STATIC AND DYNAMIC ANALYSIS OF RECTANGULAR ISOTROPIC PLATE USING MULTIQUADRIC RADIAL BASIS FUNCTION

R.K. Misra*

Abstract.

This paper presents a methodology based on the collocation multiquadric radial basis functions to analyze the static and dynamic behavior of isotropic rectangular plates. The inertia and dissipative terms are evaluated by employing Newmark implicit time marching scheme. The spatial discretization of the differential equations generates greater number of algebraic equations than the unknown coefficients. The multiple linear regression analysis, based on the least square error norm, is employed to obtain the coefficients. Convergence study has been carried out. The clamped and simply supported immovable rectangular plates subjected to uniformly distributed loadings are studied. Results have been compared with the results obtained by other numerical and analytical methods.

Notations

a, b	Dimension of plates
D	Flexural rigidity of plates
E	Young's modulus
h	Thickness of plates
C_v^*, C_v	Viscous damping, dimensionless viscous damping
q, Q	Transverse load, dimensionless transverse load
t^*, t	Time, dimensionless time

* Department of Mechanical Engg, School of Engineering, Gautam Buddha University, Gautam Buddha Nagar, Greater Noida-201310 (U.P) India.

λ	Aspect ratio (a/b)
ν	Poisson's ratio
ρ	Mass density of plates
w	Dimensionless displacement in z direction
w^*	Displacement in z^* direction
m	Mass of the plate

Key words: radial basis function, dynamic response, and isotropic rectangular plates.

1. Introduction

Uniformly loaded plate is a basic structure in mechanical engineering, building construction, civil engineering and aerospace industry. The design codes rely heavily on accurate theoretical analyses to predict the dynamic behavior. The dynamic characteristics of rectangular plates are important in engineering designs. Xiang et al. [1] used the Ritz method to solve the vibration of rectangular mindlin plates resting on elastic supports. The fundamental frequency coefficient for a rectangular plate with edges elastically restrained against both translation and rotation using polynomial coordinate functions and the Rayleigh-Ritz's method has been calculated by Laura et al. [2-3]. Non-uniform plate with the rotational springs at the edges using the Rayleigh-Ritz method is described by the Grossi et al. [4]. Gorman [5] solved the free vibration problem of shear-deformable plates resting on uniform elastic foundations using the modified Superposion-Galerkin method. Omurtag et al. [6] used mixed finite element formulation to analyze free vibration of orthotropic plate. For a uniformly loaded elastic plate undergoing small deflections, the Kirchhoff thin plate equation is adequate to describe the phenomena. In traditional methods, an approximate solution is found by using either Navier or Levy expansion series [7-8]. Analytical methods are restricted to simple geometries and boundary conditions [9], but multi quadric radial basis function (MQRBF) can be used for complex geometry and different boundary conditions. In 1990, Kansa developed domain-type meshless methods by collocating the RBFs, particularly the multiquadric (MQ), for the numerical approximation of the solution [10].

Franke [11] has ranked MQ as the best interpolation method based on its accuracy, visual aspect, and sensitivity to parameters, execution time, storage requirements and ease of implementation.

Kansa's method was recently extended to solve various ordinary and PDE. Liu et al [12, 13] has proposed the usage of RBFs for 2-D solids. This method also used to solve problems such as tissue engineering [14, 15], one-dimensional nonlinear burgers equation [16]. To the best of the authors' knowledge, this method has not been used to study the behavior of plates subjected to dynamic loading. Therefore, MQRBF method is developed here for numerical solution of fourth-order partial differential governing equation. However, the numerical results depend on the choice of radial distance between points, hence very small distance is always needed to achieve enough numerical accuracy. In the present work, MQRBF is used to obtain static and dynamic analysis of a rectangular isotropic plate. Using proper boundary conditions, greater numbers of equations are generated than the number of unknowns. Multiple linear regression analysis based on the least-square error norms is employed during the solution in order to overcome the incompatibility, i.e. greater number of equations than the unknowns, generated by Kansa method. Newmark time-marching technique is used for the temporal discretization. The results are compared and a good agreement is found. Both the static and dynamic analyses under uniformly distributed transverse load have been carried out.

2. Governing equations

The coordinate system, geometry and loading are shown in Figure 1.Neglecting the in-plane and rotary inertia. Equation of thin plate is expressed in non-dimensional form,

$$(w_{xxxx} + 2\lambda^2 w_{xxyy} + \lambda^4 w_{yyyy}) + w_{tt} + c_v w_t - Q(x,y,t) = 0 \qquad (1)$$

Where the subscript denotes the partial derivative with respect to the following suffix. The non-dimensional quantities are defined by

$$w = w^* / h, x = x^* / a, y = y^* / b, \lambda = a / b, t = t^* \sqrt{D/(\rho a^4 h)}$$

$$Q = qa^4 /(Dh), D = Eh^3 /12(1-v^2), C_v = C_v^* / m)\sqrt{(\rho a^4 h)/D}, m = \rho abh), \qquad (2)$$

The boundary conditions considered are as follows:

- For all four edges simply supported

$$x = 0,1 \qquad w = 0, w_{xx} = 0 \qquad (3)$$

$$y = 0,1 \qquad w = 0, w_{yy} = 0 \qquad (4)$$

- For all four edge clamped

$x = 0,1 \qquad w = 0, w_x = 0$ (5)

$y = 0,1 \qquad w = 0, w_y = 0$ (6)

For initial conditions it is assumed that at *t=0*

$w = 0, w_t = 0, w_{tt} = 0$ (7)

3. The Multiquadric Radial Basis Function Method

Consider a general differential equation

$Aw = f(x, y) \quad \text{in} \quad \Omega$ (8)

$Bw = g(x, y) \quad \text{on} \quad \partial\Omega$ (9)

Where A is a linear differential operator and B is a linear boundary operator imposed on boundary conditions for the composite plate.

Let $\{P_i = (x_i \,,\, y_i)\}_{i=1}^{N}$ be N collocation points in domain Ω of which $\{(x_i \,,\, y_i)\}_{i=1}^{N_I}$ are interior points; $\{(x_i \,,\, y_i)\}_{i=N_I+1}^{N}$ are boundary points. In MQRBF method, the approximate solution for differential equation (16) and boundary conditions (17) can be expressed as:

$$w(x, y) = \sum_{j=1}^{N} w_j \varphi_j(x, y) \qquad (10)$$

and multiquadric radial basis as

$$\varphi_j = \sqrt{(x - x_j)^2 + (y - y_j)^2 + c^2} = \sqrt{r_j^{\,2} + c^2} \qquad (11)$$

where $\{w_j\}_{j=1}^{N}$ are the unknown coefficients to be determined, and $\varphi_j(x_j, y_j)$ is a basis function. Other widely used radial basis functions are

$\varphi(r) = (c^2 + r^2)^{1/2}$	Multiquadrics
$\varphi(r) = (c^2 + r^2)^{-1/2}$	Inverse multiquadrics
$\varphi(r) = r^3$	Cubic
$\varphi(r) = r^2 \log(r)$	Thin plate splines
$\varphi(r) = (1 - r)^m{}_+ \, p(r)$	Wendland functions
$\varphi(r) = e^{-(cr)^2}$	Gaussian

Here c is a shape parameter, a positive constant. Ling and Kansa [17] discussed about the shape parameter

a. Multiquadric method for governing differential equation

Substituting radial basis function in equation (1), we get

$$(\sum_{j=1}^{N} w_j \frac{\partial^4}{\partial x^4}\varphi_j + 2\lambda^2 \sum_{j=1}^{N} w_j \frac{\partial^4}{\partial x^2 y^2}\varphi_j + \sum_{j=1}^{n} \lambda^4 w_j \frac{\partial^4}{\partial y^4}\varphi_j) + \sum_{j=1}^{N} w_j \frac{\partial^2}{\partial t^2}\varphi_j - C_v w_j \frac{\partial}{\partial t}\varphi_j - Q = 0 \quad (12)$$

b. Boundary Conditions

❖ *For Simple supported edge*

$$x = 0,1 \qquad \sum_{j=1}^{N} w_j \varphi_j = 0 \qquad (13)$$

$$y = 0,1 \qquad \sum_{j=1}^{N} w_j \varphi_j = 0 \qquad (14)$$

$$x = 0,1 \qquad \sum_{j=1}^{N} w_j \frac{\partial^2}{\partial x^2}\varphi_j = 0 \qquad (15)$$

$$y = 0,1 \qquad \sum_{j=1}^{N} w_j \frac{\partial^2}{\partial y^2}\varphi_j = 0 \qquad (16)$$

❖ *For all four edge clamped*

$$x = 0,1 \qquad \sum_{j=1}^{N} w_j \varphi_j = 0 \qquad (17)$$

$$y = 0,1 \qquad \sum_{j=1}^{N} w_j \varphi_j = 0 \qquad (18)$$

$$x = 0,1 \qquad \sum_{j=1}^{N} w_j \frac{\partial}{\partial x}\varphi_j = 0 \qquad (19)$$

$$y = 0,1 \qquad \sum_{j=1}^{N} w_j \frac{\partial}{\partial y}\varphi_j = 0 \qquad (20)$$

❖ *For initial conditions:* At *t=0*

w=0 and $w_t=0$ (21)

The generating equation (12) gives rise to N_I algebraic equations respectively. The 4 boundary conditions through Eqs. (13)- (16) give $2x\ (N- N_I\ )+8$ algebraic equations. The total number of equations obtained are $2N- N_I +8$ and total number of unknowns w_j are N. It can be noted that total number of equations are more than the total number of unknowns. In order to have a compatible solution, the multiple regression analysis based on least-square error norms is used which leads to compatibility. Dynamic terms are transformed to the right side so the left side matrix is invariant with respect to marching variable time t or loading Q. The set of equations (12) – (16) can be expressed in the matrix form as:

$$A.a = p \tag{22}$$

where A is (l* k) coefficient matrix, a is (k*1) vector, P is (l*1) load vector. Multiple regression analysis gives

$$a = (A^T A)^{-1} A^T P \tag{23}$$

or

$$a = B.P \tag{24}$$

Details are given in Appendix *I*. The matrix *A* is evaluated once and stored for subsequent usages. Same procedure is adopted for clamped plate.

4. Results and discussions

In this study, the governing equations of equilibrium of isotropic rectangular plate subjected to uniformly distributed transverse static and dynamic loadings are considered. The equations of motion are solved in space domain using multiquadric radial basis function and Newmark time-marching technique is used for the time domain discretization. More number of algebraic equations is generated than the unknown coefficient and a multiple linear regression analysis based on least-square error norms is employed to solve these equations. Simply supported immovable and clamped immovable boundary conditions are considered.

a. Static analysis

In order to have a check on the accuracy and stability of the method, convergence study has been carried out. It can be noted from Table 1 that 5x5 grid size is sufficient to yield quite accurate results. The static results of simply supported immovable plates are compared with the results of Timoshenko and Krieger [18] and shown in Table 2. Similarly, results for clamped immovable plates are compared with those of Timoshenko and Krieger and shown in Table 3. It can be noted that there is good agreement among the results in both simply supported and clamped immovable cases.

b. Dynamic analysis

A computer program based on the finite difference method (FDM) proposed by Chadrashekara [19] is also developed. The damped response of simple supported composite plates obtained by the present method and finite difference method has been compared and shown in Figure 2 for non-dimensional load Q = 60 and non-dimensional viscous damping factor C_v = 1.25. There is good agreement in the results. The damped dynamic analyses of clamped immovable and simply supported immovable plates have been carried out. The results for simply supported plates subjected to step loading Q = 60, and = 1.25, 6, 10 are shown in Figure 3. Similar results for clamped immovable plates subjected to step loading Q = 60, and = 1.25, 6, 10 are shown in Figure 4. It is found that the methodology is quite simple and efficient and can be extended to nonlinear problems and other areas.

5. Conclusions

Finite difference method (FDM), Finite element method (FEM) and Finite volume method (FVM) have been used by earlier researchers for solving partial differential equations. For supporting the localized approximations, these methods generate mesh. In these methods, there is continuity in function across meshes, but not in its partial derivatives. The general cases were taken to demonstrate the validity of the meshless formulation. Advantages of multiquadric radial basis functions compared to these methods are summarized as follows:

- Easy implementation for analysis of structure.
- It provides reasonably good results, at small number of grid points.
- Highly suitable for computer implementation.
- No connectivity is required for arbitrarily distributed nodes.

Above features of multiquadric radial basis functions method have been suitably demonstrated by analyzing the static and dynamic analysis of isotropic plates at various boundary conditions.

Appendix I

Multiple regression analysis

$A.a = p$

where A is $(l*k)$ coefficient matrix, a is (k*1) vector, p is $(l*1)$ load vector.

Approximating the solution by introducing the error vector e, we get

$p = Aa + e$, *where* e *is* (l*1) *vector.*

To minimize the error norm, let us define a function S as

$$S(a) = e^T e = (p - Aa)^T (p - Aa)$$

The least-square norm must satisfy

$$(\partial S / \partial a)_a = -2A^T p + 2A^T Aa = 0$$

This can be expressed as

$$a = (A^T A)^{-1} A^T P \ \ or$$

$$a = B.P$$

The matrix B is evaluated once and stored for subsequent *usages.*

Reference

1. Xiang, Y., Liew, K. M. and Kitipornchai, S., (1997): Vibration Analysis of Rectangular Mindlin Plates Resting on Elastic Edge Supports. Journal of Sound and Vibration, vol. 204, pp. 1-16.
2. Laura, P. A. A. and Grossi, R. O. (1978): Transverse Vibration of a Rectangular Plate Elastically Restrained against Rotation along Three Edges and Free on the Fourth Edge. Journal of Sound Vibration, vol.59, pp. 355-368.
3. Laura, P. A. A. and Gross, R. O. (1981): Transverse Vibrations of Rectangular Plates with Edges Elastically Restrained against Translation and Rotation. J Sound and Vibration, vol. 75, pp. 101-107.
4. Grossi, R. O. and Nallim, L. G. (1997): A Note on the Strain Energy Stored in Rotational Springs at the Plate Edges of Non-uniform Thickness. J Sound and Vibration, vol. 206, pp. 448-452.
5. Gorman, D. J., (1997): Accurate Free Vibration Analysis of Shear-deformable Plates with Torsional Elastic Edge Support. J Sound and Vibration, vol. 203, pp. 209-218.
6. Omurtag, M. H. and Kadioglu, F. (1998): Free Vibration Analysis of Orthotropic Plates Resting on Pasternak Foundation by Mixed Finite Element Formulation. Comput Struct, vol. 67, pp. 253-265.

7. Timoshenko S, Woinowsky-Krieger S. (1959): Theory of plates and shells, 2d ed. New York: McGraw-Hill.
8. Szilard R. (1974): Theory and analysis of plates. New Jersey: Prentice-Hall.
9. Rossi, R.E., Bambill, D.V., and Laura, A.A., (1998): Vibrations of a rectangular orthotropic plate with a free edge: comparison of analytical and numerical result. Ocean Engg, vol. 25, no.7, pp. 521–527.
10. Kansa E.J., (1990) Multiquadric – a scattered data approximation scheme with applications to computational fluid dynamics II. Comput Math Appl, vol. 19(8/9), pp.147-61.
11. Franken R., (1982): Scattered data interpolation: tests of some methods. Math Comput, vol.48, 181-200.
12. Liu GR, Gu YT., (2001): A local radial point interpolation method for free vibration analyses of 2-d solids. J. Sound vibration, vol. 246(1), pp. 29-46.
13. Liu GR, Wang JG., (2002): A point interpolation meshless method based on radial basis functions. Int J Num Meth Eng, vol.54, pp.1623-48.
14. Hon YC, Lu MW, Xue WM, Zhu YM., (1997): Multiquadric method for the numerical solution of a biphasic mixture model. Appl math Comput, vol. 88, pp. 153-75.
15. Hon YC, Lu MW, Xue WM, Zhou X., (1999): A new formulation and computation of the triphasic model for mechano-electrochemical mixtures. Comput Mech, vol. 24, pp. 155-165.
16. Hon YC, Mao XZ., (1998): An efficient numerical scheme for Burgers equation. Appl Math Comput, vol. 95(1), pp. 37-50.
17. Ling L, Kansa EJ., (2004): Preconditioning for radial basis functions with domain decomposition methods. Math Comput Model, vol. 40, 1413–1427.
18. Stephen P. Timoshenko and S.Woinowsky-Krieger. (1989): Theory of plates and shells: McGraw-Hill International Editions.
19. Chandrashekhara K. (2001): Theory of plates: Universities Press (India) Limited.

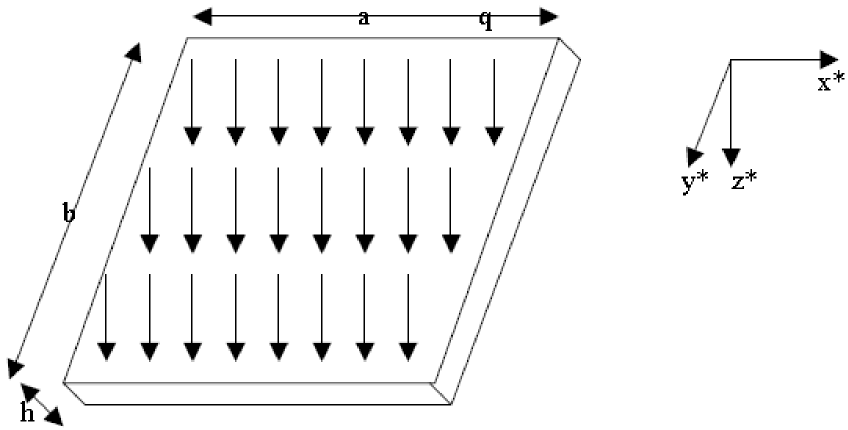

Figure 1: Geometry of the rectangular plate

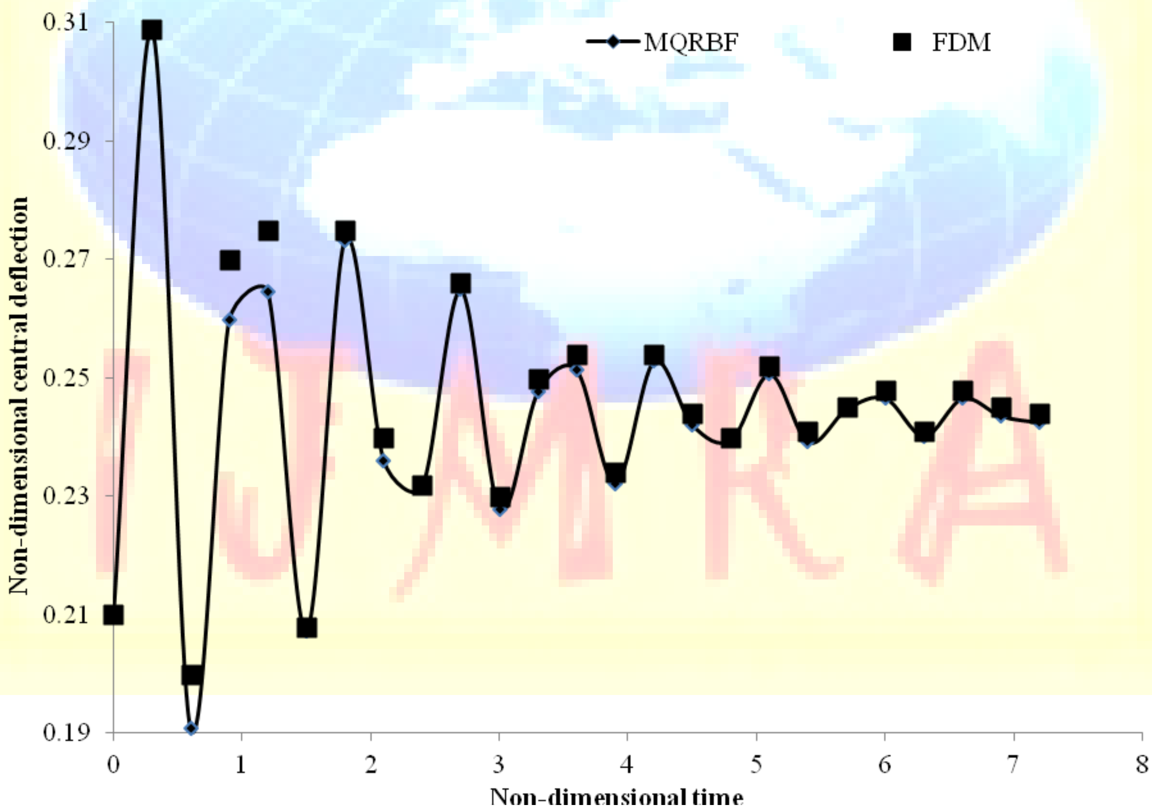

Figure 2: Damped response of the simple supported plates at non-dimensional load

Q = 60 and non-dimensional viscous damping factor C_v = 1.25.

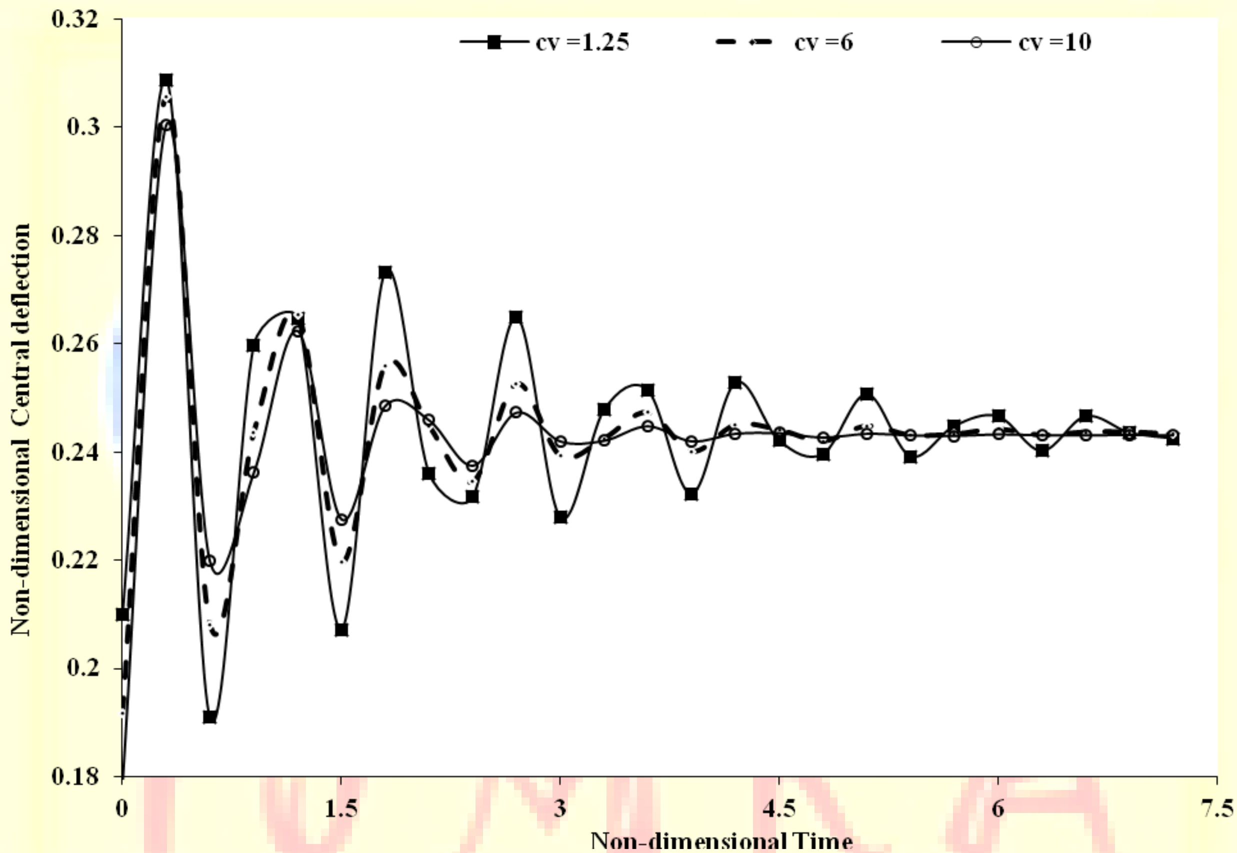

Figure 3: Damped response of the simple supported plates at non-dimensional load

Q = 60 and non-dimensional viscous damping factor C_v = 1.25, 6 and 10

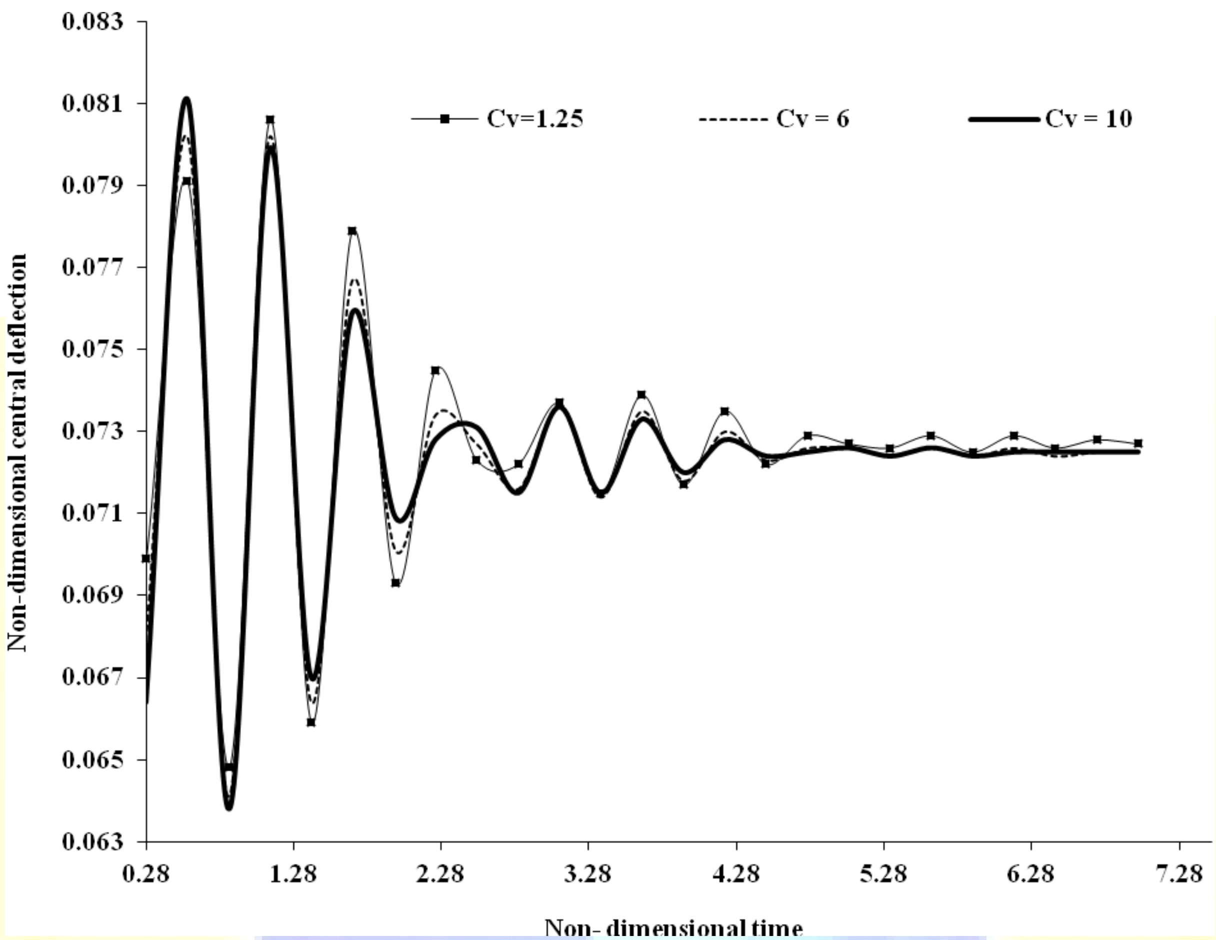

Figure 4: Damped response of the clamped plates at non-dimensional load

Q = 60 and non-dimensional viscous damping factor C_v = 1.25, 6 and 10

Table.1 Central deflection at Static Load

Grid	$w_{max} = \beta(qa^4/D)$ β
5x5	0.00410
7x7	0.00400
9x9	0.00407

Table 2. Simply Supported Rectangular Plate

b/a	w_{max} [18] $w_{max} = \alpha\ (qa^4/D)$ α	w_{max} (present method) $w_{max} = \beta(qa^4/D)$ β
1.0	0.00406	0.0041
2.0	0.01013	0.0101
3.0	0.01223	0.0122
∞	0.01302	0.0130

Table .3 Clamped Edges Rectangular Plate

b/a	w_{max} [18] $w_{max} = \alpha\ (qa^4/D)$ α	w_{max} (present method) $w_{max} = \beta(qa^4/D)$ β
1.0	0.00126	0.0012
1.5	0.00220	0.0021
2.0	0.00254	0.0025
∞	0.00260	0.0026