TOPOLOGIES ON THE UNDERLYING FUNCTION SPACE

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Abstract

For topological spaces Y and Z, $C(Y,Z)$ represent the set of continuous functions from space Y to space Z. Let A be an arbitrary intersection of non-empty open subsets of Y. In this paper, we develop the set $C(A,Z)$ of continuous functions from space A to space Z and construct topologies on this set to form the underlying function space $C_{\zeta}(A,Z)$ of the function space $C_{\tau}(Y,Z)$. We define continuous maps between the spaces X, A and $C_{\zeta}(A,Z)$, and show that topologies defined on the set $C(A,Z)$ are either $R_{A\subset Y}$-splitting or $R_{A\subset Y}$-admissible.

Keywords: function space, underlying function space, splitting topology, admissible topology, $R_{A\subset Y}$-splitting and $R_{A\subset Y}$-admissible topologies.

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1.0 Introduction

Let Y and Z be topological spaces, the set $C(Y,Z)$ endowed with topology τ will be written as $C_\tau(Y,Z)$. Let X be another topological space. A topology τ defined on $C(Y,Z)$ is called splitting topology, if the continuity of the map $h: X\times Y \to Z$ at $y \in Y$ for each fixed $x \in X$, implies the continuity of the map $h^*: X \to C_\tau(Y,Z)$ defined by $h^*(x)(y) = h(x,y) = f_x(y)$ where $f_x : Y \to Z \ \forall x \in X$ is a continuous map. A topology τ on $C(Y,Z)$ is called an admissible topology if the continuity of the map $h^*: X \to C_\tau(Y,Z)$ implies that of the associated map $h: X\times Y \to Z$ defined by $h(x,y) = h^*(x)(y) = f_x(y)$ (Arens and Dugundji, 1951, Georgiou, 2009). Initially Arens (1946) had defined an admissible topology via continuity of an evaluation mapping $e: C_\tau(Y,Z)\times Y \to Z$ defined by $e(f, y) = f(y)$.

The most widely used topology on function spaces is the set open topology. It comprises of compact open topology which is due to Fox (1945) and point open topology, which stems from the notion of convergence sequence of functions (Seymour, 1965).

Let $\{U_i : i \in I\}$ be an arbitrary family of non-empty open subsets of Y. Define $A = \bigcap_i U_i$, then $C(A,Z)$ is a set of continuous functions of the form $f \circ i = f|_A$ where $i: A \to Y$ is an inclusion mapping. We construct topologies on the set $C(A,Z)$ from set open topology defined on the set $C(Y,Z)$. For any topological space X, we show that the mappings $h|_{X\times A}: X\times A \to Z$ defined by $h|_{X\times A}(x,y) = f_x|_A(y)$ and $h^*_{C_\zeta(A,Z)}: X \to C_\zeta(A,Z)$ defined by $h^*_{C_\zeta(A,Z)}(x)(y) = f_x|_A(y)$ are continuous. The continuity of these mappings depend on the continuity of the mappings $h: X\times Y \to Z$ defined by $h(x,y) = f_x(y)$ and $h^*: X \to C_\tau(Y,Z)$ defined by $h^*(x)(y) = f_x(y)$ respectively. The topologies $R_{A\subset Y}$-splitting and $R_{A\subset Y}$-admissible on the set $C(A,Z)$ are defined.

2.0 Topologies on the Set $C(A,Z)$

The set $C(A,Z)$ inherit the topology of pointwise convergence and compact open topology from the space $C_\tau(Y,Z)$.

2.1 Topology of Pointwise Convergence or the Point-Open Topology (τ_p)

Given a point $y \in Y$ and an open set V subset of the space Z, the sets of the form $S(y,V)=\{f : f(y) \in V\}$ is the subbase for topology of pointwise convergence (Kelley, 1955). For easier reference we will write $S(y,V)=\{f \in C(Y,Z) : f(y) \in V\}$. The sets $S(y,V)$ represent the subbases for topology of pointwise convergence defined on the set $C(Y,Z)$.

Let Ω_Z be the class of open subsets of Z. The bases consist of sets of the form $\bigcap_{i=1}^{n} S(y_i, V_i) = \{f \in C(Y,Z) : f(y_i) \in V_i\}$ where $y_i \in Y$, and $V_i \in \Omega_Z$ for $i = 1, 2 \ldots n$.

For $A \subset Y$, the sets of the form $C(A,Z) \cap S(y,V) = \{f \in C(Y,Z) : f(\{y\} \cap A) \in V\} = \{f \in C(Y,Z) : f|_A (y) \in V\} = S(y,V)$ form the subbases for point open topology on $C(A,Z)$.

The bases consist of sets of the form $\bigcap_{i=1}^{n} S(y_i, V_i) = \{f \in C(Y,Z) : f|_A (y_i) \in V_i\}$, where $y_i \in A$ and $V_i \in \Omega_Z$ for $i = 1, 2 \ldots n$.

2.2 Compact Open Topology (τ_{co})

Let Y and Z be topological spaces and let $\mathcal{C}$ be the class of compact subsets of Y and Ω_Z be the class of open subsets of Z. The topology τ defined on $C(Y,Z)$ generated by $F(U,V) = \{f \in C(Y,Z) : f(U) \subset V\}$ where $U \in \mathcal{C}$ and $V \in \Omega_Z$, is called compact open topology on $C(Y,Z)$. The set $F(U,V)$ is a defining subbase for topology τ (Fox, 1945, Kelley, 1955 and Seymour, 1965).

The bases consists of sets of the form $\bigcap_{i=1}^{n} F(U_i, V_i) = \{f \in C(Y,Z) : f(U_i) \subset V_i\}$, where $U_i \in \mathcal{C}$ and $V_i \in \Omega_z$ for $i = 1, 2 \ldots n$.

For the set $C(A,Z)$, the sets of the form $C(A,Z) \cap F(U,V) = \{f \in C(Y,Z) : f(A \cap U) \subset V\} = \{f \in C(Y,Z) : f|_A (A) \subset V\} = F(A,V)$

where $U \in C$ and $V \in \Omega_Z$, defines the subbases for set open topology on the set $C(A,Z)$. If $A \in C$, then $F(A,V)$ defines the subbases for a compact open topology on the set $C(A,Z)$.

The bases consists of sets of the form $\bigcap_{i=1}^{n} F(A,V_i) = \{f \in C(Y,Z), f|_A (A) \subset V_i\}$ where $V_i \in \Omega_Z$ for $i = 1, 2 \ldots n$.

The set $C(A,Z)$ with induced topology ζ will be written as $C_\zeta(A,Z)$.

3.0 Continuity of Maps on the Underlying Function Space $C_\zeta(A,Z)$

We develop propositions which will be useful in proving $R_{A\subset Y}$-splitting and $R_{A\subset Y}$-admissible properties of topologies defined on the set $C(A,Z)$ as well as continuity of evaluation map defined on the space $C_\zeta(A,Z)$.

Topologies τ and ζ defined on $C(Y,Z)$ and $C(A,Z)$ respectively, are both set open.

3.1 Proposition

Let the map $h: X \times Y \to Z$ defined by $h(x,y) = f_x(y)$ be continuous at $y \in Y$ for each $x \in X$, where $f_x : Y \to Z$ is a continuous map defined by $f_x(y) = f(x,y)\ \forall x \in X$. The map $h|_{X\times A}: X \times A \to Z$ defined by $h|_{X\times A}(x,y) = f_x|_A(y)$ is continuous at $y \in A$ for each $x \in X$.

Proof

Let V be the open neighborhood of $z = f_x(y)$ in Z, then the set $f_x^{-1}(V)$ is open in Y. If A is a subspace of Y and $i: A \to Y$ is an inclusion map, then the composite map $f_x \circ i : A \to Z$ defined by $f_x \circ i(y) = f \circ i(x,y)\ \forall x \in X$ is continuous and $i^{-1}(f_x^{-1}(V)) = A \cap f_x^{-1}(V)$ is open in the relative topology on A. Let $f_x \circ i = f_x|_A$, then $A \cap f_x^{-1}(V) = f_x|_A^{-1}(V)$. Therefore the map $f_x|_A : A \to Z$ defined by $f_x|_A(y) = f|_A(x,y)\ \forall x \in X$ is continuous. Setting $h|_{X\times A}(x,y) = f_x|_A(y)$, we have that the map $h|_{X\times A}: X \times A \to Z$ is continuous.

3.2 Proposition

The map $\phi: C_\tau(Y,Z) \to C_\zeta(A,Z)$ defined by $\phi(f) = f|_A$ is continuous.

Proof

It suffices to show that the pre-image of any open set in the space $C_\zeta(A,Z)$, is open in the space $C_\tau(Y,Z)$.

Let U and V be open subsets of the space Y and Z respectively. Let $i:A\to Y$ be the inclusion mapping. Then $i^{-1}(U)$ is open in A and $(F(i^{-1}(U),V)$ is open in $C_\zeta(A,Z)$.

$\phi^{-1}(F(i^{-1}(U),V)=\{(f\in C_\tau(Y,Z):f|_A(i^{-1}(U)\subset V)\}=\{f\in C_\tau(Y,Z):f|_A(A\cap U)\subset V\}$

$=\{f\in C_\tau(Y,Z):f|_A(A)\subset V\}=\{f\in C_\tau(Y,Z):f(A)\subset V\}=F(A,V)$ is open in $C_\tau(Y,Z)$.

3.3 Proposition

Let $h^*:X\to C_\tau(Y,Z)$ defined by $h^*(x)(y)=f_x(y)$ be a continuous map at $y\in Y$ for each $x\in X$, where $f_x:Y\to Z$ is a continuous map defined by $f_x(y)=f(x,y)\ \forall x\in X$. The map $h^*{}_{C_\zeta(A,Z)}:X\to C_\zeta(A,Z)$ defined by $h^*{}_{C_\zeta(A,Z)}(x)(y)=f_x|_A(y)$ is continuous at $y\in A$ for each $x\in X$.

Proof

From proposition 3.1, the continuity of map $f_x:Y\to Z$ defined by $f_x(y)=f(x,y)\ \forall x\in X$ and the inclusion map $i:A\to Y$, implies that the map $f_x|_A:A\to Z$ defined by $f_x|_A(y)=f|_A(x,y)$ $\forall x\in X$ is continuous. Elements of $C_\tau(Y,Z)$ have been indexed by the set X. Therefore from proposition 3.2, the map $\phi:C_\tau(Y,Z)\to C_\zeta(A,Z)$ defined by $\phi(f_x)=f_x|_A\ \ \forall x\in X$ is continuous. The composite map $\phi\circ h^*:X\to C_\zeta(A,Z)$ defined by $\phi\circ h^*(x)(y)=\phi(h^*(x)(y))=\phi(f_x(y))=f_x|_A(y)$ is therefore continuous at $y\in A$ for each $x\in X$. Setting $h^*{}_{C_\zeta(A,Z)}=\phi\circ h^*$, we have that the map $h^*{}_{C_\zeta(A,Z)}:X\to C_\zeta(A,Z)$ is continuous.

3.4 Proposition

Let $i:C_\tau(Y,Z)\to C_\tau(Y,Z)$ be a continuous identity map defined by $i(f)=f$. The map $i|_{C_\zeta(A,Z)}:C_\zeta(A,Z)\to C_\tau(Y,Z)$ defined by $i|_{C_\zeta(A,Z)}(f|_A)=f$ is continuous.

Proof

Let U be open in Y and V be open in Z, then $F(U,V)$ is open in $C_\tau(Y,Z)$ and

$i|_{C_\zeta(A,Z)}{}^{-1}(F(U,V))=\{f\in C_\tau(Y,Z):f(U)\subset V\}=\{f\in C_\tau(Y,Z):f(U\cap A)\subset V\}=$

$\{f\in C_\tau(Y,Z):f|_A(U\cap A)\subset V\}=\{f\in C_\tau(Y,Z):f|_A(A)\subset V\}=F(A,V)$ is open in $C_\zeta(A,Z)$.

Observe that open subsets of $C_\zeta(A,Z)$ are of the form $C(A,Z)\cap F(U,V)=F(A\cap U, V\cap Z)=F(A,V)$ for U open in Y and V open in Z.

3.5 Proposition

Let the map $e{:}C_\tau(Y,Z)\to C_\tau(Y,Z)$ defined by $e(f)=f$ be continuous. The map $\varepsilon : C_\zeta(A,Z)\to C_\zeta(A,Z)$ defined by $\varepsilon(f|_A)=f|_A$ is continuous.

Proof

From proposition 3.2, the map $\phi : C_\tau(Y,Z)\to C_\zeta(A,Z)$ defined by $\phi(f)=f|_A$ is continuous. And from proposition 3.4, the map $i|_{C_\zeta(A,Z)} : C_\zeta(A,Z)\to C_\tau(Y,Z)$ defined by $i|_{C_\zeta(A,Z)}(f|_A)=f$ is also continuous. Therefore the composite map $\phi\circ i|_{C_\zeta(A,Z)} : C_\zeta(A,Z)\to C_\zeta(A,Z)$ defined by $\phi\circ i|_{C_\zeta(A,Z)}(f|_A)=\phi\left(i|_{C_\zeta(A,Z)}(f|_A)\right)=\phi(f)=f|_A$ is continuous. Setting $\phi\circ i|_{C_\zeta(A,Z)}=\varepsilon$, the map $\varepsilon : C_\zeta(A,Z)\to C_\zeta(A,Z)$ is continuous.

3.6 Theorem

Let X, Y and Z be topological spaces. If

i. the continuity of the map $h : X\times Y\to Z$ at $y\in Y$ for each $x\in X$ defined by $h(x,y)=f_x(y)$ (where $f_x : Y\to Z\ \forall x\in X$ is continuous) implies the continuity of the map $h^* : X\to C_\tau(Y,Z)$ defined by $h^*(x)(y)=f_x(y)$,

ii. topology τ on $C(Y,Z)$ is compact open,

then the continuity of the map $h|_{XxA} : X x A\to Z$ at $y\in A$ for each $x\in X$ defined by $h|_{X\times A}(x,y)=f_x|_A(y)$, implies that of the map $h^*_{C_\zeta(A,Z)} : X\to C_\zeta(A,Z)$ defined by $h^*_{C_\zeta(A,Z)_x}(y)=f_x|_A(y)$.

Proof

$$\begin{array}{ccc} h|_{X\times A} : X\times A\to Z & \longrightarrow & h^*_{C_\zeta(A,Z)} : X\to C_\zeta(A,Z) \\ \uparrow & & \uparrow \\ h : X\times Y\to Z & \longrightarrow & h^* : X\to C_\tau(Y,Z) \end{array}$$

Let $x_0 \in X$, A be a compact subset of Y, V an open subset of Z and $h^*_{C_\varsigma(A,Z)}(x_0) \in F(A,V)$. Then $h^*(x_0) \in F(A,V)$ and for the associated mapping $h: X \times Y \to Z$ which is continuous, $h(\{x_0\} \times A) \subset V$. From proposition 3.1, $h(\{x_0\} \times A) \subset V$ implies that $h|_{X\times A}(\{x_0\} \times A) \subset V$.

To show that continuity of $h^*_{C_\varsigma(A,Z)}$ depends on the continuity of $h|_{X\times A}$, it suffices to show that there exist a neighborhood W_0 of $x_0 \in X$ such that $h^*_{C_\varsigma(A,Z)}(W_0) \subset F(A,V)$. Choose a finite open cover $W_i \times U_i$ for $\{x_0\} \times A$ such that $h|_{X\times A}(W_i x U_i) \subset V$, then $h(W_i x U_i) \subset V$. We let $W_0 = \bigcap_i W_i$ and $A = \bigcap_i U_i$, then $h(W_0 x A) \subset V$ implying that $h^*(W_0) \subset F(A,V)$. From proposition 3.3, it follows that $h^*_{C_\varsigma(A,Z)}(W_0) \subset F(A,V)$. Hence the map $h^*_{C_\varsigma(A,Z)} : X \to C_\varsigma(A,Z)$ is continuous.

3.7 Theorem

Let X, Y and Z be topological spaces. If

i. the continuity of the map $h^* : X \to C_\tau(Y,Z)$ at $y \in Y$ for each $x \in X$ defined by $h^*(x)(y) = f_x(y)$ (where $f_x : Y \to Z\ \forall x \in X$ is continuous) implies the continuity of the map $h : X \times Y \to Z$ defined $h(x,y) = f_x(y)$ for a locally Hausdorff space Y,

ii. topology τ on $C(Y,Z)$ is compact open,

then the continuity of the map $h^*_{C_\varsigma(A,Z)} : X \to C_\varsigma(A,Z)$ at $y \in A$ for each $x \in X$ defined by $h^*_{C_\varsigma(A,Z)}(x)(y) = f_x|_A(y)$, implies that of the map $h|_{X\times A} : X \times A \to Z$ defined by $h|_{X\times A}(x,y) = f_x|_A(y)$.

Proof

$h^*_{C_\varsigma(A,Z)} : X \to C_\varsigma(A,Z) \longrightarrow h|_{X\times A} : X \times A \to Z$

$\uparrow \qquad\qquad\qquad\qquad \uparrow$

$h^* : X \to C_\tau(Y,Z) \longrightarrow h : X \times Y \to Z$

Let $C_\varsigma(A,Z)$ contain the image set $h^*(X)$, $(x_0, y_0) \in X \times A$ for $A \subset Y$ and V_0 be the neighborhood of the point $h|_{X\times A}(x_0, y_0) \in Z$. Then $h(x_0, y_0) \in Z$. But $h(x_0, y_0) = f_{x_0}(y_0)$, for the continuous map $f_{x_0}: Y \to Z \ \forall x_0 \in X$. Therefore y_0 has a neighborhood W in Y such that $f_{x_0}(W \cap A) \subset V$, this implies that $f_{x_0}(A) \subset V$. The set A is locally compact since open subset of a locally compact space is locally compact. Therefore y_0 has a neighborhood U with compact closure such that $\overline{U} \subset A$ and consequently $f_{x_0}(\overline{U}) \subset V$, this implies that $h^*(x_0) \in F(\overline{U}, V)$. The map h^* is continuous, therefore x_0 has a neighborhood G such that $h^*(G) \subset F(\overline{U}, V)$ and from proposition 3.3, $h^*_{C_\zeta(A,Z)}(G) \subset F(\overline{U}, V)$.

If $(x, y) \in G \times A$, and $h^*_{C_\zeta(A,Z)}(G) \subset F(\overline{U}, V)$ then $h^*(G) \subset F(\overline{U}, V)$. Now $h^*(x)(y) = f_x(y) = h(x, y) \in V$ implies that $h(G \times U) \subset V$. From proposition 3.1 $h|_{X\times A}(G \times U) \subset V$, therefore the function $h|_{X\times A}: X \times A \to Z$ is continuous.

3.8 Definition

For the topological spaces X, Y and Z,

i. Let the continuity of the map $h: X \times Y \to Z$ at $y \in Y$ for each $x \in X$ defined by $h(x, y) = f_x(y)$ (where $f_x: Y \to Z \ \forall x \in X$ is continuous) imply that of the map $h^*: X \to C_\tau(Y, Z)$ defined by $h^*(x)(y) = f_x(y)$.

ii. Let the continuity of the map $h|_{X\times A}: X \times A \to Z$ at $y \in A$ for each $x \in X$ defined by $h|_{X\times A}(x, y) = f_x|_A(y)$, imply that of the map $h^*_{C_\zeta(A,Z)}: X \to C_\zeta(A, Z)$ defined by $h^*_{C_\zeta(A,Z)}(x)(y) = f_x|_A(y)$ for $A \subset Y$.

An induced topology ζ on $C(A, Z)$ is said to be splitting restricted to $A \subset Y$ denoted by $R_{A\subset Y}$-splitting, if (i) implies (ii) as shown in theorem 3.6.

For the topological spaces X, Y and Z,

i. Let the continuity of the map $h^*: X \to C_\tau(Y, Z)$ at $y \in Y$ for each $x \in X$ defined by $h^*(x)(y) = f_x(y)$ (where $f_x: Y \to Z \ \forall x \in X$ is continuous) imply that of the map $h: X \times Y \to Z$ defined by $h(x, y) = f_x(y)$.

ii. Let the continuity of the map $h^*_{C_\zeta(A,Z)}: X \to C_\zeta(A,Z)$ at $y \in A$ for each $x \in X$ defined by $h^*_{C_\zeta(A,Z)}(x)(y) = f_x|_A(y)$, imply that of the map $h|_{X\times A}: X \times A \to Z$ defined by $h|_{X\times A}(x,y) = f_x|_A(y)$ for $A \subset Y$.

An induced topology ζ on $C(A,Z)$ is said to be admissible restricted to $A \subset Y$ denoted by $R_{A\subset Y}$-admissible if (i) implies (ii) as shown in theorem 3.7.

4.0 Continuity of the Evaluation Function on the Underlying Function Space $C_\zeta(A,Z)$

By definition, a compact open topology τ on $C(Y,Z)$ is admissible, if the evaluation map $e: C_\tau(Y,Z)\times Y \to Z$ defined by $e(f,y) = f(y)$ is continuous. We restrict the domain of the evaluation function e and show that the compact open topology ζ defined on $C(A,Z)$ is $R_{A\subset Y}$-admissible.

4.1 Theorem

Let Y and Z be topological spaces. Compact open topology ζ defined on $C(A,Z)$ is $R_{A\subset Y}$-admissible if and only if for the continuity of the evaluation map $e: C_\tau(Y,Z)\times Y \to Z$ defined by $e(f, y) = f(y)$, the map $e|_{C_\zeta(A,Z)\times A}: C_\zeta(A,Z)\times A \to Z$ defined by $e|_{C_\zeta(A,Z)\times A}(f|_A, y) = f|_A(y)$ is continuous.

Proof

Let ζ be $R_{A\subset Y}$-admissible and let the map $e^*: C_\tau(Y,Z) \to C_\tau(Y,Z)$ be the associated mapping to $e: C_\tau(Y,Z)\times Y \to Z$. Then the mapping e^* defined by $e^*(f) = f$ is continuous for compact open topology τ on $C(Y,Z)$ since it is an identity map. From proposition 3.5, the mapping $\varepsilon: C_\zeta(A,Z) \to C_\zeta(A,Z)$ defined by $\varepsilon(f|_A) = f|_A$ for an induced topology ζ on $C(A,Z)$ is also continuous. The topology ζ is $R_{A\subset Y}$-admissible, and therefore the continuity of the map $h^*_{C_\zeta(A,Z)}: X \to C_\zeta(A,Z)$ at $y \in A$ for each $x \in X$ implies that of the map $h|_{X\times A}: X\times A \to Z$, if the continuity of the map $h^*: X \to C_\tau(Y,Z)$ at $y \in Y$ for each $x \in X$ implies that of map $h: X \times Y \to Z$. We Set $X = C_\zeta(A,Z)$ for the maps $h^*_{C_\zeta(A,Z)}: X \to C_\zeta(A,Z)$ and

$h|_{X\times A}: X\times A \to Z$, and $X = C_\tau(Y,Z)$ for the maps $h^*: X \to C_\tau(Y,Z)$ and $h: X\times Y \to Z$. Then the continuity of the map $h^*_{C_\zeta(A,Z)}: C_\zeta(A,Z) \to C_\zeta(A,Z)$ implies that of the map $h|_{C_\zeta(A,Z)\times A}: C_\zeta(A,Z)\times A \to Z$ if the continuity of the map $h^*: C_\tau(Y,Z) \to C_\tau(Y,Z)$ implies that of the map $h: C_\tau(Y,Z)\times Y \to Z$. Setting $h|_{C_\zeta(A,Z)\times A} = e|_{C_\zeta(A,Z)\times A}$ and $h = e$, we have that the map $e|_{C_\zeta(A,Z)\times A}: C_\zeta(A,Z)\times A \to Z$ is continuous whenever $e: C_\tau(Y,Z)\times Y \to Z$ is a continuous map.

Conversely assume that the map $e|_{C_\zeta(A,Z)\times A}: C_\zeta(A,Z)\times A \to Z$ is continuous for an induced topology ζ on $C(A,Z)$ whenever $e: C_\tau(Y,Z)\times Y \to Z$ is a continuous map for the topology τ on $C(Y,Z)$. We show that the continuity of the map $h^*_{C_\zeta(A,Z)}: X \to C_\zeta(A,Z)$ at $y \in A$ for each $x \in X$ implies that of the map $h|_{X\times A}: X\times A \to Z$, whenever the continuity of the map $h^*: X \to C_\tau(Y,Z)$ at $y \in Y$ for each $x \in X$ implies that of the map $h: X\times Y \to Z$. Let $h: X\times Y \to Z$ be a map such that the map $h|_{X\times A}: X\times A \to Z$ has an associated continuous map $h^*_{C_\zeta(A,Z)}: X \to C_\zeta(A,Z)$ whenever $h^*: X \to C_\tau(Y,Z)$ is the associated continuous map of h. The composite map $e\circ(h^*\times id_Y): X\times Y \to Z$ is continuous at $y \in Y$ for each $x \in X$ since it is a composition of continuous maps e, h^* and the identity map $id_Y: Y \to Y$. If $id|_{A\subset Y}\ A \to A$ is also an identity map, then from proposition 3.1, the composite map $e|_{C_\zeta(A,Z)\times A} \circ (h^*_{C_\zeta(A,Z)} \times id|_{A\subset Y}): X\times A \to Z$ is also continuous at $y \in A$ for each $x \in X$. Setting $h|_{X\times A} = e|_{C_\zeta(A,Z)\times A} \circ (h^*_{C_\zeta(A,Z)} \times id|_{A\subset Y})$ and $h = e\circ(h^*\times id_Y)$ and applying theorem 3.7, we have that the continuity $h^*_{C_\zeta(A,Z)}: X \to C_\zeta(A,Z)$ at $y \in A$ for each $x \in X$ implies that of $h|_{X\times A}: X\times A \to Z$ whenever the continuity of $h^*: X \to C_\tau(Y,Z)$ at $y \in Y$ for each $x \in X$ implies that of $h: X\times Y \to Z$. Therefore topology ζ is $R_{A\subset Y}$-admissible and from theorem 3.7, it is compact.

4.2 Comparison of Topologies on the set $C(A,Z)$

We compare topologies defined on the set $C(A,Z)$ with respect to their sizes and properties of $R_{A\subset Y}$-splitting and $R_{A\subset Y}$-admissibility they satisfy.

4.2.1 Theorem

Let ζ_1 be $R_{A\subset Y}$-splitting topology on the set $C(A,Z)$. Any topology coarser than $R_{A\subset Y}$-splitting topology is also $R_{A\subset Y}$-splitting.

Proof

Let ζ_1 be $R_{A\subset Y}$-splitting topology defined on the set $C(A,Z)$ and ζ_2 be any topology defined on the set $C(A,Z)$ where $\zeta_2 \subseteq \zeta_1$. Since topology ζ_1 is $R_{A\subset Y}$-splitting on $C(A,Z)$, the continuity of the map $h|_{X\times A}: X\times A \to Z$ at $y\in A$ for each $x\in X$ implies that of the map $h^*_{C_{\zeta_1}(A,Z)}: X \to C_{\zeta_1}(A,Z)$, whenever the continuity of the map $h: X\times Y \to Z$ at $y\in Y$ for each $x\in X$ implies that of $h^*: X \to C_{\tau_1}(Y,Z)$. For a topology $\tau_2 \subseteq \tau_1$, the map $\rho: C_{\tau_1}(Y,Z) \to C_{\tau_2}(Y,Z)$ is continuous. From proposition 3.5, the map $\varepsilon: C_{\zeta_1}(A,Z) \to C_{\zeta_2}(A,Z)$ for $\zeta_2 \subseteq \zeta_1$ is also continuous. Therefore the maps $\rho\circ h^*: X \to C_{\tau_2}(Y,Z)$ and $\varepsilon\circ h^*_{C_{\zeta_2}(A,Z)}: X \to C_{\zeta_2}(A,Z)$ are continuous, since they are composition of continuous maps. Hence the continuity of $h|_{X\times A}: X\times A \to Z$ at $y\in A$ for each $x\in X$ implies continuity of $\varepsilon\circ h^*_{C_{\zeta_2}(A,Z)}: X \to C_{\zeta_2}(A,Z)$, whenever continuity of map $h: X\times Y \to Z$ at $y\in Y$ for each $x\in X$, implies that of the map $\rho\circ h^*: X \to C_{\tau_2}(Y,Z)$. Therefore ζ_2 is also R_A-splitting topology.

4.2.2 Theorem

Let ζ_1 be $R_{A\subset Y}$-admissible topology on the set $C(A,Z)$. Any topology finer than $R_{A\subset Y}$-admissible topology is also $R_{A\subset Y}$-admissible.

Proof

Let ζ_1 be $R_{A\subset Y}$-admissible topology defined on the set $C(A,Z)$ and ζ_2 be any topology defined on the set $C(A,Z)$ where $\zeta_1 \subseteq \zeta_2$. Since ζ_1 is $R_{A\subset Y}$-admissible topology, the continuity of the map $h^*_{C_{\zeta_1}(A,Z)}: X \to C_{\zeta_1}(A,Z)$ at $y\in A$ for each $x\in X$ implies that of the map $h|_{X\times A}: X\times A \to Z$, if the continuity of the map $h^*: X \to C_{\tau_1}(Y,Z)$ at $y\in Y$ for each $x\in X$ implies that of $h: X\times Y \to Z$. Let $\tau_1 \subseteq \tau_2$, then the map $e: C_{\tau_1}(Y,Z) \to C_{\tau_2}(Y,Z)$ obtained by restricting the domain of the identity map $i: C_{\tau_2}(Y,Z) \to C_{\tau_2}(Y,Z)$ is continuous. From proposition 3.5, the map $\varepsilon: C_{\zeta_1}(A,Z) \to C_{\zeta_2}(A,Z)$ for $\zeta_1 \subseteq \zeta_2$ is also continuous. The maps $e \circ h^*: X \to C_{\tau_2}(Y,Z)$ and $\varepsilon \circ h^*_{C_{\zeta_2}(A,Z)}: X \to C_{\zeta_2}(A,Z)$ are therefore continuous, since they are composition of continuous maps. We therefore have that the continuity of the map $\varepsilon \circ h^*_{C_{\zeta_2}(A,Z)}: X \to C_{\zeta_2}(A,Z)$ at $y \in A$ for each $x \in X$ implies that of the map $h|_{X \times A}: X \times A \to Z$, if the continuity of the map $e \circ h^*: X \to C_{\tau_2}(Y,Z)$ implies that of the map $h: X \times Y \to Z$. Hence ζ_2 is also $R_{A \subset Y}$-admissible topology.

4.2.3 Corollary

There exist the coarsest $R_{A \subset Y}$-splitting topology on the set $C(A,Z)$.

Proof

Let ζ_i for $i = 1,2,3........n$ be a family of $R_{A \subset Y}$-splitting topology on the set $C(A,Z)$. Let $\zeta = \bigcap_i \zeta_i$ for $i = 1,2,3........n$, then $\zeta \subset \zeta_i$ for $i = 1,2,3........n$. Since finite intersection of topologies is a topology, the topology ζ is therefore the coarsest $R_{A \subset Y}$-splitting topology on the set $C(A,Z)$.

4.2.4 Corollary

There exist the finest $R_{A \subset Y}$-admissible topology on the set $C(A,Z)$.

Proof

Let $\{\zeta_i : i \in I\}$ be a family of $R_{A \subset Y}$-admissible topology on the set $C(A,Z)$. Let $\zeta = \bigcup_i \zeta_i$, then ζ is $R_{A \subset Y}$-admissible topology on the set $C(A,Z)$ since arbitrary union of topologies is a topology. $\zeta_i \subset \zeta \; \forall i \in I$, therefore ζ is the finest $R_{A \subset Y}$-admissible topology on the set $C(A,Z)$.

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