HARDY TYPE INEQUALITIES FOR RIEMANN-LIOUVILLE AND WEYL TRANSFORMS ASSOCIATED WITH THE DIFFERENTIAL OPERATOR $S_{\alpha,\beta}$

B.B.Waphare*

Abstract:

In this paper we have constructed a differential operator $S_{\alpha,\beta}$ and it is used to give nice estimates for the kernels, which intervenes in the integral transform of the eigen function. Finally we have established Hardy type inequalities for Riemann-Liouville and Weyl transforms associated with the differential operator $S_{\alpha,\beta}$.

Key words: Differential operator, Hardy type inequalities, Hardy type inequalities, Hardy type operator, Integral transforms.

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* MITACSC, Alandi, Pune, Tal: Khed, Dist: Pune-412105

1. Introduction: We consider the differential operator on $(0,\infty)$ defined by

$$S_{\alpha,\beta} = D^2 + \frac{A'(x)}{A(x)} D + \rho^2$$

where $A(x)$ is a real function defined on $[0,\infty)$, satisfying

$$A(x) = x^{2(\alpha-\beta+1)}\ B(x);\ (\alpha-\beta) > -1$$

and B(x) is a positive, even C^∞ function on $\mathbb{R}$ such that $B(0)=1$, and $\rho \geq 0$. We assume that the function A (x) satisfies the following conditions

(i) A (x) is increasing and $\lim_{x\to+\infty} A(x) = +\infty$

(ii) $\frac{A'(x)}{A(x)}$ is decreasing and $\lim_{x\to+\infty} \frac{A'(x)}{A(x)} = 2\rho$

(iii) there exists a constant $\delta > 0$, satisfying

$$\frac{B'(x)}{B(x)} = 2\rho - \frac{2(\alpha-\beta+1)}{x} + e^{-\delta x}F(x), \text{ for } \rho > 0$$

$$\frac{B'(x)}{B(x)} = e^{-\delta x}F(x), \text{ for } \rho = 0$$

where F is C^∞ on $(0,\infty)$, bounded together with its derivatives on the interval $[x_0,\infty), x_0 > 0$.

The Bessel type operators defined by

$$\Delta_{\alpha,\beta} = D^2 + \frac{2(\alpha-\beta+1)}{x} D; (\alpha-\beta) > -1$$

is of the type $S_{\alpha,\beta}$ with

$$A(x) = x^{2(\alpha-\beta+1)}; \rho = 0.$$

The radial part of the Laplacian-Beltrami operator on the Riemanian symmetric space is of type $S_{\alpha,\beta}$. The operator $S_{\alpha,\beta}$ is studied in [13]. In particular K Trimeche [15] has proved that the differential equation

$$\Delta u(x) = -\lambda^2 u(x), \lambda \in \mathbb{C}$$

has a unique solution on $[0,\infty)$, satisfying the conditions $u(0)=1, u'(0)=0$. Inspired by this work we consider the differential equation

$$S_{\alpha,\beta} u(x) = -\lambda^2 u(x), \lambda \in \mathbb{C}\ .$$

One can easily show that this differential equation has a unique solution on $[0,\infty)$ satisfying the conditions

$$u(0)=1, u'(0)=0.$$

We extend this solution on $\mathbb{R}$ by parity denoted by φ_λ following the technique of Trimeche [15], one can easily prove that the eigenfuction φ_λ has the following Mehler type integral representation

$$\varphi_\lambda(x) = \int_o^x k(x,t)\cos\lambda t dt\ ,$$

where the kernel is defined by

$$k(x,t) = 2h(x,t) + C_{\alpha,\beta} A^{-\frac{1}{2}}(x)\ x^{-(\alpha-\beta)}(x^2-t^2)^{\alpha-\beta}, o<t<x$$

with

$$h(x,t) = \frac{1}{x}\int_o^\infty \psi(x,\lambda)\cos(\lambda t)\,d\lambda$$

$$C_{\alpha,\beta} = \frac{2\Gamma(\alpha-\beta+3/2)}{\sqrt{\pi}\ \Gamma(\alpha-\beta+1)}$$

and for all $\lambda \in \mathbb{R}$, $x \in \mathbb{R}$,

$$\psi(x,\lambda) = \varphi_\lambda(x) - x^{\alpha-\beta+1} A^{-\frac{1}{2}}(x)\ j_{\alpha-\beta+\frac{1}{2}}(\lambda x)$$

where

$$j_{\alpha-\beta+\frac{1}{2}}(z) = 2^{\alpha-\beta+\frac{1}{2}}\Gamma(\alpha-\beta+3/2)\ \frac{J_{\alpha-\beta+\frac{1}{2}}(z)}{z^{\alpha-\beta+\frac{1}{2}}}$$

where $J_{\alpha-\beta+1/2}$ is the Bessel function of the first kind and order $(\alpha-\beta+\frac{1}{2})$ (see [8]).

The Riemam-Liouville and Weyl transform associated with the differential operator $S_{\alpha,\beta}$ are respectively defined for non-negative measurable functions f by

$$\Re(f)\ (x) = \int_0^x k(x,t)\ f(t)\,dt$$

and

$$W(f)(t) = \int_t^\infty k(x,t)\ f(x)\ A(x)\,dx\,.$$

We study the operators R and W on the spaces $L^p([0,\infty), A(x)\,dx)$ consisting of measurable functions on $[0,\infty)$ such that

$$\|f\|_{p,A} = \left(\int_0^\infty |f(x)|^p\ A(x)\ dx\right)^{\frac{1}{p}} < \infty; 1<p<\infty\,.$$

The main results of this paper are the following Hardy type inequalities:

(A) For $\rho>0$ and $p > \max\ (2, 2(\alpha-\beta+1)+1)$, there exists a positive constant $C_{p,\alpha,\beta}$ such that for all $f \in L^p\ ([0,\infty), A(x)\,dx)$,

$$\|\Re(f)\|_{p,A} \le C_{p,\alpha,\beta}\ \|f\|_{p,A} \qquad (1.1)$$

and for all $y \in L^{p'}\ ([0,\infty), A(x)\,dx)$

$$\left\|\frac{1}{A(x)}W(g)\right\|_{p',A} \le C_{p,\alpha,\beta}\ \|g\|_{p',A} \qquad (1.2)$$

where , $p' = \dfrac{p}{p-1}$.

(B) For $\rho=0$ and $p>2(\alpha-\beta+1)+1$ there exists a positive constant $C_{p,\alpha,\beta}$ such that (1.1) and (1.2) hold.

2. The eigen functions of the operator $S_{\alpha,\beta}$:

The equation

$$S_{\alpha,\beta}u(t)=-\lambda^2 u(t)\ ,\lambda\in\mathbb{C} \quad (2.1)$$

has a unique solution on $[0,\infty)$ satisfying the conditions $u(0)=1, u'(0)=0$. We extend this solution on $\mathbb{R}$ by parity and we denote it ψ_λ. Equation (2.1) possesses two solutions $\varphi_{\mp\lambda}$ linearly independent having the following behavior at infinity $\varphi_{\mp}(x) \sim e^{(\mp\lambda-\rho)}$.

Then there exists a function g such that

$$\psi_\lambda(x)= g(\lambda)\ \varphi_\lambda(x)+g(-\lambda)\ \varphi_{-\lambda}(x).$$

In case of the Bessel type operator $\Delta_{\alpha,\beta}$, the functions $\psi(\lambda), \phi(\lambda)$ and $g(\lambda)$ are given respectively by

$$j_{\alpha-\beta+1/2}(\lambda x)=2^{\alpha-\beta+1/2}\ \Gamma(\alpha-\beta+3/2)\ \frac{J_{\alpha-\beta+1/2}(\lambda x)}{(\lambda x)^{\alpha-\beta+1/2}}\ ;\ \lambda x\neq 0 \quad (2.2)$$

$$K_{\alpha-\beta+1/2}(i\lambda x)=2^{\alpha-\beta+1/2}\ \Gamma(\alpha-\beta+3/2)\ \frac{K_{\alpha-\beta+1/2}(i\lambda x)}{(i\lambda x)^{\alpha-\beta+1/2}},\lambda(x)\neq 0 \quad (2.3)$$

$$g(\lambda)=2^{\alpha-\beta+1/2}\,\Gamma(\alpha-\beta+3/2)\,e^{-i(\alpha-\beta+1)\frac{\pi}{2}}\ \lambda^{-(\alpha-\beta+1)},\ \lambda>0 \quad (2.4)$$

where J_μ and K_ν are respectively Bessel function of first kind and order μ, and the modified Bessel function of third kind and of order ν.

We shall need the following proprieties (see [1], [2], [15], [16]):

1. We have
   (i) For $\rho=0,$ for all $x\geq 0,\psi_0(x)=1,$
   (ii) For $\rho\geq 0,$ there exists a constant $k>0$ such that for all $x\geq 0,$
   (iii) $e^{-\rho x}\ \leq\psi_0(x)\ \leq k(1+x)\ e^{-\rho x}$ (2.5)

2. For $\lambda\in\mathbb{R}$ and $x\geq 0$, we have
$$|\psi_\lambda(x)|\leq\psi_0(x) \quad (2.6)$$
3. For $\lambda\in\mathbb{C}$ such that $|\Im\lambda|\leq\rho$ and $x\geq 0$, $|\psi_\lambda(x)|\leq 1$
4. **(Mehler type integral representation):**
   For all $x>0,\ \lambda\in\mathbb{C}$,
$$\psi_\lambda(x)=\int_0^x k(x,t)\cos(\lambda t)\,dt \quad (2.7)$$
   where $k(x,.)$ is an even positive C^∞ function on $(-x,x)$ with support in $[-x,x]$
5. For $\lambda\in\mathbb{C}$, we have $g(-\lambda)=g(\lambda)$
6. The function $|g(\lambda)|^{-2}$ is continuous on $[0,\infty)$ and there exist positive constants k,k_1,k_2 such that

(i) If $\rho \geq 0$, for all $\lambda \in \mathbb{R}$, $|\lambda| > k$,

$$k_1 |\lambda|^{2(\alpha-\beta+1)} \leq |g(\lambda)|^{-2} \leq k_2 |\lambda|^{2(\alpha-\beta+1)}$$

(ii) If $\rho > 0$, for all $\lambda \in \mathbb{R}$, $|\lambda| \leq k$

$$k_1 |\lambda|^2 \leq |g(\lambda)|^{-2} \leq k_2 |\lambda|^2$$

(iii) If $\rho = 0$, $(\alpha-\beta) > -1/2$, for all $\lambda \in \mathbb{R}$, $|\lambda| \leq k$

$$k_1 |\lambda|^{2(\alpha-\beta+1)} \leq |g(\lambda)|^{-2} \leq k_2 |\lambda|^{2(\alpha-\beta+1)} . \qquad (2.8)$$

Now, if we take $v(x) = \sqrt{A(x)}\ \mu(x)$ then equation (2.1) becomes

$$v''(x) - [G(x) - \lambda^2] v(x) = 0$$

where

$$G(x) = \frac{1}{4}\left[\frac{A'(x)}{A(x)}\right]^2 + \frac{1}{2}\left[\frac{A'(x)}{A(x)}\right]' - \rho^2 .$$

Set

$$\xi(x) = G(x) - \frac{(\alpha-\beta)(\alpha-\beta+1)}{x^2} .$$

Now we have the following Lemmas, which follow from the hypothesis of the function A.

Lemma 2.1:

(i) The function ξ is continuous on $(0,\infty)$

(ii) There exists $\delta > 0$ and $a \in \mathbb{R}$ such that the function ξ satisfies

$$\xi(x) = \frac{a}{x^2} + e^{-\delta x} F_1(x)$$

where, F_1 is C^∞ on $(0,\infty)$, bounded together with all its derivatives on the interval $[x_0,\infty), x_0 > 0$.

Lemma2.2: Let

$$b(x,\lambda) = \psi_\lambda(x) - x^{\alpha-\beta+1} A^{-1/2}(x)\, j_{\alpha-\beta+1/2}(\lambda x) \qquad (2.9)$$

where $j_{\alpha-\beta+1/2}$ is defined by (2.2).

Then there exist positive constants C_1 and C_2 such that

$$|b(x,\lambda)| \leq C_1 A^{-1/2}(x)\, \tilde{\xi}(x)\, \lambda^{-(\alpha-\beta+2)}\, e^{(C_2\, \tilde{\xi}(x)/\lambda)} \qquad (2.10)$$

for all $x > 0$, $\lambda \in \mathbb{R}^*$

$$\tilde{\xi}(x) = \int_0^x |\xi(r)|\, dr .$$

The kernel $k(x)t)$ given by the relation (2.7) can be written as

$$k(x,t) = 2\,h(x)t) + C_{\alpha,\beta}\, A^{-1/2}(x)\, x^{-(\alpha-\beta)} (x^2 - t^2)^{\alpha-\beta}, 0 < t < x \qquad (2.11)$$

where

$$h(x,t) = \frac{1}{\pi}\int_0^\infty b(x,t)\cos(\lambda t)\, d\lambda \qquad (2.12)$$

$$C_{\alpha,\beta}=\frac{2\Gamma(\alpha-\beta+3/2)}{\sqrt{\pi}\,\Gamma(\alpha-\beta+1)}$$

and $b(x,\lambda)$ is the function defined by the relation (2.9).
Now we shall study the kernel $h(x,t)$.

3. The kernel h and related results:

In this section we will discuss the behavior of the kernel $h(x,t)$.

Lemma 3.1: For any real a>0 there exist positive constants $C_1(a)$, $C_2(a)$ such that for all $x\in[0,a]$

$$C_1(a)\,x^{2(\alpha-\beta+1)}\le A(x)\le C_2(a)\,x^{2(\alpha-\beta+1)}.$$

Lemma 3.2: There exist positive constants a_1, a_2, C_1 and C_2 such that for $|\lambda|>a$,

$$C_1(\alpha,\beta)\,\psi_\lambda(x)=\begin{cases}C(\alpha,\beta)x^{\alpha-\beta+1}A^{-1/2}(x)\left(j_{\alpha-\beta+1/2}(\lambda x)+O(\lambda x)\right);|\lambda x|\le a_2\\ C(\alpha,\beta)\lambda^{-(\alpha-\beta+1)}\,A^{-1/2}(x)\left[C_1e^{-i\lambda x}+C_2e^{i\lambda x}\right]\left[1+O(\lambda^{-1})+O(\lambda x)^{-1}\right];|\lambda x|>a_2\end{cases}$$

where

$$C(\alpha,\beta)=\Gamma(\alpha-\beta+3/2)\,A^{1/2}(1)\,e^{-\frac{1}{2}\int_0^1 B(t)\,dt}.$$

Proof: Proof follows from [16].

Theorem 3.3: For any a>0, there exists a positive constant $C_1(\alpha,\beta,a)$ such that

$$|h(x,t)|\le C_1(\alpha,\beta,a)\,x^{\alpha-\beta}\,A^{-1/2}(x),\text{for all } 0<t<x\le a.$$

Proof: For $0<t<x$ by using (2.12), we have

$$|h(t,x)|\le\frac{1}{\pi}\int_0^\infty|b(x,\lambda)|d\lambda$$
$$=\frac{1}{\pi}\int_0^{a_1}|b(x,\lambda)|d\lambda\,\frac{1}{\pi}\int_{a_1}^\infty|b(x,\lambda)|d\lambda$$
$$=I_1(x)+I_2(x) \qquad (3.1)$$

where a_1 is a constant given by Lemma 3.2 and

$$I_1(x)=\frac{1}{\pi}\int_0^{a_1}|b(x,\lambda)|d\lambda,$$
$$I_2(x)=\frac{1}{\pi}\int_{a_1}^\infty|b(x,\lambda)|d\lambda.$$

Set

$$f_\lambda(x)=x^{-(\alpha-\beta)}\,A^{1/2}(x)\,|\psi(x,\lambda),|\;0<x<a,\lambda\in\square$$

From lemma 2.2 the function

$$(x,\lambda)\to f_\lambda(x)$$

is continuous on $[0,a]\times[0,a_1]$. Then

$$I_1(x)=\frac{1}{\pi}\int_0^{a_1}|b(x,\lambda)|d\lambda\le C'_{\alpha,\beta}\,x^{\alpha-\beta}\,A^{-1/2}(x) \qquad (3.2)$$

where

$$C'_{\alpha,\beta} = \frac{a_1}{\pi} \sup_{(x,\lambda)\in[0,a]\times[0,a_1]} \left| f_\lambda(x) \right|.$$

Now consider the second term

$$I_2(x) = \frac{1}{x} \int_{a_1}^{\infty} \left| b(x,\lambda \right| d\lambda .$$

Case I: $-1 \le (\alpha-\beta) \le 0$

Now from inequality (2.9) we have

$$I_2(x) \le \frac{C_1}{\pi} A^{-1/2}(x)\tilde{\xi}(x) \int_{a_1}^{\infty} \lambda^{-(\alpha-\beta+2)} e^{C_2 \frac{\tilde{\xi}}{|\lambda|}} d\lambda$$

$$\le \tilde{C}_1 A_1^{-1/2}(x)\tilde{\xi}(x)\, e^{C_2 \tilde{\xi}(x)/a_1} x^{\alpha-\beta}.$$

As $\tilde{\xi}$ is bounded on $[0,\infty)$, we can deduce that

$$I_2(x) \le C_{2,\alpha,\beta}\ x^{\alpha-\beta} A^{-\frac{1}{2}}(x) \tag{3.3}$$

this completes the proof of case I.

Case II: $(\alpha-\beta) > 0$

Let a_1, a_2 be the constants given in Lemma 3.2. Thus from this Lemma we can deduce that there exists a positive constant $C_1(\alpha,\beta)$ such that

$$\left|\psi_\lambda(x)\right| \le C_1(\alpha,\beta) A^{-\frac{1}{2}}(x)\ \lambda^{-(\alpha-\beta+1)} \text{ , for all } x > \frac{a_2}{a_1},\ \lambda > a_1. \tag{3.4}$$

Note that the function

$$s \to s^{\alpha-\beta+1} j_{\alpha-\beta+1/2}(s)$$

is bounded on $[0,\infty)$.

For $x > \dfrac{a_2}{a_1}$, from equality (2.9), we have

$$\frac{1}{\pi}\int_{a_1}^{\infty} \left| b(x,\lambda) \right| d\lambda \le \frac{1}{\pi}\int_{a_1}^{\infty} \left|\psi_\lambda(x)\right| d\lambda + \frac{1}{\pi} x^{\alpha-\beta+1} A^{-1/2}(x) \int_{a_1}^{\infty} \left| j_{\alpha-\beta+1/2}(x) \right| d\lambda$$

$$\le \frac{C_1(\alpha,\beta)}{\pi} A^{-1/2}(x) \int_{a_1}^{\infty} \lambda^{-(\alpha-\beta+1)} d\lambda + \frac{1}{\pi} x^{\alpha-\beta} A^{-1/2}(x) \int_{a_2}^{\infty} \left| j_{\alpha-\beta+1/2}(u) \right| du$$

$$\le \frac{C_1(\alpha,\beta)}{(\alpha-\beta)\pi} A^{-1/2}(x)(1/a_1)^{\alpha-\beta} + \frac{1}{\pi} x^{\alpha-\beta} A^{-1/2}(x) \int_{a_2}^{\infty} \left| j_{\alpha-\beta+1/2}(u) \right| du$$

$$\le \frac{C_1(\alpha,\beta)}{(\alpha-\beta)\pi} A^{-1/2}(x)(x/a_2)^{\alpha-\beta} + \frac{1}{\pi} x^{\alpha-\beta} A^{-1/2}(x) \int_{a_2}^{\infty} \left| j_{\alpha-\beta+1/2}(u) \right| du$$

$$\le C_2(\alpha,\beta)\, x^{\alpha-\beta}\ A^{-1/2}(x) \tag{3.5}$$

where,

$$C_2(\alpha,\beta)=\frac{C_1(\alpha,\beta)}{\pi(\alpha-\beta)}(a_2)^{-\alpha+\beta}+\frac{1}{\pi}\int_{a_2}^{\infty}\left|j_{\alpha-\beta+1/2}(u)\right|du .$$

Now for $0<x<a_2/a_1$, from Lemma 3.2 and the fact that

$$\left|j_{\alpha-\beta+1/2}(\lambda x)\right|\le 1 \text{, for all } x\in\mathbb{R} ,$$

we can deduce that there exists a positive constant $M_1(\alpha,\beta)$ such that

$$|b(x,\lambda)|\le M_1(\alpha,\beta)\ x^{\alpha-\beta+1}A^{-1/2}(x);\ \text{for all } 0<x,\frac{a_2}{a_1},\ 0\le\lambda\le\frac{a_2}{a_1}.$$

This involves

$$\frac{1}{\pi}\int_{a_1}^{a_2}|b(x,\lambda)|\,d\lambda\le\frac{M_1(\alpha,\beta)}{\pi}x^{\alpha-\beta+1}A^{-\frac{1}{2}}(x)\left(\frac{a_2}{a_1}-a_1\right)$$

$$\le\frac{a_2}{\pi}M_1(\alpha,\beta)\,x^{\alpha-\beta}A^{-\frac{1}{2}}(x)\ . \qquad (3.6)$$

But

$$\frac{1}{\pi}\int_{\frac{a_2}{x}}^{\infty}|b(x,\lambda)|\,d\lambda$$

$$\le\frac{C_1(\alpha,\beta)}{\pi}A^{-1/2}(x)\int_{a_2/x}^{\infty}\lambda^{-(\alpha-\beta+1)}d\lambda+\frac{1}{\pi}x^{\alpha-\beta}\ A^{-1/2}(x)\int_{a_2}^{\infty}\left|j_{\alpha-\beta+1/2}(u)\right|du$$

$$\le\frac{C_1(\alpha,\beta)}{(\alpha-\beta)\pi}A^{-1/2}(x)\ a_2^{-(\alpha-\beta)}+\frac{1}{\pi}x^{\alpha-\beta}\ A^{-1/2}(x)\int_{a_2}^{\infty}\left|j_{\alpha-\beta+1/2}(u)\right|du$$

$$\le C_2(\alpha,\beta)\,x^{\alpha-\beta}\ A^{-1/2}(x). \qquad (3.7)$$

Thus by using (3.6) and (3.7), we have

$$\frac{1}{\pi}\int_{a_1}^{\infty}|b(x,\lambda)|\,d\lambda\le M_2(\alpha,\beta)\,x^{\alpha-\beta}\ A^{-\frac{1}{2}}(x) \qquad (3.8)$$

where

$$M_2(\alpha,\beta)=\frac{a_2}{\pi}M_1(\alpha,\beta)+C_2(\alpha,\beta)\ ;\ \text{for all } 0<x<\frac{a_2}{a_1}.$$

Now by using (3.5) and (3.8) it follows that

$$I_2(x)\le M_2(\alpha,\beta)\,x^{\alpha-\beta}\ A^{-\frac{1}{2}}(x);\ \text{for all } 0<x<\frac{a_2}{a_1}.$$

Thus the proof is complete.

We need the following Lemmas to provide estimates or the Kernel h.

Lemma 3.4

(i) $A(x)\sim e^{2\rho x},(x\to+\infty);$ for $\rho>0$

(ii) $A(x)\sim x^{2(\alpha-\beta+1)},(x\to+\infty);$ for $\rho=0$

Proof: Proof follows from the hypothesis of the function A.

Lemma 3.5: For $\rho = 0$ and $(\alpha-\beta) > 0$ there exists two positive constants $D_1(\alpha,\beta)$ and $D_2(\alpha,\beta)$ satisfying

(i) $|\psi_\lambda(x)| \le D_1(\alpha,\beta)\, x^{\alpha-\beta+1} A^{-1/2}(x),\ \ x>0,\ \lambda \ge 0$

(ii) $|\psi_\lambda(x)| \le D_2(\alpha,\beta)|g(\lambda)| A^{-1/2}(x),\ \ x>1,\ \lambda x > 1$

where

$$\lambda \to g(\lambda)$$

is the spectral function given by (2.8).

Now we will give the behavior of the function h for large valves of the variable x

Theorem 3.6: For $\rho = 0, (\alpha-\beta) > 0$ and $a > 0$, there exists a positive constant $K_{\alpha,\beta,a}$ such that for $0 < t < x, x > a$

$$|h(x,t)| \le \mathrm{K}_{\alpha,\beta,a}\, x^{\alpha-\beta} A^{-1/2}(x).$$

Proof: From (2.12), we have

$$h(x,t) = \frac{1}{\pi}\int_0^\infty b(x,\lambda 0\cos(\lambda t)\,d\lambda$$

$$|h(x,t)| \le \frac{1}{\pi}\int_0^\infty |b(x,\lambda)|\,d\lambda = \frac{1}{\pi}\int_0^1 |b(x,\lambda)|\,d\lambda + \frac{1}{\pi}\int_1^\infty |b(x,\lambda)|\,d\lambda. \qquad (3.9)$$

Now from Lemma 2.2 and the fact that $(\alpha-\beta) > 0$, we have

$$\frac{1}{\pi}\int_1^\infty |b(x,\lambda)|\,d\lambda \le \frac{C_1}{\pi} A^{-1/2}(x)\tilde{\xi}(x)\, e^{C_2\tilde{\xi}(x)} \int_1^\infty \lambda^{-(\alpha-\beta+2)}\,d\lambda.$$

As the function $\tilde{\xi}$ is bounded on $[0,\infty)$, there exists $d_{\alpha,\beta}$>0 such that

$$\frac{1}{\pi}\int_1^\infty |b(x,\lambda)|\,d\lambda \le d_{\alpha,\beta}\, x^{\alpha-\beta} A^{-1/2}(x). \qquad (3.10)$$

Note that

$$\frac{1}{\pi}\int_1^\infty |b(x,\lambda|\,d\lambda \le \frac{1}{\pi}\int_0^1 |\psi_\lambda(x)|\,d\lambda + \frac{1}{\pi} x^{\alpha-\beta+1} A^{-1/2}(x)\int_0^1 |j_{\alpha-\beta+1/2}(\lambda x)|\,d\lambda$$

But

$$\frac{1}{\pi}\int_0^1 |\psi_\lambda(x)|\,d\lambda = \frac{1}{\pi}\int_0^{\frac{1}{x}} |\psi_\lambda(x)|\,d\lambda + \frac{1}{x}\int_{\frac{1}{x}}^1 |\psi_\lambda(x)|\,d\lambda.$$

By using (i) of Lemma 3.5, we can obtain

$$\frac{1}{\pi}\int_0^{1/x} |\psi_\lambda(x)|\,d\lambda \le \frac{C_1}{\pi} x^{\alpha-\beta} A^{-1/2}(x). \qquad (3.11)$$

From (ii) of Lemma 3.5 and the relation (2.8), there exists $d_2(\alpha,\beta) > 0$ such that

$$\frac{1}{\pi}\int_{\frac{1}{x}}^1 |\psi_\lambda(x)|\,d\lambda \le \frac{d_2(\alpha,\beta)}{\pi} A^{-\frac{1}{2}}(x)\int_{\frac{1}{x}}^1 \lambda^{-(\alpha-\beta+1)}\,d\lambda$$

$$\leq \frac{d_2(\alpha,\beta)}{\pi} A^{-\frac{1}{2}}(x) \int_{\frac{1}{x}}^{\infty} \lambda^{-(\alpha-\beta+1)} d\lambda$$

$$\leq \frac{d_2(\alpha,\beta)}{\pi(\alpha-\beta)} x^{\alpha-\beta} A^{-\frac{1}{2}}(x). \qquad (3.12)$$

By using relations (3.9), (3.10), (3.11) and (3.12), the proof will be completed.

Theoram 3.7: For $\rho > 0$ and $a > 1$ there exists a positive constant $C_{\alpha,\beta,a}$ such that

$$|h(x,t)| \leq C_2(\alpha,\beta,a) x^{\delta} A^{-1/2}(x); \text{ for all } 0 < t < x;\ x \geq a,$$

where,

$$\delta = \max(1, \alpha-\beta+1).$$

Proof: By using properties (2.5) and (2.6) and proceeding as in Theorem 3.6, the proof can be completed.

4. Hardy type operators T_ψ :

In this section, we shall define a class of integral operators and we recall some of their properties, which we use in the next section to obtain the main result of this paper:

Let $\psi:(0,1) \to (0,\infty)$ be a measurable function, then we associate the integral operator T_ψ defined for all non-negative measurable functions f by

$$\mathrm{T}_\psi(f)(x) = \int_0^x \psi(\frac{1}{x}) f(t) \nu(t)\, dt, \quad \text{for all } x > 0$$

where

(i) ν is a measurable non-negative function on $(0,\infty)$ such that

$$\int_0^a \nu(t)\, dt < \infty, \quad \text{for all } a > 0 \qquad (4.1)$$

and

(ii) μ is a non-negative function on $(0,\infty)$ satisfying

$$\int_a^b \mu(t)\, dt < \infty, \quad \text{for all } 0 < a < b. \qquad (4.2)$$

Following [5],[6], [10], [11], we have the following results:

Theorem 4.1 : Let p,q be two real numbers such that $1 < p \leq q < \infty$

Let ν and μ be two measurable non-negative functions on $(0,\infty)$, satisfying (4.1) and (4.2). Suppose that the function

$$\psi:(0,1) \to (0,\infty)$$

is continuous non increasing and satisfies

$$\psi(xy) \leq D(\psi(x)+\psi(y)), \text{ for all } x, y \in (0,1)$$

where D is a positive constant. Then the following assertions are equivalent:

(i) There exists a positive constant $C_{p,q}$ such that for all non-negative measurable functions f,

$$\left(\int_0^\infty (\mathrm{T}_\psi(f)(x))^q \mu(x) dx\right)^{\frac{1}{q}} \leq C_{p,q} \left(\int_0^\infty (f(x))^p \nu(x) dx\right)^{\frac{1}{p}}.$$

(ii) The functions

$$F(r)=\left(\int_r^\infty \mu(x)dx\right)^{\frac{1}{q}}\left(\int_0^r (\psi(x/r))^{p'}\nu(x)dx\right)^{\frac{1}{p'}}$$

and

$$G(r)=\left(\int_r^\infty (\psi(r/x))^q \mu(x)dx\right)^{\frac{1}{q}}\left(\int_0^r \nu(x)dx\right)^{\frac{1}{p'}}$$

are bounded on $(0,\infty)$, where $p'=\dfrac{p}{p-1}$.

Theorem 4.2: Let p and q be two real numbers such that $1<p\le q<\infty$ and μ,ν two measurable non-negative functions on $(0,\infty)$ satisfying the hypothesis of theorem 4.1.

Let

$$\psi:(0,1)\to(0,\infty)$$

be a measurable non-decreasing function.

If there exists $b\in[0,1]$ such that the function

$$r\to\left(\int_r^\alpha (\psi(r/x))^{bq}\mu(x)dx\right)^{\frac{1}{q}}\left(\int_0^r (\psi(x/r))^{p'(1-b)}\nu(x)dx\right)^{\frac{1}{p'}}$$

is bounded on $(0,\infty)$, then there exists a positive constant $C_{p,q}$ such that for all non-negative measurable functions f, we have

$$\left(\int_r^\infty (\psi(r/x))^{bq}\mu(x)\right)^{\frac{1}{q}}\left(\int_0^r (\psi(x/r))^{p'(1-b)}\nu(x)dx\right)^{\frac{1}{p'}}$$

where $p'=\dfrac{p}{p-1}$.

Corollary 4.3: With the hypothesis of Theorem 4.1 and $\psi=1$, the following assertions are equivalent:

(i) There exists a positive constant $C_{p,q}$ such that for all non-negative measurable functions f we have

$$\left(\int_0^\infty (\mathrm{H}(f)(x))^q \mu(x)dx\right)^{\frac{1}{q}}\le C_{p,q}\left(\int_0^\infty (f(x))^p\nu(x)\,dx\right)^{\frac{1}{p}} ,$$

(ii) The function

$$\mathrm{I}(r)=\left(\int_r^\infty \mu(x)dx)\right)^{\frac{1}{q}}\left(\int_0^r \nu(x)dx\right)^{\frac{1}{p'}}$$

is bounded on $(0,\infty)$, where H is the hardy operator defined by

$$\mathrm{H}(f)(x)=\int_0^x f(t)\,\nu(t)dt,\text{ for all } x>0.$$

5. The Riemann-Liouville and Weyl transforms Associated with the operator $S_{\alpha,\beta}$

In this section we give the proof of inequalities (1.1) and (1.2) stated in first section (introduction).

First we define the following:

(i) $L^p([0,\infty), A(x)\,dx); 1<p<\infty$ the space of measurable functions on $[0,\infty)$ such that

$$\|f\|_{P,A}=\left(\int_0^\infty (f(x)^p\,A(x)\,dx\right)^{\frac{1}{p}}<\infty$$

(ii) $\Re_0$ the operator defined for all non-negative measurable functions f by

$$\Re_0(f)(x)=\int_0^x h(x,t)\,f(t)dt,\ \text{for all}\ x>0$$

where h is the Kernel studied in the third section .

(iii) $\Re_1$ the operator defined for all non-negative measurable functions f by

$$\Re_1(f)(x)=\frac{2\Gamma(\alpha-\beta+3/2)}{\sqrt{\pi}\,\Gamma(\alpha-\beta+1)}x^{\alpha-\beta}\,A^{-1/2}(x)\int_0^x (x^2-t^2)^{\alpha-\beta} f(t)dt\ .$$

Definition 5.1:

(a) The Riemanm-Liouville transform associated with the operator $S_{\alpha,\beta}$ is defined for all non-negative measurable functions f on $(0,\infty)$ by

$$\Re(f)(x)=\int_0^x k(x,t)\,f(t)dt$$

(b) The Weyl tranform associated with operator $S_{\alpha,\beta}$ is defined for all non-negative measurable functions f by

$$W(f)(t)=\int_t^x k(x,t)\,f(x)\,A(x)dx$$

where k is the kernel given by the relation (2.7).

Lemma 5.1:

(i) For $\rho>0$, $(\alpha-\beta)>-1$ and $p>\max(2,2(\alpha-\beta)+3)$ there exists a positive constant $C_1(\alpha,\beta,p)$ such that for all $f\in L^p([0,\infty), A(x)\,dx)$

$$\|\Re_0(f)\|_{P,A}\le C_1(\alpha,\beta,p)\|f\|_{P,A}.$$

(ii) For $\rho=0$, $(\alpha-\beta)>0$ and $(p-3)>2(\alpha-\beta)$ there exists a positive constant $C_2(\alpha,\beta,p)$ such that for all $f\in L^p([0,\infty), A(x)dx)$

$$\|\Re_0(f)\|_{P,A}\le C_2(\alpha,\beta,p)\|f\|_{P,A}.$$

Proof: (i) Suppose that $\rho>0$ and $p>\max(2,2(\alpha-\beta)+3)$. Let $\nu(x)=A^{1-p'}(x)$ and

$$\mu(x)=C_1(\alpha,\beta,a)\,x^{p(\alpha-\beta)}\,A^{1-\frac{p}{2}}(x)1_{(0,a]}(x)+C_2(\alpha,\beta,a)\,x^{p\delta}\,A^{1-\frac{p}{2}}(x)1_{[a,\infty)}(x)$$

with $a>1$, $C_1(\alpha,\beta,a), C_2(\alpha,\beta,a), C_2(\alpha,\beta,a)$ and δ are the constants given in Theorem 3.3 and Theorem 3.7.Then

$$v(x)\le m_1(\alpha,\beta,p)\,x^{2(\alpha-\beta+1)(1-p')}$$

and

$$\mu(x)\le m_2(\alpha,\beta,p)\,x^{2(\alpha-\beta+1)-p}.$$

These inequalities imply that

$$\int_0^n v(x)dx<\infty,\ \text{for all } n>0,$$

$$\int_{n_1}^{n_2}\mu(x)dx<\infty,\ \text{for all } 0<n_1<n_2$$

and

$$I(r)=\left(\int_r^\infty \mu(x)dx\right)^{1/p}\left(\int_0^r v(x)dx\right)^{1/p'}$$

$$\le\left(m_2(\alpha,\beta,p)\int_r^\infty x^{2(\alpha-\beta+1)}dx\right)^{\frac{1}{p}}\left(m_1(\alpha,\beta,p)\int_0^\gamma x^{2(\alpha-\beta+1)(1-p')}dx\right)^{\frac{1}{p'}}$$

$$\le\frac{(m_2(\alpha,\beta,p))^{\frac{1}{p}}\,m_1(\alpha,\beta,p)^{\frac{1}{p'}}}{(p-2(\alpha-\beta)-3)^{\frac{1}{p}}\left[(2(\alpha-\beta+1)(1-p')+1\right]^{\frac{1}{p'}}}$$

$$=\frac{(m_2(\alpha,\beta,p))^{\frac{1}{p}}\left[(p-1)\,m_1(\alpha,\beta,p)\right]^{\frac{1}{p'}}}{p-2(\alpha-\beta)-3}.$$

Now from Corollary 4.3, there exists a positive constant $C_{p,\alpha,\beta}$ such that for all non-negative measurable functions g, we have

$$\left(\int_0^\infty (\mathrm{H}(g)(x))^p\,\mu(x)dx\right)^{\frac{1}{p}}\le C_{p,\alpha,\beta}\left(\int_0^\infty (g(x))^p\,v(x)dx\right)^{\frac{1}{p}} \qquad (5.1)$$

with

$$\mathrm{H}(g)(x)=\int_0^\infty g(t)v(t)dt.$$

Put

$$T(f)(x)=\left(\frac{\mu(x)}{A(x)}\right)^{\frac{1}{p}}\int_0^x f(t)dt,$$

we have,

$$\mathrm{H}(g)(x)=\left(\frac{\mu(x)}{A(x)}\right)^{-\frac{1}{p}}T(f)(x),$$

where

$$g(x) = f(x)\, A^{p'-1}(x).$$

By using inequality (5.1), we can infer that for all non-negative measurable functions f, we have

$$\left(\int_0^\infty (T(f)(x))^p\, A(x)dx \right)^{\frac{1}{p}} \le C_{p,\alpha,\beta} \left(\int_0^\infty (f(x))^p\, A(x)dx \right)^{\frac{1}{p}}. \qquad (5.2)$$

From Theorem 3.3 and Theorem 3.7, we can infer that the function

$$\Re_0(f)(x) = \int_0^x h(x,t)\, f(t)\, dt$$

is well defined and we have

$$|\Re_0(f)(x)| \le T(|f|(x). \qquad (5.3)$$

From inequalities (5.2) and (5.3), we can obtain

$$\left(\int_0^\infty |\Re_0(f)(x)|^p\, A(x)dx \right)^{\frac{1}{p}} \le C_{p,\alpha,\beta} \left(\int_0^\infty |f(x)|^p\, A(x)dx \right)^{\frac{1}{p}}$$

which proves (i).

(ii) Suppose that $\rho = 0$ and $(\alpha-\beta) > 0$. From Theorem 3.3 and Theorem 3.6, we have

$$|h(t,x)| \le C\, x^{\alpha-\beta}\, A^{-1/2}(x); \text{ for all } 0 < t < x.$$

If we take $\mu(x) = x^{(\alpha-\beta)p}\, A^{1-p/2}(x)$ and $\nu(x) = A^{1-p'}(x)$ and proceeding as in the proof of (i) above, we can obtain the result in (ii).This completes the proof.

Lemma 5.2: Suppose that $-1 < (\alpha-\beta) \le 0$, $\rho = 0$ and that there exists a positive constant a such that

$$h(x,t) = 0, \text{ for all } 0 < t < x,\ x > a.$$

Then for all $(p-3) > 2(\alpha-\beta)$, we can find a positive constant $C_{\alpha,\beta,a}$ such that

$$\|\Re_0(f)\|_{p,A} \le C_{\alpha,\beta,a}\, \|f\|_{p,A}; \text{ for all } f \in L^p\left([0,\infty), A(x)dx\right).$$

Proof: By Theorem 3.3 and the hypothesis, there exists a positive constant a such that

$$|h(x,t)| \le C(\alpha,\beta,a)\, x^{\alpha-\beta}\, A^{-1/2}(x) 1_{(0,a]}(x) \text{ ;for all } 0 < t < x.$$

Thus by taking

$$\mu(x) = C(\alpha,\beta,a)\, x^{p(\alpha-\beta)}\, A^{1-p/2}(x) 1_{(0,a]}(x), \quad \nu(x) = A^{1-p'}(x)$$

and following a similar procedure as Lemma 5.1 and Lemma 5.2, we can obtain the required result.

Lemma 5.3

(i) For $(\alpha-\beta) > -1, \rho = 0$ and $p > \max(2, 2(\alpha-\beta)+3)$,there exists a positive constant $C_{p,\alpha,\beta,a}$ such that for all $f \in L^p\left([0,+\infty), A(x)dx\right)$, we have

$$\|\Re_1\|_{p,A} \le C_{p,\alpha,\beta}\, \|f\|_{p,A}$$

(ii) For $(\alpha-\beta) > -1, \rho = 0$ and $(p-3) > 2(\alpha-\beta)$,there exists a positive constant $C_{p,\alpha,\beta,a}$ such that for all $f \in L^p\left([0,\infty), A(x)dx\right)$, we have

$$\|\Re_1(f)\|_{p,A} \le C_{p,\alpha,\beta}\, \|f\|_{p,A}.$$

Proof: Let T_ψ be the Hardy type operator defined for all non-negative measurable functions f by

$$T_\psi(f)(x)=\int_0^x \psi(t/x)\, f(t)\nu(t)\,dt\,,$$

where

$$\psi(x)=(1-x^2)^{\alpha-\beta}$$

and

$$\nu(x)=A^{1-p'}(x)\,.$$

Then for all non-negative measurable functions f, we have

$$\Re_1(f)(x)=C_{\alpha,\beta}\, x^{\alpha-\beta}\, A^{-1/2}(x)T_\psi(g)(x) \qquad (5.4)$$

where

$$g(x)=f(x)\,A^{p'-1}(x)\,.$$

Take

$$\mu(x)=x^{p(\alpha-\beta)}\, A^{1-p/2}(x)\,.$$

Then according to the hypothesis satisfied by the function A, there exist positive constants $C_{1,}\, C_2$ such that for all $(\alpha-\beta)>-1$ and $\rho>0$ we have

$$0\le\mu(x)\le C_1\, x^{2(\alpha-\beta+1)-p} \qquad (5.5)$$

$$0\le\nu(x)\le C_2\, x^{2(\alpha-\beta+1)(1-p^1)}\,. \qquad (5.6)$$

Now for $(\alpha-\beta)\ge 0,\ \rho>0$ and $(p-3)>2(\alpha-\beta)$, from inequalities (5.5) and (5.6) we can infer that

(a) the function ψ is continuous and non-increasing on (0,1)

(b) the functions ψ,ν and μ satisfy the hypothesis of Theorem 4.1

(c) the functions

$$F(r)=\left(\int_r^\infty \mu(x)dx\right)^{\frac{1}{p}}\left(\int_0^r (\psi(x/r)^{p'}\, \nu(x)dx\right)^{\frac{1}{p'}}$$

and

$$G(r)=\left(\int_r^\infty (\psi(r/x)^{p}\, \mu(t)dt\right)^{\frac{1}{p}}\left(\int_0^r \nu(t)dt\right)^{\frac{1}{p'}}$$

are bounded on $[0,\infty)$.

Hence from Theorem 4.1, there exists $C_{p,\alpha,\beta}>0$ such that for all measurable non-negative functions f we have

$$\left(\int_0^\infty (T_\psi(f(x))^p\, \mu(x)dx\right)^{\frac{1}{p}}\le C_{p,\alpha,\beta}\left(\int_0^\infty (f(x)^p\, \nu(x)dx\right)^{\frac{1}{p}}.$$

This inequality together with the relation (5.4) lead to

$$\left(\int_0^\infty (\Re_1(f(x))^p\, A(x)\,dx\right)^{\frac{1}{p}}\le C_{p,\alpha,\beta}\left(\int_0^\infty (f(x))^p\, A(x)dx\right)^{\frac{1}{p}}$$

which proves (i) in the case $(\alpha-\beta)\geq 0$.

For $-1<(\alpha-\beta)<0$ and $p>2$, we have

(d) the function ψ is continuous and non-decreasing on (0,1)

(e) if we pick

$$b\in\left(\max\left(0,\frac{1-p(\alpha-\beta+1)}{-p(\alpha-\beta)}\right),\min\left(1,\frac{1}{-p(\alpha-\beta)}\right)\right)$$

and using relations (5.5) and (5.6) we can obtain that the function

$$\mathrm{H}(r)=\left(\int_r^\infty(\psi(r/x))^{bp}\,\mu(x)dx\right)^{\frac{1}{p}}\left(\int_0^r(\psi(x/r)^{(1-b)p'}\,\nu(x)dx\right)^{\frac{1}{p'}}$$

is bounded on $(0,\infty)$.

Finally by using Theorem 4.2 and equation (5.4), proof of (i) can be completed (ii) can be proved in the same manner as (i).

Theorem 5.4:

(i) For $(\alpha-\beta)>-1,\rho>0$ and , $p>\max(2,2(\alpha-\beta)+3)$ there exists a positive constant $C_{p,\alpha,\beta}$ such that for all $f\in L^{p'}([0,\infty),A(x)dx)$

$$\|R(f)\|_{p,A}\leq C_{p,\alpha,\beta}\|f\|_{p,A}$$

(ii) For $(\alpha-\beta)>-1,\rho>0$ and $p>\max(2,2(\alpha-\beta)+3$ there exists a positive constant $C_{p,\alpha,\beta}$ such that for all $g\in L^{p^1}([0,\infty),A(x)dx)$

$$\left\|\frac{1}{A(x)}W(g)\right\|_{p',A}\leq C_{p,\alpha,\beta}\|g\|_{p',A}$$

where $p'=\dfrac{p}{p-1}$.

Proof: (i) Follows from first parts of Lemma 5.1, Lemma 5.3 and the fact that

$$\Re(f)=\Re_0(f)+\Re_1(f)$$

(ii) follows from (i) and the relations

$$\|g\|_{p',A}=\max_{\|f\|_{p,A}\leq 1}\int_0^\infty f(x)\,g(x)\,A(x)\,dx \qquad (5.7)$$

for all measurable non-negative functions f and g and

$$\int_0^\infty\Re(f)(x)\,g(x)\,A(x)\,dx=\int_0^\infty W(g)(x)\,f(x)dx.$$

Theorem 5.5:

(i) For $(\alpha-\beta)>0,\rho=0$ and $(p-3)>2(\alpha-\beta)$ there exists a positive constant $C_{p,\alpha,\beta}$ such that for all $f\in L^p([0,\infty),A(x)dx)$

$$\|\Re(f)\|_{p,A}\leq C_{p,\alpha,\beta}\|f\|_{p,A}$$

(ii) For $(\alpha-\beta)>0,\rho=0$ and $(p-3)>2(\alpha-\beta)$ there exists a positive constant $C_{p,\alpha,\beta}$ such that for all $g\in L^{p'}([0,\infty)),A(x)dx)$

$$\left\| \frac{1}{A(x)} W(g) \right\|_{p',A} \leq C_{p,\alpha,\beta} \|g\|_{p',A}$$

where , $p' = \dfrac{p}{p-1}$.

(iii) For $-1 < (\alpha - \beta) \leq 0, \rho = 0, (p-3) > 2(\alpha - \beta)$ and under the hypothesis of Lemma 5.2, the previous results hold.

Proof: Proof can be completed by using above Lemmas.

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