAGRANGE'S PRINCIPLE

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Abstract

A category is an algebraic structure made up of a collection of objects linked together by morphisms. As a foundation of mathematics, categories were created as a way of relating algebraic structures and systems of topological spaces In this paper we define a derivative using cones in the category of topological modules and use the Lagrange's principle to obtain optimization results in the category.

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1. Introduction

The main problem that seems to make the study of optimization in categories difficult is the fact that it is algebraic in nature yet most of the optimization is studied in classical analysis. Since the innovative use of infinitesimals by Lawvere (1963) and Kock (1981) it has been possible to study some parts of analysis in such toposes.Most of the extrema problems involve the order properties of the real line and this explains why extrema properties of a complex variable do not exist.Sukhinin (1982) introduced the idea of extrema in spaces without norm that is applicable even to functions of complex variables. It is this idea we seek to adopt in the ordered category of topological modules. The systematic study and use ordered vector spaces and cones in mathematics begun around the world after 1950 mainly through the efforts of Russian, Japanese, German and Dutch mathematicians. The notion of cones is important in many areas; two notable areas are optimization theory and the fixed point theory. Since cones are being employed to solve optimization problems, the theory of ordered vector spaces is an indispensable tool for solving a variety of applied problems in diverse areas such as engineering, econometrics and the social sciences (Charambos and Rabee, 2007).

A subset C of a vector space V is a cone if $\alpha x + \beta y$ belongs to C for any positive scalars α and β and any x,y in C.This concept is meaningful for any vector space that allows the concept of positive scalars such as spaces over rational ,algebraic and even the real numbers. It follows hence that the empty set, the space V and any linear sub space of V including the trivial sub space $\{\varnothing\}$ are convex cones by this definition. The intersection of two convex cones in the same vector space is again a convex cone but their union may fail to be one. The family of convex cones is closed under arbitrary linear maps and particularly if C is a convex cone so is its opposite C' and $C' \cap C$ is the largest linear subspace contained in C (Rockafellar, 1997).

If X is within the Euclidean space, the cone on X is homeomorphic to the union of lines from X to another point. That is, the topological cone agrees with the geometric cone when defined. All cones are path connected since every point can be connected to the vertex by the homotopy $h_t(x,s) = \{x,(1-t)s\}$.In algebraic topology, cones are used precisely because they embed a space as a subspace of a contractible space (Allen,2002).Since cones can be topologically closed, open or both then continuity being a central concept in topology can be defined using open neighbouhoods that is if X and Y are topological spaces , a function f: $X\rightarrow Y$ is continuous if and only if for all open neighbouhoods of B of f(x) there exists a neighbouhood A of X such that$A \subseteq f^{-1}(B)$, (Gaal and Steven, 2009).

2. Literature review

A category is an algebraic structure that comprises 'objects' that are linked by 'arrows'. A category has two basic properties, the ability to compose the arrows associatively and the existence of an identity arrow for each object. A simple example is the category of sets whose objects are sets and whose arrows are functions. On the other hand, a monoid can be understood as a special sort of category and so can any pre-order. Generally objects and arrows may be abstract entities of any kind and the notion of category provide a fundamental and abstract way to describe mathematical entities and their relationships.This is the central idea of category theory, a branch of mathematic which seeks to generalize all of mathematics in terms of objects and arrows independent of what the object and arrows represent.

Virtually every branch of modern mathematics can be defined in terms of categories and in doing so revealing deep insights and similarities between seemingly different areas of mathematics. Categories were introduced by Eilenberg and Mac Lane (1945).

A topological module is a module over a topological ring such that scalar multiplication and addition are continuous. In abstract algebra, a module over a ring is a generalized notion of a vector space, wherein the corresponding scalars lie in an arbitrary ring. An abelian topological group that is a module over a topological ring R, in which the multiplication mapping $R \times A \rightarrow A$, taking (r, α) to $r\alpha$ is required to be continuous. A right topological module is defined analogously. Every sub module B of a topological module A is a topological module. If the module A is separated and B is closed in A, then A/B is a separated module (Bourbaki, 1966). The left R- modules together with their module homomorphisms form a category written as R-Mod which is an abelian category, (Anderson and Fuller, 1992).

Given a ring R and an R- module, a descending filtration of M is a decreasing sequence of sub modules M_n.This is therefore a special case of the notion of groups with the additional condition that the sub groups be sub modules. The associated topology is defined as for groups. An ascending filtration is defined in the same way (Oksendal and Bent, 2003).In topology and analysis, filters are used to define convergence in a manner similar to the role of sequences in metric spaces. Sequences are usually indexed by natural numbers which are a totally ordered set.Thus, limits in spaces can be defined using sequences (Victor, 2012).

2.2 Boundary and Extrema

Given two real axes X and Y and a function f: X $\rightarrow$ Y, that takes a closed bounded set A $\subset$ X to a closed bound set B $\subset$ Y then problems of maxima and minima involves finding points in A that are mapped by to either the maximum or minimum points of B. If the closed bounded set is not the real line then there will be no maximum or minimum. It is possible however to introduce a related idea of extremal point as a point in A that is mapped by the function f to the boundary of B. This is done using locally convex topological vector spaces with bounded topology. This is done because classical calculus operations work well up to the abstraction of Banach spaces but not beyond. Further developments have shown that classical calculus can still work well in topological vector spaces more general than Banach spaces provided the topology used is bounded. Another good reason for this topological space is that it is amenable to category theoretic approach of study (Andreas and Peter, 1997).

2.3 Classical Lagrange Multiplier

One of the most common problems in calculus is that of finding maxima and minima of a function, but it is often difficult to find a closed form for the function being extremized. Consider the optimization problem: maximize f(x, y) subject to g(x, y). We introduce a new variable λ called the Lagrange function defined by:

$\omega(x, y, \lambda) = f(x, y) + \lambda \{g(x, y) - c\}$ Where the λ term may added or subtracted. If f(x, y) is a maximum for the original constrained problem then $\exists\ \lambda$ such that (x, y, λ) is a stationary point for the Lagrange function. However, not all stationary points yield a solution to the original problem. Thus, this method yields a necessary condition for optimality in constrained problems (Vapnyarskii, 2001). The role of Lagrange multipliers in locally convex spaces and the concept of extrema can be extended to topological spaces without norm and even to functions of complex variables.

2.4 Lagrange Multiplier in locally convex spaces without norm

Sukhinin studied Lagrange multipliers in vector spaces with bounded topology and it is this idea we intend to generalize to categories of topological modules. To obtain results similar to those found in classical analysis, Sukhinin (1982) used the idea of cones in linear topological spaces without norm. We now follow Sukhinin (1982) to get these important definitions;

2.4.1Cone in a Topological Vector Space without Norm

Let X be a linear topological vector space without norm. $S \subset X, x_o \in X$, then $\beta S(x_o)=\{h\in X:\exists U\exists\delta\forall B\forall t\in(0,\delta):x_o+t(h+U\cap B)\subset S\}$ and $\beta S_+(x_o)=X\backslash\{\beta(X\backslash S\text{-}(x_o)\}$are called cones. U is neighbourhood of zero, B is a bounded set and β is a certain system of bounded, convex sets in U containing zero.

2.4.2 Small Map

Let $x_o\in S \subset X$ and let Z be a topological vector space. A map r: $S \rightarrow Z$ will be called β small relative to the cone K at the point x_o if $\forall h\in K,\forall V\exists U,\exists\delta,\forall B :(t<\delta,x\in h+U\cap B,x_o+tx\in S)$,V is a neighborhood of zero in Z.

2.4.3 Differentiable Map

A map f: $S \rightarrow Z$ is called β differentiable relative to K at the point x_o if $f(x_o + h) - f(x_o) = f'(x_o) h + r(h)$. Where $f'(x_o)$ is a linear continuous operator and the map r: $S \rightarrow Z$ is β small relative point x_o.

2.4.4 Critical Map

Let $S \subset X$ and let Z be a linear space. We say that a map f: $S \rightarrow Z$ is β critical relative to the cone K at the point $x_o\in S$ if $\exists z\in Z,\forall h\in K,\exists U,\exists\delta\forall,B\forall t\in(0,\delta):tz\notin f\{[x_o+t(h+U\cap B)]\cap S\}\backslash f(x_o)$.

2.4.5 Conditional Minimum of Maps

Let $x_o\in S \subset X$, let Z be ordered linear space and let f: $S \rightarrow Y$ be a map satisfying $f(x_o) = 0$.We say that x_o is a point of conditional β minimum of the map F: $S \rightarrow Z$ relative to the cone K under the condition $f(x) = 0$ provided

$\forall h\in K\exists U\exists\delta\forall B\forall t\in(0,\delta):x\in[x_o+t(h+U\cap B)]\cap S,f(x_o) = 0\Rightarrow F(x)\geq F(x_o)$.

2.4.6 Theorem 1: Let $x_o \in S \subset X$, let K_1 be a cone in X with vertex at zero, and suppose that the map f: S $\rightarrow$Y is such that

$\forall h \in K_1, \forall U \exists B \exists \delta \forall t \in (0,\delta) \exists x \in t(U \cap B):[x_o + th + x \in S] \wedge [f(x_o + th + x) = f(x_o)]$ and

$f(x_o) = 0$.Further, let Z be an ordered topological vector space, suppose that the map F: S $\rightarrow$ Z is β differentiable relative to K_1 at the point x_o.If x_o is a point of conditional β minimum of the map F relative to K_1 and under the condition $f(x_o) = 0$ then $F'(x_o)(h) \geq 0$ for $h \in K_1$.

Sometimes, the theorem above can be given the form of the rule of Lagrange multipiers, as shown below;

Let Z be an ordered topological vector space, g be a continuous linear operator from X into Z, let A be a linear (not necessary continuous) operator from X into Y, K be a cone in X, and suppose that$\forall h \in$ (Ker A) $\cap$K: g (h) $\geq$0…...*.

Proposition1:Ssuppose (*) above holds, g is a linear is open (AX=Y), and that K = X, then $\exists$ an operator $\lambda \in \omega(Y, Z)$ such that $(g+\lambda o\ A)\ x = 0$ for some $x \in X$. Indeed to maximize f(x, y) subject g(x, y), we introduce a new variable λ and

$$\omega (x, y,\lambda) = f(x, y) +\lambda\, [g (x, y) - c].$$

If f(x, y) is maximum for constrained problem then $\exists\lambda$: (x, y, λ) is a stationary point for ω(x, y, z) i.e. $\omega(x, y,\lambda) \in \partial Y$. since K = X, K is cone, AX = Y, then Y is a cone. In this case $\omega(x, y,\lambda) \in$ vertex of Y which is zero. In our case f(x, y) = g(x, y), [g(x, y) – c] = Ax hence $(g + \lambda oA)\ x = 0$ implies optimality

of ω.

3. Preliminaries and Definitions

To come up with a criterion for finding extrema in categories with infinitesimals using a modification of Sukhinin's definition of extrema in topological vector space without norms. We will need the idea of boundedness and boundary in categories. These will be defined in the required categories. The next step will be to define the extremal object in an ordered category of topological modules with infinitesimals and finally prove a result on conditions for existence of extrema. In this regard we intend to formulate and prove theorem 1 in an ordered category of topological modules.

3.2 Methodology

The method which we intend to use to get results in optimization in such categories hinges on the following definitions;

3.2.1 Amazingly tiny object models (a.t.o.m)

Let R be a ring, then $D = \{d \in R: d^2 = 0\}$ is called the collection of a.t.o.m. (amazingly tiny object models in R).A map $r: D \to D$ will be called a homomorphism of infinitesimals.

3.2.2 Modules with a.t.o.m

X/D is a factor module with a.t.o.m. if X is a module.

3.2.3 Bounded Module

A module X is bounded if $\forall$ $x \in X$, $d \in D$, then $d.x \in D$.

3.2.4 Boundary

Let X, Y be topological modules with infinitesimals such that $X - Y \in D$, then X-Y is the boundary of Y and we write ∂Y.

3.2.5 Extremal Object

Let X be a category of topological modules with infinitesimals, f be morphism in X, $x \in X$ is an extremal object of the morphism $f(x) \subset \partial X$.

Using the definitions, it should be possible to get a result similar to Lagrange's method of multipliers in the category of topological modules by modifying Sukhinin's method.

3.2.6 Cones in Topological Modules

Let X be a topological module, U a neighbourhood of zero, B a bounded neighbourhood of zero in X. Then $\{h \in X: x_o + t(h + U \cap B) \subset S\}$where $S \subset X$ is a cone.

4. Main Results

Theorem 2: Let X and Y be ordered topological modules, $x_0 \in S \subset X, K_1$ be a cone in X with vertex at zero and suppose that the map $f: S \to Y$ satisfies the condition $(x_0 + dh + x \in S) \wedge [f(x_0 + dh + x) = f(x_0)]$ and $f(x_0) = 0$. Further, let Z be an ordered topological module, $F: S \to Z$ be synthetically differentiable relative to K_1 at x_0. If x_0 is a point of conditional minimum of the map F relative to K_1 and under the condition $f(x_0) = 0$, then $F'(x_0)(h) \notin t(h+U\cap B)$ for $h\in K_1$ ....………………**(1)**

Proof: Assume that $x_0 = 0$, $F(0) = 0$. Let $h\in K_1$ and $V\subset D$ in Z. Set $r(x) = F(x) - F'(0)(x)$ and set $V'\subset D$ in D satisfying $V' + V'\subset V$. Then $\exists$ U_1 and δ_1 such that $F'(0)(U_1) \subset V'$ and $\forall B\forall d\in D$ such that

$(x'\in[d(h + U_1\cap B)] \cap S, f(x') = 0$. Which implies that $[r(x')\in dV']\wedge[F(x') \in t(h + U\cap B)]$.

Further $\exists$ $x\in d(U_1\cap B)$ for which $(dh + x) \in S$ and $f(dh + x) = 0$.

Here, $F(dh + x) \in U_1\cap B$ and $F(dh + x) \in C$, where C is a positive cone in Z. Further,

$F'(0)(dh + x) + r(dh + x) \subseteq d F'(0)(h) + F'(0)(x) + r(dh + x)$. Then,

$F'(0)(h) + F'(0)(d^{-1}x) + d^{-1}r(dh + x) \in[F'(0)(h) + V' + V'] \cap C[F'(0)(h) + V] \cap C$, that is

$[F'(0)(h) + V] \cap C \neq \emptyset$, since C is closed and V is arbitrary, $F'(0)(h) \in C$, that is $F'(0)(h) \notin d(h + U\cap B)$.

Sometimes, the condition (1) can be given the form of the rule of Lagrange multipliers.

Let Z be an ordered topological module, g be a continuous linear operator from X into Z, A be a linear operator from X into Y, K be cone in X and suppose that;

$\forall h\in(\mathrm{Ker}\ A) \cap K: g(h)\in d(h + U\cap B)$…………………….……………**(2)**

Proposition 2: Suppose that (2) holds, g is linear, A is open (AX = Y), and that A = X then there exists on operator

$\lambda\in \Re(Y, Z)$ such that $(g + \lambda oA) = 0$ for some $x\in X$. Where $\Re(Y, Z)$ is a class of continuous linear maps from Y into Z.

Proof:

From (2) it follows that $\ker A \subset \mathrm{Ker}\ g$. Now we set $\lambda y = -gA^{-1}$ for some $y \in Y$. The continuity of λ is a consequence of the continuity of g and the openness of A.

Proposition 3

If f is a linear and continuous function, then it maps a cone to a cone.

Proof: {d (h + U∩B)} is a cone since f (x_0 + d (h + U∩B)} - f(x_0) = f′(x_0){d(h + U∩B)} + r(h + U∩B),

And since the function f is linear, then it follows that;

f (x_0 + d (h + U∩B)} – f(x_0) = d {f′(x_0) (h + U∩B)} + r (h + U∩B), since the left hand side is a cone, then it follows that the right hand side is also a cone with vertex at zero. Hence the function f maps a cone to a cone.

5. Conclusion

The results obtained in this work is a great contribution to the study of optimization in the category of ordered topological modules

6. References

[1]. Allen, H., (2002).Algebraic Topology, Cambridge University Press, Cambridge.

[2]. Anderson, F.W and Fuller, K.R., (1992).Graduate Texts in mathematics, (second ed.) Springer-Verlag.

[3]. Andreas, K and Peter, W.M., (1997).The convenient setting of Global Analysis, American Mathematical Society, U.S.A.

[4]. Bourbaki, N., (1996). Elements of Mathematics, General topology, Addison-Wesley Education Publishers, Boston.

[5]. Charambos, D.A and Rabee, T., (2007).Cones and duality, American Mathematical Society, U.S.A.

[6]. Kock, A., (1981). Synthetic Differential Geometry. Cambridge University Press, Cambridge.

[7]. Lawvere, F.W., (1963). Sematics of Algebraic Theories, Ph. D Thesis. Columbia University, Columbia.

[8]. Mac Lane, S and Eilenberg, S., (1945). Transactions of the American Mathematical Society, **No.2** pp.231-294.

[9]. Oksendal and Bernt, K., (2003).Stochastic Differential Equations, an Introduction with applications, Springer – Verlag.

[10]. Rockafellar, R.T., (1997).Convex Analysis, Princeton University Press, Princeton.

[11]. Sukhinin,M.F., (1982).The role of Lagrange Multipliers in locally Convex Spaces, Siberian Journal of mathematics,**No.23** pp.153 – 165.

[12]. Vapnyarskii, I.B., (2001).Lagrange multipliers, Springer – Verlag.

[13]. Victor, P., (2012).Filters on partially ordered sets and generalizations, International Journal of Pure and Applied Mathematics, **No.74 (1)** pp.55 – 119.