OBSERVATIONS ON PELL NUMBER AND PELL LUCAS NUMBER

M.A.GOPALAN*

V.GEETHA**

ABSTRACT :

Pell number together with Pell Lucas number have been analysed. Some identities among these numbers are presented.

KEYWORDS: Pell number, Pell Lucas number.

INTRODUCTION:

Number is the essence of Mathematical calculation. Varieties of numbers have variety of range and richness. Many integers exhibit fascinating properties, they form sequences, they form patterns and so on [1-23]. In this communication we consider twin sequences Pell numbers and Pell Lucas numbers and some identities relating themselves.

Properties:

1. $PL_n^2 = PL_{2n} + 2(-1)^n$

Proof:

$$PL_n^2 = \left[(1+\sqrt{2})^n + (1-\sqrt{2})^n\right]^2$$

$$= (1+\sqrt{2})^{2n} + (1-\sqrt{2})^{2n} + 2(-1)^n$$

$$= PL_{2n} + 2(-1)^n$$

$$PL_n^2 = PL_{2n} + 2(-1)^n$$

2. $PL_{2n} = 8P_n^2 + 2(-1)^n$

* Department of Mathematics, Shrimathi Indira Gandhi College,Trichirappalli, Tamilnadu, India

** Department of Mathematics, Cauvery College For Women,Trichirappalli, Tamilnadu, India

Proof:

$$PL_{2n} = (1+\sqrt{2})^{2n} + (1-\sqrt{2})^{2n}$$

$$= \left[(1+\sqrt{2})^{n} - (1-\sqrt{2})^{n}\right]^{2} + 2(-1)^{n}$$

$$= 8P_n^2 + 2(-1)^n$$

$$PL_{2n} = 8P_n^2 + 2(-1)^n$$

3. $PL_{3n} = PL_n^3 - 3PL_n(-1)^n$

Proof:

$$PL_{3n} = (1+\sqrt{2})^{3n} + (1-\sqrt{2})^{3n}$$

$$= \left[(1+\sqrt{2})^{n} + (1-\sqrt{2})^{n}\right]^{3} + 3(-1)^{n}\left[(1+\sqrt{2})^{n} + (1-\sqrt{2})^{n}\right]$$

$$= PL_n^3 - 3(-1)^n PL_n$$

$$PL_{3n} = PL_n^3 - 3PL_n(-1)^n$$

4. $PL_{4n} = PL_n^4 - 4PL_n^2(-1)^n + 2$

Proof:

$$PL_{4n} = (1+\sqrt{2})^{4n} + (1-\sqrt{2})^{4n}$$

$$= PL_n^4 - 4(-1)^n PL_{2n} - 6(-1)^n$$

$$= PL_n^4 - 4(-1)^n PL_n^2 + 2 \qquad \text{(Using Property 2)}$$

$$PL_{4n} = PL_n^4 - 4PL_n^2(-1)^n + 2$$

5. $PL_{n+2} - 3PL_n = 8P_n$

Proof:

$$PL_{n+2} - 3PL_n = (1+\sqrt{2})^{n+2} + (1-\sqrt{2})^{n+2} - 3\left[(1+\sqrt{2})^{n} + (1-\sqrt{2})^{n}\right]$$

$$= 2\sqrt{2}\left[(1+\sqrt{2})^{n} - (1-\sqrt{2})^{n}\right]$$

$$= 8P_n$$

$$PL_{n+2} - 3PL_n = 8P_n$$

6. $PL_{n+2} - 2PL_{n+1} - PL_n = 0$

Proof:

$$PL_{n+2}-2PL_{n+1}-PL_n=(1+\sqrt{2})^{n+2}+(1-\sqrt{2})^{n+2}-2\left[(1+\sqrt{2})^{n+1}+(1-\sqrt{2})^{n+1}\right]-\left[(1+\sqrt{2})^{n}+(1-\sqrt{2})^{n}\right]$$

$$=(1+\sqrt{2})^n(0)+(1-\sqrt{2})^n(0)$$

$$=0$$

$$PL_{n+2}-2PL_{n+1}-PL_n=0$$

7. $2(3P_{n+1}-P_{n+2})=PL_n$

Proof:

$$2(3P_{n+1}-P_{n+2})=2\left\{\frac{3}{2\sqrt{2}}\left[(1+\sqrt{2})^{n+1}-(1-\sqrt{2})^{n+1}\right]-\frac{1}{2\sqrt{2}}\left[(1+\sqrt{2})^{n+2}-(1-\sqrt{2})^{n+2}\right]\right\}$$

$$=\frac{1}{2\sqrt{2}}2\sqrt{2}\left[(1+\sqrt{2})^n+(1-\sqrt{2})^n\right]$$

$$=PL_n$$

$$2(3P_{n+1}-P_{n+2})=PL_n$$

8. $PL_nPL_{n+1}-8P_nP_{n+1}=4(-1)^n$

Proof:

$$PL_nPL_{n+1}=\left[(1+\sqrt{2})^n+(1-\sqrt{2})^n\right]\left[(1+\sqrt{2})^{n+1}+(1-\sqrt{2})^{n+1}\right]$$

$$=f(f+\sqrt{2}g)$$

$$8P_nP_{n+1}=8\left\{\frac{1}{2\sqrt{2}}\left[(1+\sqrt{2})^n-(1-\sqrt{2})^n\right]\frac{1}{2\sqrt{2}}\left[(1+\sqrt{2})^{n+1}-(1-\sqrt{2})^{n+1}\right]\right\}$$

$$=g(g+\sqrt{2}f)$$

$$PL_nPL_{n+1}-8P_nP_{n+1}=f(f+\sqrt{2}g)-g(g+\sqrt{2}f)$$

$$=f^2-g^2$$

$$=4(-1)^n$$

where $f=\left[(1+\sqrt{2})^n+(1-\sqrt{2})^n\right]$ and $g=\left[(1+\sqrt{2})^n-(1-\sqrt{2})^n\right]$

Hence $PL_nPL_{n+1}-8P_nP_{n+1}=4(-1)^n$

9. $PL_nPL_{n+1}+8P_nP_{n+1}$ is written as a difference of two squares.

Proof:

$$PL_nPL_{n+1}+8P_nP_{n+1}=f^2+\sqrt{2}fg+g^2+\sqrt{2}fg$$

$$=(f+\sqrt{2}g)^2-g^2$$

where $f=\left[(1+\sqrt{2})^n+(1-\sqrt{2})^n\right]$ and $g=\left[(1+\sqrt{2})^n-(1-\sqrt{2})^n\right]$

Hence $PL_nPL_{n+1}+8P_nP_{n+1}$ is written as a difference of two squares.

10. $6(PL_nPL_{n+1}+8P_nP_{n+1}+8P_n^2)$ is a nasty number.

Proof:

$$PL_nPL_{n+1}=\left[(1+\sqrt{2})^n+(1-\sqrt{2})^n\right]\left[(1+\sqrt{2})^{n+1}+(1-\sqrt{2})^{n+1}\right]$$

$$=f(f+\sqrt{2}g)$$

$$8P_nP_{n+1}=8\left\{\frac{1}{2\sqrt{2}}\left[(1+\sqrt{2})^n-(1-\sqrt{2})^n\right]\frac{1}{2\sqrt{2}}\left[(1+\sqrt{2})^{n+1}-(1-\sqrt{2})^{n+1}\right]\right\}$$

$$=g(g+\sqrt{2}f)$$

$$8P_n^2=8\left[\frac{1}{8}\left[(1+\sqrt{2})^n-(1-\sqrt{2})^n\right]^2\right]$$

$$=g^2$$

$$6(PL_nPL_{n+1}+8P_nP_{n+1}+8P_n^2)=6\left[(f+\sqrt{2}g)^2-g^2+g^2\right]$$

where $f=\left[(1+\sqrt{2})^n+(1-\sqrt{2})^n\right]$ and $g=\left[(1+\sqrt{2})^n-(1-\sqrt{2})^n\right]$

$6(PL_nPL_{n+1}+8P_nP_{n+1}+8P_n^2$ is a nasty number.

11. $6\left[PL_nPL_{n+1}+8P_nP_{n+1}+\frac{(PL_{n+1}-PL_n)^2}{2}\right]$ is a nasty number.

Proof:

$$PL_nPL_{n+1}=\left[(1+\sqrt{2})^n+(1-\sqrt{2})^n\right]\left[(1+\sqrt{2})^{n+1}+(1-\sqrt{2})^{n+1}\right]$$

$$=f(f+\sqrt{2}g)$$

$$8P_nP_{n+1}=8\left\{\frac{1}{2\sqrt{2}}\left[(1+\sqrt{2})^n-(1-\sqrt{2})^n\right]\frac{1}{2\sqrt{2}}\left[(1+\sqrt{2})^{n+1}-(1-\sqrt{2})^{n+1}\right]\right\}$$

$$=g(g+\sqrt{2}f)$$

$$PL_{n+1}-PL_n=\left[(1+\sqrt{2})^{n+1}+(1-\sqrt{2})^{n+1}\right]-\left[(1+\sqrt{2})^n+(1-\sqrt{2})^n\right]$$

$$= \sqrt{2}g$$

$$\frac{(PL_{n+1} - PL_n)^2}{2} = g^2$$

$$6\left[PL_n PL_{n+1} + 8P_n P_{n+1} + \frac{(PL_{n+1} - PL_n)^2}{2}\right] = 6\left[(f + \sqrt{2}g)^2 - g^2 + g^2\right]$$

$$6\left[PL_n PL_{n+1} + 8P_n P_{n+1} + \frac{(PL_{n+1} - PL_n)^2}{2}\right] = 6\left[(f + \sqrt{2}g)^2 - g^2 + g^2\right]$$

$$= 6\left[(f + \sqrt{2}g)^2\right]$$

where $f = \left[(1+\sqrt{2})^n + (1-\sqrt{2})^n\right]$ and $g = \left[(1+\sqrt{2})^n - (1-\sqrt{2})^n\right]$

$6\left[PL_n PL_{n+1} + 8P_n P_{n+1} + \frac{(PL_{n+1} - PL_n)^2}{2}\right]$ is a nasty number.

12. $PL_{m+n} PL_{m-n} = \begin{cases} PL_{2m} + PL_{2n} \\ PL_{2m} - PL_{2n} \end{cases}$

According as $m-n$ is even or $m-n$ is odd.

Proof:

$$PL_{m+n} PL_{m-n} = \left[(1+\sqrt{2})^{m+n} + (1-\sqrt{2})^{m+n}\right]\left[(1+\sqrt{2})^{m-n} + (1-\sqrt{2})^{m-n}\right]$$

$$= (1+\sqrt{2})^{2m} + (1-\sqrt{2})^{2m} + (1+\sqrt{2})^m (1+\sqrt{2})^n (1-\sqrt{2})^m (1-\sqrt{2})^{-n} + (1+\sqrt{2})^m (1+\sqrt{2})^{-n} (1-\sqrt{2})^m (1-\sqrt{2})^n$$

$$= PL_{2m} + (-1)^{m-n} PL_{2n}$$

Hence $PL_{m+n} PL_{m-n} = \begin{cases} PL_{2m} + PL_{2n} \\ PL_{2m} - PL_{2n} \end{cases}$ According as $m-n$ is even or $m-n$ is odd.

13. $\left(\frac{PL_n^2 - 8P_n^2}{2}, 2\sqrt{2}P_n PL_n, \frac{PL_n^2 + 8P_n^2}{2}\right)$ forms a Pythagorean triplet.

Proof:

$$\frac{PL_n^2-8P_n^2}{2}=4(1+\sqrt{2})^n(1-\sqrt{2})^n$$

$$2\sqrt{2}P_nPL_n=((1+\sqrt{2})^n)^2-((1-\sqrt{2})^n)^2$$

$$\frac{PL_n^2+8P_n^2}{2}=((1+\sqrt{2})^n)^2+((1-\sqrt{2})^n)^2$$

Hence . $\left(\frac{PL_n^2-8P_n^2}{2},2\sqrt{2}P_nPL_n,\frac{PL_n^2+8P_n^2}{2}\right)$ forms a Pythagorean triplet.

14. $PL_{n+1}+PL_n=4P_{n+1}$

Proof:

$$PL_{n+1}+PL_n=\left[(1+\sqrt{2})^{n+1}+(1-\sqrt{2})^{n+1}\right]+\left[(1+\sqrt{2})^n+(1-\sqrt{2})^n\right]$$
$$=2\left[(1+\sqrt{2})^n+(1-\sqrt{2})^n\right]+\sqrt{2}\left[(1+\sqrt{2})^n-(1-\sqrt{2})^n\right]$$
$$=\sqrt{2}\left[(1+\sqrt{2})^{n+1}+(1-\sqrt{2})^{n+1}\right]$$
$$=4P_{n+1}$$
$$PL_{n+1}+PL_n=4P_{n+1}$$

15. $PL_{n+1}-PL_n=4(P_{n+2}-2P_{n+1})$

Proof:

$$PL_{n+1}-PL_n=\left[(1+\sqrt{2})^{n+1}+(1-\sqrt{2})^{n+1}\right]-\left[(1+\sqrt{2})^n+(1-\sqrt{2})^n\right]$$
$$=\sqrt{2}g \quad (1)$$

$$4(P_{n+2}-2P_{n+1})=4\frac{1}{2\sqrt{2}}\left\{\left[(1+\sqrt{2})^{n+2}-(1-\sqrt{2})^{n+2}\right]-2\left[(1+\sqrt{2})^{n+1}-(1-\sqrt{2})^{n+1}\right]\right\}$$
$$=\sqrt{2}g \quad (2)$$

$$g=\left[(1+\sqrt{2})^n-(1-\sqrt{2})^n\right]$$

From (1) and (2)

$$PL_{n+1}-PL_n=4(P_{n+2}-2P_{n+1})$$

16. $PL_n^4-64P_n^4=64(-1)^nP_n^2+16$

Proof:

$$PL_n^4 - 64P_n^4 = \left[(1+\sqrt{2})^n + (1-\sqrt{2})^n\right]^2 - 64\left[\frac{1}{2\sqrt{2}}\left((1+\sqrt{2})^n - (1-\sqrt{2})^n\right)\right]^4$$

$$= 8(-1)^n PL_{2n}$$

$$= 8(-1)^n \left[8P_n^2 + 2(-1)^n\right] \qquad \text{(By Property (2))}$$

$$= 64(-1)^n P_n^2 + 16$$

$$PL_n^4 - 64P_n^4 = 64(-1)^n P_n^2 + 16$$

Also $PL_n^4 - 64P_n^4 = 8(-1)^n PL_n^2 - 16$

$$PL_n^4 - 64P_n^4 = 8(-1)^n PL_{2n}$$

$$= 8(-1)^n \left[PL_n^2 - 2(-1)^n\right] \qquad \text{(By Property (1))}$$

$$= 8(-1)^n PL_n^2 - 16$$

Hence $PL_n^4 - 64P_n^4 = 8(-1)^n PL_n^2 - 16$

CONCLUSION:

To conclude one may search for other patterns and their properties.

REFERECNCES:

[1].L.E. Dickson, History of the Theory of Number, Chelsea Publishing Company, New York, 1952.

[2].J.H. Conway and R.K. Guy, The Book of Numbers, Spring Verlag New York,41.

[3].T .Pappas , Triangular, Square & Pentagonal Numbers, The joy of Mathematics, Sacarlos, C.A. Wide World Publication , Tetra, 214.

[4].B.L. Bhatia, and Mohanty, Supriya , Nasty numbers and their characterization, Mathematical Education, II (1) :34-37, 1985.

[5].Carol number –Wikipedia, the free encyclopedia.

[6].S. Harvey, http//www.geocites.com/harvey563/carol_kynea txt.

[7].A. F. Horadam, Jacobsthal Representation Numbers, Fib.Quart.34, 40-54,1996.

[8].A000108 Catalan numbers on the On-line encyclopedia of integer sequences.

[9]. Z. Cerin, Properties of odd and even terms of the Fibonacci Sequences, Demonstratio Math., 39,55-60,2006.

[10].Z. Cerin, Some alternating sums of Lucas numbers, Central European J Math. 3, 1-13,2005.

[11].Z. Cerin and G.M. Gianella, On sums of squares of numbers, INTEGERS: Electronic Journal of combinatorial Number theory, 6,# A15,2006.

[12].M.A. Gopalan and S.Devibala, On Lanekal number, Antartica J Math.4(1), 35-39, 2007.

[13].M.A.Gopalan and A.Gnanam, A notable Integer Sequences, Math .Sci.Res.J.12(1), 7-15,2008.

[14].M.A.Gopalan, Manju Somanath and N. Vanitha, A Remarkable sequences, Proc.nat.accd.sci.india, 77(A),II,2007.

[15]. M.A.Gopalan and A.Gnanam, Star numbers, Math.Sci.Res.J.12(12):303-308,2008.

[16]. M.A.Gopalan and A.Gnanam, Magna Numbers, Indian journal of Mathematical Sciences, 5(1) 33-34 june 2009.

[17]. M.A.Gopalan and A.Gnanam, A Remarkable Integer sequences, ATAM,67-70,2007.

[18]. M.A.Gopalan and G.Srividhya, Diophantine Quadruples for Pell numbers with Property D(1), Antartica J Math. 7(3),357-362,2010.

[19].V.E. Hoggatt Jr, Fibonacci and Lucas numbers, Houghton-Mifflin, Boston 1969.

[20]. Mc Danial, Wayne, The existence of infinitely many k-smith numbers, Fibonacci Quarterly, 25(1), 76-80,1987.

[21].V.Rajesh and G.Leversha , Some Properties of odd terms of Fibonacci sequences, Math.Gazette, 88,85-86,2004.

[22].A.G.Shannon, A.F. Horadom, R.P. Loh, Proper divisors of Lenakel numbers, The Mathematics student , 45(2) , 63-73.,1997.

[23].Stanley, R.P. Enumerative Combinatories, 2, Cambridge University Press,219-229,1999.