THE CONCEPTUAL FRAMEWORK ON DIFFERENTIAL EQUATIONS

K. Thanesh Kumar*

Abstract

The area of differential equations is a very broad field of study. The versatility of differential equations allows the area to be applied to a variety of topics from physics to population growth to the stock market. They are a useful tool for modeling and studying naturally occurring phenomena such as determining when beams may break as well as predicting future outcomes such as the spread of disease or the changes in populations of different species over time. Anytime an unknown phenomenon is changing with respect to time or space, a differential equation is involved. The current paper discuss the conceptual underpinnings of differential equations and this study discussed on two basic theories IVP and BVP.

Keywords: Differential equations, IVP, BVP, ODE and PDE

* M.Sc, M.Phil, (PhD) Assistant Professor Institute of Aeronautical Engineering Dundigal, Hyderabad

INTRODUCTION

The area of differential equations is a very broad field of study. The versatility of differential equations allows the area to be applied to a variety of topics from physics to population growth to the stock market. They are a useful tool for modeling and studying naturally occurring phenomena such as determining when beams may break as well as predicting future outcomes such as the spread of disease or the changes in populations of different species over time. Anytime an unknown phenomenon is changing with respect to time or space, a differential equation is involved.

In more general terms, a differential equation is simply an equation involving an unknown function and its derivatives. To be more technical, a differential equation is a "mathematical equation for an unknown function of one or several variables that relates the values of the function itself and its derivatives of various orders"to a particular phenomena [2]. Differential equations generally fall into two categories: ordinary differential equations (ODE) or partial differential equations (PDE), the distinction being that ODEs involve unknown functions of one independent variable while PDEs involve unknown functions of more than one independent variable.

In this paper we will focus on ordinary differential equations.

Some defining characteristics of a differential equation are its order and if it is linear. The order of the equation refers to the highest order derivative present. A differential equation is said to be linear if it is linear in its dependent variable and its derivatives, i.e. in the case of an ODE, if it can be written in the form

$$a_n(x)y^n + a_{n-1}(x)y^{n-1} + \cdots + a_1(x)y' + a_0(x)y = Q(x) \ [1].$$

In addition, we say that linear differential equations are homogeneous when Q(x) = 0. A very important property of homogeneous linear ordinary differential equations says that there are n linearly independent solutions for an n_{th} order equation and that all solutions can be written as a linear combination of these solutions.

Initial conditions are when y and its derivatives are evaluated at a single point.

Typically an nth order ODE will have

$$y(x_0), y'(x_0), y''(x_0), \ldots, y^{(n-1)}(x_0)$$

given. A differential equation together with these initial conditions is called an initial value problem (IVP).

If y and/or its derivatives are evaluated at two different points we say that we have a boundary condition. A boundary value problem (BVP) is a differential equation with boundary conditions. An example of a BVP is the equation

$$y'' + p(x)y' + q(x)y = g(x)$$

with the boundary condition

$$y(\alpha) = a, \quad y(\beta) = b.$$

When g(x) = 0 and a = b = 0, the BVP is said to be homogeneous.

The following are examples of common differential equations. The first two examples are IVPs and the last two are examples of IBVPs.

Example- 1. The Mass-Spring Equation. The motion of a mass on the end of a spring can be given by

$$my'' + cy' + ky = f(t),$$

$$y(0) = a, y'(0) = b,$$

Where y is the position of the mass from equilibrium, m is the mass, c is a damping constant, k is the spring constant, and f(t) is an outside forcing function. Usually y(0) and y'(0) are given and represent the initial position and velocity, respectively.

Example-2. The population of mosquitoes in a certain area increases at a rate proportional to the current population, and, in the absence of other factors, the population doubles each week. If there are 200,000 mosquitoes in the area initially, and predators eat 20,000 mosquitoes each day. What is the population of mosquitoes in the area at any time [1]?

$$p'(t) = r(t) - q(t),$$

$$p(0) = 200,000,$$

Where r(t) is the rate of growth for the population and q(t) is the death rate for the population. The solution for this problem is

$$p(t) = 201,977.31 - 1977.31e^{t\ln(2)},$$

Where t ≥ 0 is time measured in weeks.

This type of problem requires us to model a differential equation to fit the stipulations of the growth rate of the population as well as the death rate. For more applied situations, such as developing a disease model, there will typically be a system of differential equations to solve as opposed to a single ODE.

Example - 3. The Heat Equation. Consider a one-dimensional bar of length L.
The transfer of heat in this bar is given by

$$\alpha^2 u_{xx} = u_t, \quad \text{for } 0 < x < L, t \geq 0$$

with boundary conditions

$$u(0,t) = 0, \quad u(L,t) = 0, \quad \text{for } t > 0$$

and initial condition

$$u(x,0) = f(x), \quad \text{for } 0 \leq x \leq L$$

Where u represents the heat in the bar for every x 2 [0, L] and t _ 0 and f is the initial temperature distribution [1]. It should be noted that the boundary conditions can be interpreted as holding the temperature at the ends of the bar at zero degrees.

Example - 4. The Wave Equation. The motion of a string of length L can be described by

$$\alpha^2 u_{xx} = u_{tt}, \quad \text{for } 0 < x < L,\ t \geq 0$$

with boundary conditions

$$u(0,t) = 0, \quad u(L,t) = 0, \quad \text{for } t \geq 0$$

and initial conditions

$$u(x,0) = f(x), \quad \text{for } 0 \leq x \leq L$$

$$u_t(x,0) = g(x), \quad \text{for } 0 \leq x \leq L,$$

Where u represents the position of the string from equilibrium at any x 2 [0, L] and t ≥ 0, f is the initial position and g is the initial velocity [1]. Here the boundary conditions represent the string being held at the equilibrium position.

This paper examines the application of boundary value problems in determining the buckling load of an elastic column. The boundary conditions are determined by the length of the column and how it is supported (i.e. clamped end, hinged end, etc.). The following sections will discuss some of the theory behind these boundary value problems, some typical problems and their solutions, and how to interpret these results in the context of our problem.

DEFINITIONS AND THEOREMS

In this chapter we introduce some basic theory of IVPs and BVPs as applicable to our area of study. For simplicity, we are considering first and second order differential equations.

1. INITIAL VALUE PROBLEMS

Initial value problems, and linear problems in particular, can be separated from boundary value problems. There is a rich literature involving linear IVPs.

The following theorem concerns the existence of solutions to IVPs.

Theorem1.1. [1] Consider the initial value problem

$$y'' + p(t)y' + q(t)y = g(t),$$

$$y(t_0) = a, y'(t_0) = b,$$

Where p, q, and g are continuous on an open interval I that contains the point t_0.

Then there is exactly one solution $y = \phi(t)$ of this problem, and the solution exists throughout the interval I.

In general, for nonlinear ODEs the existence of a solution will be given in some interval which contains the initial value. The above theorem not only talks about a solution existing but also tells us the interval in which it does exist. The next theorem highlights another difference between linear and nonlinear problems.

Theorem1.2. (Principle of Superposition)
If y_1 and y_2 are two solutions of the differential equation

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

then the linear combination $c_1y_1 + c_2y_2$ is also a solution for any values of the constants c_1 and c_2.

This theorem does not hold for nonlinear problems and highlights one of the main differences between linear and nonlinear problems. It can be illustrated by many examples in ODE books. The following definition is extremely important when dealing with solutions to linear functions.

Definition. [1] Suppose y_1 and y_2 are solutions of a differential equation. We define the Wronskian, W, as

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2.$$

This definition along with Theorem 1.2 gives us the following.

Theorem1.3. [1] Suppose that y_1 and y_2 are two solutions of the differential equation

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

and that the Wronskian

$$W = y_1y_2' - y_1'y_2$$

is not zero at the point t_0 where the initial conditions,

$$y(t_0) = a, y'(t_0) = b$$

are assigned. Then there is a choice of the constants c_1, c_2 for which

$$y = c_1y_1(t) + c_2y_2(t)$$

satisfies the differential equation and its initial conditions.

It is this choice of constants that will aid in simpifying the process of solving the problems of this paper. Finally, we have

Theorem 1.4. [1] If y_1 and y_2 are two solutions of the differential equation

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

and if there is a point t_0 where the Wronskian of y_1 and y_2 is nonzero, then the family of solutions

$$y = c_1y_1(t) + c_2y_2(t)$$

with arbitrary coefficients c_1 and c_2 includes every solution of the differential equation.

Notice that the last theorem deals only with a linear ODE and does not involve an IVP. In particular, it is extremely important in that it tells us first, that every second order ODE has two solutions, second, that these solutions are linearly independent, third, that these are the only solutions to the ODE, and finally, that all solutions can be written as a linear combination of these two solutions.

2 BOUNDARY VALUE PROBLEMS

Suppose we have the differential equation

$$y'' + p(x)y' + q(x)y = 0$$

with boundary conditions

$$y(\alpha) = a, \quad y(\beta) = b.$$

In order to solve this BVP, we need to find a function y = ' such that ' satisfies the differential equation on the interval $\alpha < x < \beta$ and takes on the values of a and b at the endpoints of the interval [1]. To find ', we first examine the general solution to the ODE and then use the boundary conditions to determine if there are constants to solve the problem.

Although the idea of finding solutions to linear IVPs and BVPs is fairly straight forward, the results can be vastly different. As we saw in the previous section, linear IVPs have the existence of a unique solution in an interval which is well defined.

Boundary value problems, on the other hand, may have a unique solution, no solution, or infinitely many solutions depending on the conditions of the problem.

Consider the problem

$$y'' + y = 0,$$

subject to the boundary conditions

$$y(0) = a, \quad y\left(\frac{\pi}{2}\right) = b.$$

Here, one solution exists. If we change the BC to

$$y(0) = a, \quad y(2\pi) = b$$

then we have no solution if a 6= b. However, if a = b we have an infinite number of solutions.

In this sense, we may relate BVPs to systems of linear algebraic equations.

Consider the linear system

$$\mathbf{Ax} = \mathbf{b}$$

Where **A** is an n × n matrix, **x** is an n × 1 vector to be determined, and **b** is a given n × 1 vector. The solution to the system is dependent on the matrix **A**. If **A** is nonsingular, the system will have a unique solution. On the other hand, if **A** is singular, the system may have no solution or an infinite number of solutions. In the case of the homogeneous linear system

$$\mathbf{Ax} = \mathbf{0},$$

the trivial solution **x**= **0** always exists. Moreover, if **A** is nonsingular, the trivial solution is the only solution. However, if **A** is singular, there are infinitely many non-trivial solutions. This homogeneous linear system is similar to the differential equation

$$y'' + p(x)y' + q(x)y = 0$$

with boundary conditions

$$y(\alpha) = 0 \text{ and } y(\beta) = 0,$$

where α and β are the endpoints of our interval. Thus we need to solve

$$\begin{pmatrix} y_1(\alpha) & y_2(\alpha) \\ y_1(\beta) & y_2(\beta) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

where y_1 and y_2 are solutions to the ODE. This corresponds exactly to the algebraic problem mentioned above. It is important to note that, from the previous section, we know that there are exactly two linearly independent solutions to the ODE.

We may take the relation between BVPs and linear algebraic systems further by considering the linear system

$$\mathbf{Ax} = \lambda\mathbf{x}.$$

This sytem has the solution **x** = **0** for all values of λ, but for certain values of λ, the solution has non-trivial solutions. We call these values of λ eigenvalues and their respective solutions, **x**, the corresponding eigenfunctions. The problem as a whole is called an eigenvalue problem. An example of an eigenvalue problem in ODE's would be

$$y'' + \lambda y = 0$$

with boundary conditions

$$y(0) = 0 \text{ and } y(\beta) = 0 \text{ [1]}.$$

The relationship between boundary value problems and systems of linear equations provides a very useful tool in determining the type of solution for a BVP as well as solving for the constant values of the general homogeneous solution to a linear ODE. The idea is the same as it is for an algebraic system. We want to determine all values of _ for which the nontrivial solution y exists.

CONCLUSION

The theory of differential equations is closely related to the theory of difference equations, in which the coordinates assume only discrete values, and the relationship involves values of the unknown function or functions and values at nearby coordinates. Many methods to compute numerical solutions of differential equations or study the properties of differential equations involve approximation of the solution of a differential equation by the solution of a corresponding difference equation.

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