ALMOST CONTINUOUS FUNCTIONS AND MANIFOLDS BASED ON COFINITE SPACES

H. S. Were*

K. C. Sogomo *

M. N. Gichuki*

Abstract

Manifolds have been modeled in various spaces with both Hausdorff and non-Hausdorff properties. In this paper, we define pseudoderivative on almost continuous functions and get some of its properties. Thereafter we model a cofinite topological manifold $\mathcal{W}$ using almost continuous functions.

* Egerton University, Egerton, Kenya

1. Introduction and Literature Review

Manifolds have been looked at as topological spaces locally homeomorphic to Euclidean spaces $\mathbb{R}^n$ of a fixed dimensions without assuming the Hausdorff separation axioms (Baillif and Gabard, 2008). Topological manifold M of dimension n has been defined by Lee (2000) based on the properties of Hausdorffness, second countability and as a locally Euclidean of dimension n. This has been shown in much of the mathematical literature. In 1950s and 1960s, a broad definition of a manifold was given which omitted the point set axioms and allowed higher cardinalities and non-Hausdorff manifolds to be modeled. It also omitted finite dimension which allowed structures such as Hilbert manifolds, Banach manifolds and Fréchet manifolds to be modeled on Hilbert spaces, Banach spaces and Fréchet spaces respectively. Manifolds have also been modeled on convenient vector spaces, (Kriegl and Michor, 1997). A non-Hausdorff manifold was defined by Kent *et al*., (2009) as a topological space which has a countable base of open sets and locally Euclidean of dimension n. They reviewed some topological properties of non-Hausdorff manifolds where they introduced the notion of compatible apparition points for the non-Hausdorff manifolds and studied the properties of these points.

Almost continuity generalizes the notion of continuity and every continuous function is an almost continuous function even though the reciprocal may not hold. It had been defined differently by Stallings (1959), Frolik (1961), Husain (1966) and Singal and Singal (1968). Among many generalization of notion of continuity, almost continuous functions have been defined on Hausdorff spaces as well as on T_1- spaces. Properties and several results concerning almost continuous functions have been studied and proved, (Long and McGehee, 1970). A remarked was made that most of the interesting results with almost continuous functions are obtained with the class of T_1- spaces, (Naimpally 1966). On the class of T_1- spaces, Gichuki (1996) investigated the invariance and inverse invariance of some topological properties with respect to continuous functions and almost continuous functions. On separable locally convex topological vector spaces, one variant of the inverse function theorem have been developed where $D_{\beta,\gamma}$ derivative at a point in a given open subset of a locally convex space have been shown, (Sukhinin and Sogomo, 1985).

2. Preliminaries

The results in this paper are obtained based on infinite cofinite spaces, almost continuous functions as defined by Stallings (1959), Topological manifolds as defined by Lee (2000), inverse function theorem by Sukhinin and Sogomo (1985), closed graph theorem by Rudin (1991) and implicit function theorem.

Definition 1.1

A piecewise linear map is a map composed of some number of linear segments defined over an equal number of neighborhoods.

Theorem 1.1

Given any compact space $P \subset X$ and any $f \in C[P]$, there exists a map $\Phi_\varepsilon : X \longrightarrow Y$ such that $f(x) - \Phi_\varepsilon(x) \subset V_0\ \forall x \in X$ where V_0 is a neighborhood of zero in Y……………..…………..…. (**)

Proof

Since f is uniformly continuous on the compact set P and given any neighborhood of zero $V_0 \subset Y$, there exists a neighborhood of zero $U_0 \subset X$ such that, for any $x, x' \in U_0$ with $x - x' \subset U_0$, we have $f(x) - f(x') \subset V_0$. Now pick $P_n : n \in \mathbb{N}$ such that $\bigcup P_n = P$ and let $x_m \coloneqq P_n$ where $m = 0, 1, \dots, n$. Define Φ_ε as follows: $\Phi(e) \coloneqq f(e)\ \forall e \in P_1$ and $\phi_i(x) \coloneqq f(x_i)\ \forall x \in P_i,\ 1 \leq i \leq n$ so that, the map Φ_i is constantly equal to the value of f at P_i hence $f(x) - \Phi_\varepsilon(x) \subset V_0$ and $f(x) - f(x_i) \subset V_0$ since $x - x_i \subset U_0$. Therefore, (**) is satisfied on P.

Implicit function theorem

It has been shown by Lerman that inverse function theorem implies implicit function theorem. His proof however was in finite dimensional spaces. Before we prove the inverse function theorem in cofinite spaces, we show below that the implicit function theorem is valid for such spaces.

Theorem 1.2

Assume conditions of the inverse function theorem are fulfilled. Then there is a neighborhood U of point x_0 in X, V of y_0 in Y and a function $g: U \longrightarrow V$ so that $f\big(x, g(x)\big) = c$ for $x \in U$. Here $c = f(x_0, y_0)$.

Proof

Consider the map $H(x,y)=\big(x,f(x,y)\big)$. Since f satisfies conditions of closed graph theorem, then $f(x,y)$ is invertible near (x_0,y_0) and by the inverse function theorem, $H(x,y)$ is invertible near (x_0,y_0). Denote by G the inverse of H then for u,v near (x_0,c), we have $(u,v)=H\big(G(u,v)\big)=H\big(G_1(u,v),G_2(u,v)\big)=\Big(G_1(u,v),f\big(G_1(u,v),G_2(u,v)\big)\Big)$. Therefore $u=G_1(u,v)$…………….. (i) and $v=f\big(G_1(u,v),G_2(u,v)\big)$………….. (ii). Substituting (i) into (ii), we get $v=f\big(u,G_2(u,v)\big)$ for all u near x_0, v near c. Let $v=c$, $u=x$, we get $c=f\big(x,G_2(x,c)\big)$. Let $g(x)=G_2(x,c)$, we then have $c=f\big(x,g(x)\big)$.

Definition 1.2

Let $\mathcal{P}$ be an open set in $\phi(U(x_0))$. The map $f:\mathcal{P}\longrightarrow\psi(V(x_0))$ is $D_{\beta,\gamma}$ differentiable at $x_0\in\mathcal{P}$, if there exists a linear continuous operator $f'(x_0):\phi(U(x_0))\longrightarrow\psi(V(x_0))$, such that $R(x,h)\equiv f(x+h)-f(x)-f'(x_0)h$ satisfies the condition:

$\exists C\forall B\exists U:(h\in B+U, ah\in U,\ x-x_0\in U)\implies R(x,ah)\in aC$…..……………………. (i)

Theorem 1.3

Let $\mathcal{P}$ be an open set in $\phi(U(x_0))$, $f:\mathcal{P}\longrightarrow\psi(V(x_0))$ is strictly $D_{\beta_{\mathcal{C}},\gamma_{\mathcal{C}}}$ differentiable at $x_o\in\mathcal{P}$ and $f'(x_0)$ a linear homeomorphism of $\phi(U(x_0))$ onto the subspace (respectively onto a closed subspace) $\phi(U(x_0)\cap V(x_0))\subset\psi(V(x_0))$, with induced topology. Then there exists such an open neighborhood $\mathcal{N}$ of x_0, such that $f|_{\mathcal{N}}:\mathcal{N}\longrightarrow\psi(V(x_0))$ is injective.

Proof

We look at the case of $D_{\beta_{\mathcal{C}},\gamma_{\mathcal{C}}}$ differentiability. Without reducing generality, assume that $x_0=0$, $y_0=0$, $\phi(x_0)=\phi(U(x_0)\cap V(x_0))$ and $f'(x_0)$ is identity operator. On the contrary we look at the map $\mathcal{F}:\phi(U(x_0)\cap V(x_0))\longrightarrow\phi(U(x_0)\cap V(x_0))$, defined by the formula $\mathcal{F}(\mathcal{y})=f'(x_0+[f'(x_0)]^{-1}\mathcal{y})-f(x_0)$ for $\mathcal{y}\in f^{-1}(x_0)(\mathcal{P}-x_0)$.

Note that a linear homeomorphism takes a bounded set to a bounded set and a compact to a compact. In view of the above, f can be expressed as $f(x+h)-f(x)=h+R(x,h)$ where R satisfies (i) for $x_0=0$, $\gamma=\gamma_{\mathcal{C}}$ and $\beta=\beta_{\mathcal{C}}$. For $B\supset(2C)\cap\phi(x_0)$, where C is taken from (i), we find such a U, that $(h\in B+2U, ah\in 2U, x\in U)\Longrightarrow R(x,ah)\in aC$………..…………. (ii)

We show that $f|_U: U \longrightarrow \psi(x_0)$ is injective. Let $x, x+h \in U$ and $f(x+h) = f(x)$. Then $h \in U - x \subset 2U$ and from (ii) follows that $R(x,h) \in C$.

Further $h + R(x,h) = f(x+h) - f(x) = 0$ i.e. $h = -R(x,h) \in C$. Let $\mu > 0$. Then since C is closed, $h \in (\mu C) \cap \phi(x_0) = \frac{1}{2}\mu[(2C) \cap \phi(x_0)] \subset \frac{1}{2}\mu B$. In this regard from (ii) follows that $-h = R(x,h) = R\left(x, \frac{1}{2}\mu(2\mu^{-1}h)\right) \in \frac{1}{2}\mu C$, since $2\mu^{-1}h \in B \subset B + 2U$.

If $\mathcal{K} \subset \psi(x_0)$, $y \in \psi(x_0)$, $P_{\mathcal{K}}(y) = \begin{cases} \infty, & y \notin \mu C (\forall \mu > 0) \\ \inf\{\mu > 0 : y \in \mu\mathcal{K}\} \end{cases}$

Then it implies that $\mu = P_c(h) \leq \frac{1}{2}\mu$ and this is impossible since $\mu > 0$. Therefore $P_c(h) = 0$ and hence $h = 0$ in view of boundedness of C. From this instead of $\mathcal{N}$ in the theorem, U can be used. The closure of $\phi(x_0) = \phi(U(x_0) \cap V(x_0))$ in the case of D_{β_c,γ_c}differentiability of f was used in the choice of $B \supset (2C) \cap \phi(x_0)$, since $2C \cap \phi(x_0)$ is compact in $\phi(x_0)$, if C is compact in $\psi(x_0)$.

Theorem 1.4

Assume conditions of the theorem 1.3 are fulfilled (except closure of $\phi(U(x_0) \cap V(x_0))$ where R satisfies the condition: $\exists C \; \forall C' \; \exists V$:

$(h \in (C' + V) \cap \phi(U(x_0)),\ ah \in V \cap \phi(U(x_0)),\ x - x_0 \in V \cap \phi(U(x_0)) \Longrightarrow R(x, ah) \in aC$ .... (iii). (Here $\phi(U(x_0)) = \phi(U(x_0) \cap V(x_0))$, $f'(x_0) = \text{id}$). Then there exists such an open neighborhood $\mathcal{H}$ of x_0 such that $f|_{\mathcal{H}}: \mathcal{H} \longrightarrow f(\mathcal{H})$ is bijective and uniformly continuous together with the inverse map.

Proof

Under conditions of the proof of theorem 1.4, for $C' \supset 2C$ where C is taken from (iii) we look for such a V that the following condition is fulfilled:

$(h \in (C' + V) \cap \phi(x_0),\ ah \in (2V) \cap \phi(x_0),\ x \in V \cap \phi(x_0)) \Longrightarrow R(x, ah) \in aC$......(iv). Choose an arbitrary V'. Then there exists such a V'', that $V'' \subset V' \cap V$ and $Cl(C' + V'') \subset C' + V$. Let $x, x+h \in V \cap \phi(x_0)$. Then $h \in (2V) \cap \phi(x_0)$. Let $\mu = P_{C'+V''}(h)$ where $\mu > 0$, then from (iv) follows that $R(x,h) = R(x, \mu(\mu^{-1}h) \in \mu C \subset \frac{1}{2}\mu C' \subset \frac{1}{2}\mu(C' + V'')$.

Since $\mu^{-1}h \in Cl(C' + V'') \cap \phi(x_0) \subset (C' + V) \cap \phi(x_0)$.

In this case $P_{C'+V''}(R(x,h)) \leq \frac{1}{2}\mu = \frac{1}{2}P_{C'+V''}(h)$, hence

$P_{C'+V''}(f(x+h)-f(x)) = P_{C'+V''}(h+R(x,h)) \geq \frac{1}{2}P_{C'+V''}(h)$.................................. (v)

From convexity and balancedness of the set $C'+V''$, it follows that if $\mu = P_{C'+V''}(h) = 0$, then (iv) is also fulfilled. From boundedness of C' and arbitrariness of V' and the definition of induced uniform structure on $\mathcal{H} = V \cap \phi(x_0)$ and (v) follows that $f|_{\mathcal{H}}: \mathcal{H} \longrightarrow f(\mathcal{H})$ is bijective and uniform continuity of inverse mapping.

Main results

a) Almost continuous functions

Definition 1

A function g is an almost continuous function at x_0 if there exists a continuous function f such that $(f-g)(x_0) \subset U_0$ for every neighborhood of zero U_0.

Theorem 1

Let $X_1 = (X, \mathcal{T})$ be an infinite space with finite complement topology and let g be an almost continuous function. Then $X_2 = g(X_1)$ is an infinite space with finite complement topology.

Proof

Given that X_1 is an infinite set with finite complement topology and if $\mathfrak{U} \subset X_1$, then $\mathfrak{U}'$ consists of finite elements. If f is a continuous function from $X_1 \longrightarrow X_2$, then $f(\mathfrak{U}') \subset X_2$ consists of finite elements since f is a one to one map. Note that g is an almost continuous functions if for every neighborhood of zero $U_0 \subset X_2$, there exists a neighborhood $V \subset X_1$ such that $(f-g)(V) \subset U_0 \; \forall \, V \subset X_1$. Since $f-g$ is one to one, then U_0 consists of infinite elements and $(f-g)(V')$ must consist of finite elements since V consists of infinite elements and V' consists of finite elements. This is due to the fact that X_1 is an infinite set with finite complement topology.

Theorem 2

Let X and Y be linear topological spaces without norm and $f \in C[X]$, then there exists an almost continuous function $\Psi_\varepsilon: X \longrightarrow Y$ such that $f(x) - \Psi_\varepsilon(x) \subset V_0 \; \forall \, x \in X$ where V_0 is a neighborhood of zero in Y.

This follows from the fact that almost continuous functions are subsets of piecewise linear map and continuous functions are subsets of almost continuous functions. The result of this theorem follows from theorem 1.1 above.

b) Pseudoderivative on almost continuous functions

Definition 2

A linear operator T is $D_{\beta,\gamma}$ pseudoderivative on an almost continuous function g at x_0 if there exists a $D_{\beta,\gamma}$ differentiable function f such that $(T - f')(x_0) \subset U_0$ for every neighborhood of zero,U_0.

Properties of pseudoderivative on almost continuous functions

Given that g is an almost continuous function and f is a continuous function both defined in an open neighborhood U_0 of a point x, then the pseudoderivative on almost continuous functions satisfies the following properties:

Linearity which consists of two parts:

a) $[\lambda(g - f)](x + h) - [\lambda(g - f)](x) = \lambda g(x + h) - \lambda g(x) - \lambda f(x + h) + \lambda f(x)$

$$= \lambda\big(g'(x) - f'(x)\big) \subset U_0$$

$$\therefore \big[\lambda\big(g(x) - f(x)\big)\big]' = \lambda\big(g(x) - f(x)\big)' \subset U_0$$

From the above, pseudoderivative of a constant times an almost continuous function can as well be given by $(\lambda g)'(x) = \lambda g'(x)$

b) Given two almost continuous functions g_1 and g_2 and a continuous function f we have the sum of pseudoderivatives given as

$$[(g_1 + g_2) - f](x + h) - [(g_1 + g_2) - f](x)$$

$$= g_1(x + h) + g_2(x + h) - f(x + h) - g_1(x) - g_2(x) + f(x)$$

$$= g_1(x + h) - g_1(x) + g_2(x + h) - g_2(x) - \big(f(x + h) - f(x)\big)$$

$$= \big[\big(g_1'(x) + g_2'(x)\big) - f'(x)\big] \subset U_0$$

$$\therefore (g_1 + g_2)'(x) = g_1'(x) + g_2'(x)$$

The product of pseudoderivatives on almost continuous functions can as well be shown by considering two almost continuous functions g and h and two differentiable functions f_1 and f_2 as follows:

$$[gh - f_1 f_2](x + \alpha) - [gh - f_1 f_2](x)$$

$$= gh(x + \alpha) - g(x)h(x + \alpha) + g(x)h(x + \alpha) - g(x)h(x) - f_1 f_2(x + \alpha) + f_1(x) f_2(x + \alpha)$$
$$- f_1(x) f_2(x + \alpha) + f_1(x) f_2(x)$$

$$= h(x+\alpha)[g(x+\alpha)-g(x)] - f_2(x+\alpha)[f_1(x+\alpha)-f_1(x)] + g(x)[h(x+\alpha)-h(x)] - f_1(x)[f_2(x+\alpha)-f_2(x)]$$

$$= [h(x+\alpha)g'(x) - f_2(x+\alpha)f_1'(x)] + [g(x)h'(x) - f_1(x)f_2'(x)] \subset U_0$$

From above such a product can be indicated as $(gh)'(x) = g(x)h'(x) + g'(x)h(x)$.

c) Manifolds based on cofinite spaces

Definition 3

A topological space $\mathcal{W}$ is a topological manifold on a cofinite space if it has a countable base of open sets and for every neighborhoods $U(x_0)$ and $V(x_0)$ of a point $x_0 \in \mathcal{W}$ there exists almost continuous functions ϕ and ψ such that:

1. $\phi(U(x_0))$ and $\psi(V(x_0))$ maps $\mathcal{W}$ to cofinite spaces.
2. $\exists$ a bijective almost continuous function $h: \phi(U(x_0) \cap V(x_0)) \longrightarrow \psi(U(x_0) \cap V(x_0))$.

Theorem 3

If for any point x_0 in a topological space $\mathcal{W}$ and any neighborhoods $U(x_0)$ and $V(x_0)$ of x_0 and almost continuous functions ϕ and ψ are such that they map $\mathcal{W}$ to cofinite spaces $\phi(U(x_0) \cap V(x_0))$ and $\psi(U(x_0) \cap V(x_0))$. Then $\exists$ a bijective almost continuous function $h: \phi(U(x_0) \cap V(x_0)) \longrightarrow \psi(U(x_0) \cap V(x_0))$.

Proof

Assuming that ϕ and ψ satisfy condition 1 of definition 3, then definition 1.3 together with theorems 1.3 and 1.4 proves theorem 3.

4. References

Baillif, M. and Gabard, A. (2008). Manifolds: Hausdorffness versus Homogeneity. *Proceedings of American Mathematical Society*. **136**: 1105 – 1111.

Deshpande, J. V. (1990). *Introduction to Topology*, Tata McGraw-Hill publishing Company Limited, New Delhi.

Frolik, Z. (1961). Remarks concerning the invariances of Baire spaces under mappings. *Czechoslovak Mathematical journal*.**86**: 381 – 385.

Gichuki, M. N. (1996). *Invariance of topological properties with respect to continuous and almost continuous functions*, Masters of philosophy. Thesis, Moi University, Kenya.

Husain, T. (1966). Almost continuous mappings. *Prace Matematyczne*.**10**: 1 – 7.

Kent, S. L., Mimna, R. A. and Tartir, J. K. (2009). A note on topological properties of Non-Hausdorff manifolds. *International Journal of Mathematics and Mathematical Sciences*. **2009**: 1 – 4.

Kriegl, A. and Michor, P. W. (1997). *The convenient setting of global analysis*, America Mathematical Society, USA.

Lee, J. M. (2000). *Introduction to topological manifolds*, Springer-Verlag, New York, USA.

Lerman, E. (2005). *Inverse function theorem implies implicit function theorem*. University of Llinois.

Long, P. E. and McGehee, E. P. (1970). Properties of almost continuous functions. *Proceedings of the American Mathematical Society*.**24**: 175 – 180.

Naimpally, S. A. (1966). Graph topology for function spaces. *Transactions of the American Mathematical Society*. **123**: 267 – 272.

Rudin, W. (1991). *Functional Analysis (2nd edition)*. McGraw-Hill book company Inc. Singapore.

Singal, M. K. and Singal, A. R. (1968). Almost continuous mappings. *Yokohama Mathematical Journal*. **16**: 63 – 73.

Stallings, J. (1959). Fixed point theorems for connectivity maps. *Fundamenta Mathematicae* **47**: 249 – 263.

Sukhinin, M. F. and Sogomo, K. C. (1985). One variant of the inverse function theorem in Locally convex spaces. *Moscow U.D.N Sbornik Statei* pp.119 – 124.