INTEGRAL SOLUTIONS OF $2(X^2+Y^2)+3XY=(\alpha^2+7)^n Z^4$

M.A.Gopalan*

S.Vidhyalakshmi*

S.Mallika*

Abstract

The non-homogeneous quartic equation with three unknowns represented by $2(x^2+y^2)+3xy=(\alpha^2+7)^n z^4$ *is analyzed for finding its non-zero distinct integral solutions. Two different methods have been presented for determining the integral solutions of the ternary non-homogeneous biquadratic equation under consideration.The recurrence relations satisfied by the values of x and y are of degree four with three unknowns are exhibited. Knowing an integer solution of the given equation, triples of non-zero distinct integers generating an presented. A few interesting relations among the solutions of the considered non-homogeneous diophantine equation infinite number of integer solutions for odd ordered and even ordered solutions satisfying the given equation are presented respectively.*

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* Department of Mathematics, Shrimati Indira Gandhi college,Trichirappalli.620002

1. Introduction

In [1-3] various attempts have been made to arrive at the general solution of some typical quartic equations with two or more unknowns .In particular, one may refer [4-15] for special quartic equations with three unknowns analyzed for their integral solutions in general forms. This communication concerns with yet another ternary quartic equation $2(x^2+y^2)+3xy=(\alpha^2+7)^n z^4$ for determining non-zero integral solutions. Two different methods of solutions are presented. The recurrence relations for x and y are given.

2.Method of analysis

The equation under consideration to be solved is

$$2(x^2+y^2)+3xy=(\alpha^2+7)^n z^4 \quad (1)$$

Two different methods of solving the above equation are illustrated below.

2.1: Method : I

The values of x and y are obtained by applying the method of induction presented below. To start with consider the equation

$$2(x^2+y^2)+3xy=z^4 \quad (2)$$

Introducing the transformations

$$x=u+v, y=u-v, u\neq v \quad (3)$$

in (2), it is written as

$$7u^2+v^2=z^4 \quad (4)$$

Assume

$$z=7a^2+b^2 \qquad a,b\neq 0 \quad (5)$$

Now

$$z^4=(7a^2+b^2)^4=7(4b^3a-28a^3b)^2+(b^4-42a^2b^2+49a^4)^2 \quad (6)$$

comparing (4) and (6), we get

$$u=4b^3a-28a^3b$$

$$v=b^4-42a^2b^2+49a^4$$

In view of (3) and (5) the solutions $(x_{0,}y_0,z_0)$ of (2) are found to be

$$x_0=4b^3a-28a^3b+b^4-42a^2b^2+49a^4$$

$$y_0=4b^3a-28a^3b-b^4+42a^2b^2-49a^4$$

$$z_0=7a^2+b^2$$

Now consider the equation

$$2(x^2+y^2)+3xy=(\alpha^2+7)z^4 \qquad (7)$$

Again substituting (3) in (7), it becomes,

$$7u^2+v^2=(\alpha^2+7)z^4 \qquad (8)$$

Using (5) in (8) and employing the method of factorization, define

$$(v+i\sqrt{7}u)=(\alpha+i\sqrt{7})(b+i\sqrt{7}a)^4$$

Equating the real and imaginary parts, we get,

$$u=(b^4-42a^2b^2+49a^4)+\alpha(4b^3a-28a^3b)$$

$$v=\alpha(b^4-42a^2b^2+49a^4)-7(4b^3a-28a^3b)$$

Substituting the above values of u and v in (3) the pair (x_1,y_1) satisfying (7) are given by,

$$x_1=(\alpha-3)x_0-4y_0$$

$$y_1=4x_0+(\alpha+3)y_0$$

In matrix notation, we write,

$$\begin{pmatrix} x_1 & 0 \\ y_1 & 0 \end{pmatrix}=\begin{pmatrix} \alpha-3 & -4 \\ 4 & \alpha+3 \end{pmatrix}\begin{pmatrix} x_0 & 0 \\ y_0 & 0 \end{pmatrix}$$

By the method of induction we can find sequence of solutions of (1) as

$$\begin{pmatrix} x_n & 0 \\ y_n & 0 \end{pmatrix}=\begin{pmatrix} \alpha-3 & -4 \\ 4 & \alpha+3 \end{pmatrix}^n\begin{pmatrix} x_0 & 0 \\ y_0 & 0 \end{pmatrix}$$

where

$$M=\begin{pmatrix} \alpha-3 & -4 \\ 4 & \alpha+3 \end{pmatrix}$$

The eigen values of M are got by

$$\begin{vmatrix} \alpha-3-\lambda & -4 \\ 4 & \alpha+3-\lambda \end{vmatrix}=0$$

which implies

$$\lambda=\alpha\pm i\sqrt{7}$$

Taking

$$\alpha_1=\alpha+i\sqrt{7}\,,\ \beta_1=\alpha-i\sqrt{7}$$

and using the formula for A^n ,namely,

$$A^n=\frac{\alpha_1{}^n(A-\beta I)}{\alpha_1-\beta_1}+\frac{\beta_1{}^n(A-\alpha I)}{\beta_1-\alpha_1}$$

we get

$$A^n = \frac{1}{2i\sqrt{7}} \begin{pmatrix} -3(\alpha_1{}^n - \beta_1{}^n) + i\sqrt{7}(\alpha_1{}^n + \beta_1{}^n) & -4(\alpha_1{}^n - \beta_1{}^n) \\ 4(\alpha_1{}^n - \beta_1{}^n) & 3(\alpha_1{}^n - \beta_1{}^n) + i\sqrt{7}(\alpha_1{}^n + \beta_1{}^n) \end{pmatrix}$$

Hence , the solutions of (1) are given by

$$x_n = \frac{1}{2i\sqrt{7}} \left[(-3x_0 - 4y_0)(\alpha_1{}^n - \beta_1{}^n) + i\sqrt{7}(\alpha_1{}^n + \beta_1{}^n) \right]$$

$$y_n = \frac{1}{2i\sqrt{7}} \left[(4x_0 + 3y_0)(\alpha_1{}^n - \beta_1{}^n) + i\sqrt{7}(\alpha_1{}^n + \beta_1{}^n) \right]$$

$$z_n = 7a^2 + b^2$$

The solutions x_n and y_n satisfy the following recurrence relations.

$$x_{n+2} - 2\alpha x_{n+1} + (\alpha^2 + 7)x_n = 0$$

$$y_{n+2} - 2\alpha y_{n+1} + (\alpha^2 + 7)^n y_n = 0$$

2.2;Method:II

In view of (3) , (1) is written as

$$7u^2 + v^2 = (\alpha^2 + 7)^n z^4 \qquad (9)$$

Assuming (5) in (9) and employing the method of factorization, define

$$(v + i\sqrt{7}u) = (\alpha + i\sqrt{7})^n (b + i\sqrt{7}a)^4$$

Expanding binomially and equating real and imaginary parts ,we have

$$v = f(\alpha)[b^4 - 42a^2b^2 + 49a^4] - 7g(\alpha)[4ab^3 - 28a^3b]$$

$$u = f(\alpha)[4ab^3 - 28a^3b] + g(\alpha)[b^4 - 42a^2b^2 + 49a^4]$$

where

$$f(\alpha) = \sum_{r=0}^{\left[\frac{n}{2}\right]} nC_{2r}(-1)^r \alpha^{n-2r} 7^r$$

$$g(\alpha) = \sum_{r=1}^{\left[\frac{n+1}{2}\right]} nC_{2r-1}(-1)^{r-1} \alpha^{n-2r+1} 7^{r-1}$$

The non-zero distinct integral solutions of (1) are given by

$$x = [g(\alpha) + f(\alpha)][b^4 - 42a^2b^2 + 49a^4] + [f(\alpha) - 7g(\alpha)][4ab^3 - 28a^3b]$$

$$y = [g(\alpha) - f(\alpha)][b^4 - 42a^2b^2 + 49a^4] + [f(\alpha + 7g(\alpha)][4ab^3 - 28a^3b]$$

$$z = 7a^2 + b^2$$

3.Remarkable observations:

If (x_0, y_0, z_0) is any given non-zero solution of the (1), then each of the triples

$$(x_{2n-1}, y_{2n-1}, z_{2n-1}) = (7^{4n-2} y_0, 7^{4n-2} x_0, 7^{2n-1} z_0)$$

$$(x_{2n}, y_{2n}, z_{2n}) = (7^{4n} x_0, 7^{4n} y_0, 7^{2n} z_0)$$

also satisfies (1)

A few interesting relations observed from the above triples are presented below

1) $\frac{x_{2n-1}}{y_0} = \frac{y_{2n-1}}{x_0} = \left(\frac{z_{n-1}}{z_0}\right)^2$

2) $\left(\frac{7^4 - 1}{7^4}\right)\sum_{n-1}^{N} \frac{x_{2n}}{x_0} + 1 = \frac{z_{4N}}{z_0}$

3) The triple $\left(\frac{x_{2n-1}}{y_0}, \frac{z_{2n-1}^2}{z_0^2}, \frac{y_{2n-1}}{x_0}\right)$ forms an arithmetic progression.

4) $\frac{48}{7}\sum_{n=1}^{N} z_{2n-1} + z_0 = z_{2N}$

5) $7^3 y_{2n-1} z_{2n-1} = x_{2n} z_{2n}$

6) Each of the following expression is a nasty number

i) $\frac{6x_{2n}y_0}{x_{2n-1}x_0}$

ii) $6[\frac{z_{4n}}{z_0} + \frac{2z_{2n}}{z_0} + 1]$

iii) $6\left[(x_0+2y_0)x_{2n}+y_0y_{2n}\right]$

iv) $6\left[x_{2n-1}y_0+(x_0+2y_0)y_{2n-1}\right]$

v) $6\left[z_{4n}z_0+(y_0+2z_0)y_{2n}\right]$

4. Conclusion:

In this paper the non-homogeneous biquadratic equation with three unknowns is studied for determining its non-zero distinct integral solutions .Two different approaches have been presented. To start with, the method of induction has been introduced to find the corresponding integer solutions of the given ternary biquadratic equation.Secondly, the process of factorizations and the method of cross multiplication have been considered to determining the non-zero distinct integral solutions of the Diophantine equation under consideration.

It is worth to mention here that, when n takes even values, the equation (9) can be written as a system of double equations. Employing the method of cross-multiplication, an infinite number of integer solutions of the given ternary quartic Diophantine equation may be obtained, To conclude one may search for other choices of solutions to (1) along with the corresponding properties.

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