# ON CHARACTERIZATIONS OF W-TYPE SPACES

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## Abstract:

In this paper we obtain new characterization of certain spaces of W-type.

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**1. Introduction:** The spaces of W-type were studied by B.L. Gurevich [5] and I. M. Gelfand and G.E. Shilov [4]. The investigations of the behaviour of the Fourier transformation on the W-spaces are done in [4] and [5]. Also W-spaces are applied to the theory of partial differential equations. These spaces are generalizations of spaces of S-type [3].

Pathak [6] and Eijndhoven and Kerkhof [2] introduced new spaces of W-type and investigated the behaviour of the Hankel transformation over them.

Motivated by the work of Pathak and Upadhyay [7], we give new characterizations of the spaces of W-type introduced in [2]. In our investigation the Hankel type transformation defined by

$$h_{\alpha,\beta}\,(\phi)\,(x) = \int_0^\infty y^{4\alpha}\,(xy)^{-(\alpha-\beta)}\,J_{\alpha-\beta}\,(xy)\;\;\phi\,(y)\,dy, \qquad x \in\,(0,\infty),$$

plays an important role, where as usual $J_\lambda$ denotes the Bessel type function of the first kind and order $\lambda$. Throughout this paper $(\alpha-\beta)$ will always represent a real number greater than $-1/2$.

From [1, Corollary 4.8], it is known that $h_{\alpha,\beta}$ is an automorphism of the space $S_e$ constituted by all those complex valued even smooth functions $\phi = \phi(x),\ x\ \in\ \mathbb{R}$, such that

$$\rho_{m,n}\,(\phi) = \ \mathrm{Sup}_{x\,\in\,\mathbb{R}}|x^m D^n\ \phi(x)| < \infty,\ \text{for every M}, n\ \in\ \mathbb{N}.$$

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Moreover $h_{\alpha,\beta}^{-1}$ , the inverse of $h_{\alpha,\beta}$ , coincides with $h_{\alpha,\beta}$ on $S_e$. Throughout this paper K will always denote the following set of functions.

$K = \{M \in C^2([0\infty)): M(0) = M'(0) = 0,\ M'(\infty) = \infty \ and \ M''(x) > 0,\ x \in (0,\infty)\}$.

$M^X$ will represent the Young dual function of M ([4, p.19]).

Interesting and useful properties of the functions in $K$ can be found in [2] and [4]. Following [4], we define the W-spaces as follows:

Let $M,\Omega \in K$ and $a,b > 0$. The space $W_{m,a}$ consists of all those complex valued and smooth functions $\phi$ on ℝ such that for every $m \in \mathbb{N} - \{0\}$ and $k \in \mathbb{N}$ there exists $C_{m,k} > 0$ for which

$$|D^k\phi(x)| \leq C_{m,k}\, e^{-M(a(1-1/m|x|))},\ x \in \mathbb{R}.$$

The space $W^{\Omega,b}$ consists of all entire functions $\phi$ such that for every $m \in \mathbb{N} - \{0\}$ and $k \in \mathbb{N}$ there exists $C_{m,k} > 0$ for which

$$|z^k\phi(z)| \leq C_{m,k}\, e^{\Omega\left(b\left(1+\frac{1}{m}\right)|I(z)|\right)}, \qquad z \in \mathbb{C}.$$

Ejndhoven and Kerkhof [2] investigated the behaviour of the transformation $h_{\alpha,\beta}$ on the subspaces of the $W$ −spaces defined as follows :

A function $\phi$ is in $We_{M,a}$ (respectively, $W^{\Omega,b}$ and $W_{M,a}^{\Omega,b}$). We now introduce new spaces of $W$ −type.

Let $\Omega, M \in K,\ a,b > 0$ and $1 \leq p \leq \infty$. A complex valued and smooth function $\phi = \phi(x)$, $x \in I = (0,\infty)$ is in $W\, e^p_{\alpha,\beta,M,a}$ if and only if $\phi$ belongs to $S_e$ and

$$\left\| e^{M[a(1-1/m)x]\Delta^k_{\alpha,\beta}\phi(x)} \right\|_p < \infty \text{ for every } m \in \mathbb{N} - \{0\} \text{ and } k \in \mathbb{N}.$$

Here and in the sequel $\|\cdot\|_p$ denotes the norm in the Lebesgue space $L_p\,(0,\infty)$. By $\Delta_{\alpha,\beta}$ we denote the Bessel type operator

$$x^{4\beta-2}D\, x^{4\alpha}\, D.$$

The space $W\, e^{p,\Omega,b}$ consists of $\phi \in S_e$ that admit a holomorphic extension to the whole complex plane and that satisfy the following two conditions:

(i) there exists $\epsilon > 0$ such that for every $k \in \mathbb{N}$, we find $C_k > 0$ for which

$$|z^k\, \phi\,(z)| \leq C_k\, e^{(\Omega(b\,\epsilon\,|I(z)|))},\ z \in \mathbb{C},$$

(ii) $Sup_{y \in \mathbb{R}} \left\| e^{-\Omega\left(b\left(1+\frac{1}{n}\right)|y|\right)} (x+iy)^m\, \phi\,(x+iy) \right\|_p < \infty$, for every $n \in \mathbb{N} - \{0\}$ and $m \in \mathbb{N}$.

A complex valued and smooth function $\phi = \phi(x),\ x \in I$ is in $W\ e_{M,a}^{p,\Omega,b}$ if and only if, $\phi$ is in $S_e$ admitting a holomorphic extension to the whole complex plane and $\phi$ satisfies (i) and (iii) $Sup_{y\in\mathbb{R}} \left\| e^{\left(M\,[a(1-1/m)x]-\Omega\left[b\left(1+\frac{1}{n}\right)|y|\right]\right)}\,\phi\,(x+iy) \right\|_p < \infty$ for every $m,n\ \in\ \mathbb{N}-\{0\}$.

In Section 2 we establish that $W\ e_{\alpha,\beta,M,a}^{p} = W\ e_{M,a}$ , $W\ e^{p,\Omega,b} = W\ e^{\Omega,b}$ and $W\ e_{M,a}^{p,\Omega,b} = W\ e_{M,a}^{\Omega,b}$ for every $(\alpha-\beta) > -1/2$ and $1 \le p \le \infty$.

Throughout this paper for every $1 \le p \le \infty$ we denote by $p'$ the conjugate of $p$ $\left(i.e.\ p' = \frac{p}{p-1}\right)$. Also by $C$ we always represent a suitable positive constant, not necessarily the same in each occurrence.

**2. Characterizations of $We$ −spaces:** In this section we prove by using the Hankel type transformation $h_{\alpha,\beta}$ , that $W\ e_{\alpha,\beta,M,a}^{p} = W\ e_{M,a}$ , $W\ e^{p,\Omega,b} = W\ e^{\Omega,b}$ *and* $W\ e_{M,a}^{p,\Omega,b} = W\ e_{M,a}^{\Omega,b}$ for every $(\alpha-\beta) > -1/2$ and $1 \le p \le \infty$.

**Lemma 2.1:** Let $1 \le p \le \infty$ and $(\alpha-\beta) > -1/2$ . Then $W\ e_{\alpha,\beta,M,a}^{p}$ is contained in $W\ e_{M,a}$.

**Proof:** First assume that $1 \le p \le \infty$. Let $\phi$ be in $W\ e_{\alpha,\beta,Ma}^{p}$. Define

$$\psi\,(y) = h_{\alpha,\beta}\,(\phi)\,(y) = \int_0^\infty (xy)^{-(\alpha-\beta)}\,J_{\alpha-\beta}\,(xy)\,\phi(x)\,x^{4\alpha}\,dx,\ \ y\ \in\ \mathbb{C}\,. \qquad (2.1)$$

According to Corollary 4.8 in [1], $\psi$ is in $S_e$ . Moreover, the last integral is defined for every $y\ \in\ \mathbb{C}$ . In fact, for every $y\ \in\ \mathbb{C}$ and $n\ \in\ \mathbb{N}-\{0\}$, by virtue of (5.3b) of [2] and Holder's inequality

we have

$$\int_0^\infty \left|(xy)^{-(\alpha-\beta)}\,J_{\alpha-\beta}\,(xy)\right||\phi(x)|\,x^{4\alpha}\,dx\ \le C\int_0^\infty e^{x|I(y)|}\,|\phi(x)|\,x^{4\alpha}\ dx$$

$$\le C\int_0^\infty e^{x|I(y)|-M\,(a(1-1/n)x)}e^{M\,(a(1-1/n)x)}\,|\phi(x)|\,x^{4\alpha}\,dx$$

$$\le C\left(\int_0^\infty \left|e^{x|I(y)-M(a(1-1/n)x)|}x^{4\alpha}\right|^{p'}dx\right)^{1/p'}$$

$$\times\left(\int_0^\infty \left|e^{M(a(1-1/n))x}\phi(x)\right|^{p}dx\right)^{1/p}$$

$$\leq C\left(\int_0^\infty \left|e^{x|I(y)-M(a(1-1/n))}x^{4\alpha}\right|^{p'} dx\right)^{1/p'} .$$

Moreover, denoting as usual by $M^X$ the yong dual of $M$, according to well-known properties of $M^x$ ([4]) we obtain for every $x \in I,\ y \in \mathbb{C},\ n,m \in \mathbb{N}-\{0\},\ where\ 1<m<n$,

$$x|I(y)| - M(a(1-1/n)x) \ = \frac{x|I(y)|}{a(1-1/m)}\ a(1-1/m) - M(a(1-1/n)x)$$

$$\leq M(a(1-1/m)x) - M\left(a\left(1-\frac{1}{n}\right)x\right)$$

$$+ M^X\left(\frac{|I(y)|}{a(1-1/m)}\right)$$

$$\leq\ -M\ \left(a\left(\tfrac{1}{m}-1/n\right)x\right)+\ M^X\ \left(\tfrac{I(y)}{a(1-1/m)}\right).$$

Hence for every $m,n \in \mathbb{N}-\{0\}$ with $1<m<n$ we can write

$$\int_0^\infty \left|(xy)^{-(\alpha-\beta)}\, J_{\alpha-\beta}\,(xy)\right| |\phi(x)|\, x^{4\alpha}\, dx$$

$$\leq C\left(\int_0^\infty e^{-M\left(a\left(\frac{1}{m}-1/n\right)x\right)}x^{4\alpha}\right)^{p'} dx)^{1/p'}\, e^{M^X\left(\frac{|I(y)|}{a(1-1/m)}\right)}$$

$$\leq C\, e^{M^X\left(\frac{|I(y)|}{a(1-1/m)}\right)},\qquad y\ \in\ \mathbb{C}\,,\qquad because\ \lim_{x\to\infty} M'(x) = \infty\,.$$

If $p=1$ or $p=\infty$ we can argue in a similar way.

Thus we conclude that the integral in the right hand side of (2.1) is a continuous extension of $\psi$ to the whole complex plane. Moreover, by proceeding in a similar way we can see that it also is entire. Such an extension will be denoted again by $\psi$. Note that $\psi$ is an even function.

We prove that $\psi\ \in W\, e^{M^{X,1/a}}$. It is simple to deduce from Lemma 5-4-1 of [9] that for every $k\ \in\ \mathbb{N}$

$$y^{2k}\,\psi\,(y) =\ (-1)^k\ \int_0^\infty (xy)^{-(\alpha-\beta)} J_{\alpha-\beta}\,(xy)\, \Delta^k_{\alpha,\beta}\,[\phi(x)]\, x^{4\alpha}\, dx,\ \ y\ \in\ \mathbb{C}\,.$$

Then, proceeding as above, we get for every $k,m\ \in\ \mathbb{N},\ m>1$,

$$|y^{2k}\,\psi\,(y)|\ \leq\ \int_0^\infty \left|(xy)^{-(\alpha-\beta)}\, J_{\alpha-\beta}\,(xy)\right| \left|\Delta^k_{\alpha,\beta}[\phi(x)]\, x^{4\alpha}\, dx\right.$$

$$\leq C \int_0^\infty e^{x|I(y)|}\, x^{4\alpha} \left|\Delta_{\alpha,\beta}^k [\phi(x)]\right| dx$$

$$\leq C\, e^{M^X\left(\frac{|I(y)|}{a(1-1/m)}\right)}, \; y \in \mathbb{C}. \qquad (2.2)$$

Hence $\psi \in W\, e^{M^X,\; 1/a}$.

Since $h_{\alpha,\beta} = h_{\alpha,\beta}^{-1}$ on $S_e$, according to Lemma 7.4 of [2], we conclude that $W\, e_{\alpha,\beta,M,a}^p$ is contained in $W\, e_{M,a}$.

**Lemma 2.2:** Let $1 \leq p \leq \infty$ and $(\alpha-\beta) > -1/2$. Then $W\, e_{M,a}$ is contained in $W\, e_{\alpha,\beta,M,a}^p$.

**Proof:** By virtue of Lemma 7.3 of [2], $h_{\alpha,\beta}\left(W\, e_{M,a}\right) \subset W\, e^{M^X,\; 1/a}$.

Since $h_{\alpha,\beta} = h_{\alpha,\beta}^{-1}$ on $S_e$, our result will be established when we see that $h_{\alpha,\beta}(\phi)$ is in $W\, e_{\alpha,\beta,M,a}^p$.

Note first that according to Corollary 4.8 of [1], $h_{\alpha,\beta}\, \phi$ is in $S_e$. Let $k \in \mathbb{N}$. By involving Lemma 5-4-1 of [9] we can obtain that

$$\Delta_{\alpha,\beta}^k\, h_{\alpha,\beta}(\phi)(x) = (-1)^k\, h_{\alpha,\beta}\left(z^{2k}\, \phi(z)\right)(x), \; x \in I. \qquad (2.3)$$

A procedure similar to the one developed in the proof of Lemma 6.1 of [2] allows us to write, for every $x > 1$ and $\tau > 0$,

$$\Delta_{\alpha,\beta}^k\, h_{\alpha,\beta}(\phi)(x) = \frac{1}{2}\int_{-\infty}^{\infty} \left(x(\sigma+i\tau)\right)^{-(\alpha-\beta)} H_{\alpha,\beta}^{(1)}\left(x(\sigma+i\tau)\right) \phi(\sigma+i\tau)$$

$$\times\, (\sigma+i\tau)^{2(\alpha-\beta)+2k+1}\, d\sigma,$$

where $H_{\alpha,\beta}^{(1)}$ denotes the Hankel type functions ([8], p.73).

Now for every $x > 1$ and $\tau > 0$ we divide the last integral as follows :

$$\int_{-\infty}^{\infty} \left(x(\sigma+i\tau)\right)^{-(\alpha-\beta)} H_{\alpha,\beta}^{(1)}\left(x(\sigma+i\tau)\right) \phi(\sigma+i\tau)(\sigma+i\tau)^{2(\alpha-\beta)+2k+1}\, d\sigma$$

$$\left(\int_{|x(\sigma+i\tau)|\leq 1} + \int_{|x(\sigma+i\tau)|>1}\right) \left(x(\sigma+i\tau)\right)^{-(\alpha-\beta)} H_{\alpha,\beta}^{(1)}$$

$$\times\, \left(x(\sigma+i\tau)\right) \phi\, (\sigma+i\tau)\, (\sigma+i\tau)^{2(\alpha-\beta)+2k+1}\, d\sigma.$$

We will analyze each of the integrals separately.

Assume first that $(\alpha-\beta)\geq 1/2$. On one side by using (5.3 c) of [2], we get for every $n\in \mathbb{N}-\{0\}$

$$\int_{|x(\sigma+i\tau)|\leq 1} \left|\left(x(\sigma+i\tau)\right)^{-(\alpha-\beta)} H^{(1)}_{\alpha,\beta}\left(x(\sigma+i\tau)\right)\phi(\sigma+i\tau)(\sigma+i\tau)^{2(\alpha-\beta)+2k+1}\right| d\sigma$$

$$\leq C\, e^{-x\tau}\int_{-\infty}^{\infty}|\phi(\sigma+i\tau)|\, d\sigma$$

$$\leq C\, e^{-x\tau+M^{X}[1/a(1+1/n)\tau]}\,,\ x>1\ and\ \tau>0;$$

On the other hand, by using again (5.3c) of [2], for every $n\in \mathbb{N}-\{0\}$

$$\int_{|x(\sigma+i\tau)|>1} \left|\left(x(\sigma+i\tau)\right)^{-(\alpha-\beta)} H^{(1)}_{\alpha,\beta}\left(x(\sigma+i\tau)\right)\phi(\sigma+i\tau)\,(\sigma+i\tau)^{2(\alpha-\beta)+2k+1}\right|\, d\,\sigma$$

$$\leq C\, e^{-x\tau}\int_{-\infty}^{\infty}\left|\phi\,(\sigma+i\tau)\,(\sigma+i\tau)^{2(\alpha-\beta)+2k+1}\right|d\sigma \qquad (2.5)$$

$$\leq C\, e^{-x\tau+M^{X}[1/a(1+1/n)\tau]}\,,\qquad x>1\ and\ \tau>0.$$

For fixed $n\in \mathbb{N}-\{0\}$, we choose $\tau>0$ such that

$$M^{X'}\left(\frac{1}{a}\left(1+\frac{1}{n}\right)\tau\right)=\frac{ax}{(1+1/n)}.$$

Then from Lemma 2.4 of [2] we have

$$-x\tau+\ M^{X}(1/a(1+{}^{1}/_{n})\tau)-M\left(\frac{ax}{1+1/n}\right). \qquad (2.6)$$

Hence by combining (2.4), (2.5) and (2.6), it follows that

$$\left|\Delta^{k}_{\alpha,\beta}\, h_{\alpha,\beta}\,(\phi)(x)\right|\leq C\, e^{-M\left[ax\left(1-\frac{1}{n+1}\right)\right]},\ x>1\,,and\ n\in\mathbb{N}.$$

Note also that, if $-1/2<(\alpha-\beta)<1/2$, by involving (5.3.d) of [2] one has

$$\left|\Delta^{k}_{\alpha,\beta}\, h_{\alpha,\beta}\,\phi\,(x)\right|\leq C\, e^{-x\tau}\int_{-\infty}^{\infty}\left|\phi(\sigma+i\tau)(\sigma+i\tau)^{\alpha-\beta+2k+1/2}\right|d\sigma,\ \ \tau>0$$

and $x>1$.

Proceeding as above, we conclude that

$$\left|\Delta^{k}_{\alpha,\beta}\, h_{\alpha,\beta}\,(\phi)(x)\right|\leq C\, e^{-M[ax(1-1/n)]},\qquad x>1\ and\ m\in\mathbb{N}-\{0\}.$$

Now let $x\in(0,1)$ and $m\in\mathbb{N}-\{0\}$. According to (5.3b) of [2] we have

$$\left|e^{M[ax(1-1/m)]}\,\Delta^{k}_{\alpha,\beta}[h_{\alpha,\beta}\,(\phi)\,(x)]\right|=\left|e^{M[ax(1-1/m)]}h_{\alpha,\beta}\left(z^{2k}\phi(z)\right)(x)\right|$$

$$\leq C \int_0^{\infty} \sigma^{2(\alpha-\beta)+2k+1} |\phi(\sigma)|\, d\sigma$$

because $M$ is an increasing function on $(0, \infty)$ .

Hence, for every $k \in \mathbb{N}$ *and* $m \in \mathbb{N} - \{0\}$ ,

$$\left|e^{M[ax(1-1/m)]} \Delta_{\alpha,\beta}^{k}\, h_{\alpha,\beta}\,(\phi)(x)\right| \leq C, \qquad x > 0,$$

and, if $m \in \mathbb{N} - \{0\},\ k \in \mathbb{N}$ *and* $1 \leq p < \infty$, *then*

$$\left\{\int_0^{\infty} \left|e^{M[ax(1-1/m)]} \Delta_{\alpha,\beta}^{k}\, h_{\alpha,\beta}\,(\phi)\,(x)\right|^{p} dx\right\}^{1/p} \leq C$$

because

$$\int_0^{\infty} e^{-pM\left[ax\left(1/m - \frac{1}{(m+1)}\right)\right]} dx < \infty\,.$$

Thus we establish that $h_{\alpha,\beta}\, \phi \in W\, e^{p}_{\alpha,\beta,M,a}$ , $1 \leq p \leq \infty$ , and the proof is completed.

From Lemmas 2.1 and 2.2 we deduce

**Theorem 2.1:** For every $1 \leq p \leq \infty$ and $(\alpha - \beta) > -1/2$ , $W\, e^{p}_{\alpha,\beta,M,a} = W\, e_{M,a}$ .

**Lemma 2.3:** Let $1 \leq p \leq \infty$ . Then $W\, e^{p,\Omega,b}$ is contained in $W\, e^{\Omega,b}$ .

**Proof:** Let $\phi$ be in $W\, e^{p,\Omega,b}$. Assume that $(\alpha - \beta) > -1/2$. Proceeding as in the proof of Lemma 2.2 we can establish that for every $k \in \mathbb{N}$ there exists $\ell = \ell(k)$ such that

$$\left|\Delta_{\alpha,\beta}^{k}\, h_{\alpha,\beta}\,(\phi)\,(x)\right| \leq C\, e^{-x\tau} \int_{-\infty}^{\infty} |\phi\,(\sigma + i\tau)|\,\left(|\sigma + i\tau|^{\ell} + 1\right) d\sigma,\ \ \tau,\ \ x \in (0, \infty).$$

Hence, according to Holder's inequality and (2.6), we obtain for each $k \in \mathbb{N},\ m \in \mathbb{N} - \{0\}$ and suitable $\tau > 0$

$$e^{\Omega^{\mathrm{X}}\left[\frac{1}{\mathrm{b}}(1-1/\mathrm{m})\mathrm{x}\right]} \left|\Delta_{\alpha,\beta}^{k}\, h_{\alpha,\beta}\,(\phi)\,(x)\right|$$

$$\leq C\, e^{\Omega^{\mathrm{X}}\left[\frac{1}{\mathrm{b}}\left(1-\frac{1}{\mathrm{m}}\right)\mathrm{x}\right] - \Omega^{\mathrm{X}}\left[\frac{1}{\mathrm{b}}\left(1-\frac{1}{\mathrm{m}+1}\right)\right)\mathrm{x}} \left\{\int_{-\infty}^{\infty} \frac{d\sigma}{(1+\sigma^2)^{p'}}\right\}^{1/p'}$$

$$\times \left\{\int_{-\infty}^{\infty} e^{-\Omega[b\left(1+\frac{1}{m}\right)\tau]} \left(|\sigma + i\tau| + 1\right)\left(|\sigma + i\tau|^{\ell} + 1\right)|\phi(\sigma + i\tau)|\right)^{p} d\sigma\right\}^{1/p}$$

$\leq C, x \in (0, \infty)$ ,

provided that $1 \le p \le \infty$. When $p = 1 \; or \; p = \infty$ we can proceed in a similar way. Thus we prove that $h_{\alpha,\beta}(\phi) \in W\, e^{\infty}_{\alpha,\beta,\Omega^{X},\,1/p}$. Therefore Theorem 2.1 shows that $h_{\alpha,\beta}\,(\phi) \in W\, e_{\Omega^{X},\,1/b}$ .

Since $h_{\alpha,\beta} = h^{-1}_{\alpha,\beta}$ on $S_e$, it is sufficient to take into account Lemma 7.3 of [2] to see that $\phi \in W\, e^{\Omega,b}$, and the proof of this lemma is complete.

The next result is not difficult to see.

**Lemma 2.4:** Let $1 \le p \le \infty$. Then $W\, e^{\Omega,b}$ is contained in $W\, e^{p,\Omega,b}$.

As an immediate consequence of Lemmas 2.3 and 2.4 we obtain the following

**Theorem 2.2:** Let $1 \le p \le \infty$. Then $W\, e^{p,\Omega,b} = W\, e^{\Omega,b}$.

**Lemma 2.5:** Let $1 \le p \le \infty$. Then $W\, e^{p,\Omega,b}_{M,a}$ is contained in $W\, e^{\Omega,b}_{M,a}$ .

**Proof:** Let $\phi \in W\, e^{p,\Omega,b}_{M,a}$. Choose $(\alpha - \beta) \ge 1/2$ . Since $h_{\alpha,\beta} = h^{-1}_{\alpha,\beta}$ on $S_e$, by virtue of Lemma 7.5 of [2], to prove this lemma it is sufficient to see that $h_{\alpha,\beta}\,\phi$ is in $W\, e^{M^{X},\;1/a}_{\Omega^{X},\;1/b}$.

The Hankel type transformation $h_{\alpha,\beta}\,\phi$ of $\phi$ is in $S_e$ (Corollary 4.8 [1]). Moreover proceeding as in the proof of Lemma 2.1, we can see that $h_{\alpha,\beta}\,\phi$ can be holomorphically extended to the whole complex plane.

Let $\tau > 0$. An argument similar to the one developed in Lemma 6.1 of [2] allows us to write.

$$\left(h_{\alpha,\beta}\phi\right)(x) = \frac{1}{2}\int_{-\infty}^{\infty}\left(x(\sigma+i\tau)\right)^{-(\alpha-\beta)} H^{(1)}_{\alpha,\beta}\left(x(\sigma+i\tau)\right)\phi\,(\sigma+i\tau)$$

$$\times\;(\sigma+i\tau)^{4\alpha}\,d\sigma\,,\qquad |x| > 1.$$

As in the proof of Lemma 2.2,

$$\left(h_{\alpha,\beta}\,\phi\right)(x) = \frac{1}{2}\left(\int_{|x(\sigma+i\tau)|\le 1} + \int_{|x(\sigma+i\tau)>1|}\right)\left(x(\sigma+i\tau)\right)^{-(\alpha-\beta)} H^{(1)}_{\alpha,\beta}$$

$$\times\;\left(x(\sigma+i\tau)\right)\phi\,(\sigma+i\tau)\,(\sigma+i\tau)^{4\alpha}\,d\sigma,\qquad |x| > 1.$$

We must analyze each of the two integrals.

According to (5.3c) of [2] we have, for every $n, m \in \mathbb{N} - \{0\}$,

$$\int_{|x(\sigma+i\tau)|>1}\left|\left(x\,(\sigma+i\tau)\right)^{-(\alpha-\beta)} H^{(1)}_{\alpha,\beta}\left(x\,(\sigma+i\tau)\right)\phi(\sigma+i\tau)\,(\sigma+i\tau)^{4\alpha}\right|\,d\sigma$$

$$\le C\,|x|^{2\beta-1}\int_{-\infty}^{\infty} e^{-(R(x))\tau-(I(x))\sigma}\,|\phi\,(\sigma+i\tau)\,(\sigma+i\tau)^{2\alpha}|\,d\sigma,$$

$$\leq C|x|^{2\beta-1}\left\{\int_{-\infty}^{\infty} e^{(R(x))\tau+|I(x)||\sigma|-M\left(a\left(1-\frac{1}{n}\right)\sigma\right)+\Omega\left(b\left(1+\frac{1}{m}\right)\tau\right)}\left(|\sigma+i\tau|^{2\alpha}\right)^{p'} d\sigma\right\}^{1/p'},$$

where $|x| > 1$, provided that $1 < p < \infty$. By Lemma 2.4 of [2],

$$|I(x)||\sigma| \leq M^X\left(\frac{|I(x)|}{a\left(1-\frac{1}{\ell}\right)}\right) + M\left(a(1-1/\ell)|\sigma|\right), \sigma \in \mathbb{R},\ x \in \mathbb{C})\ and\ \ell \in \mathbb{N},$$

$\ell > 1$.

Then

$$|R(x)||\sigma| - M\left(a(1-1/n)|\sigma|\right) \leq M^X\left(\frac{|R(x)|}{a(1-1/\ell)}\right) - M\left(a\left(\frac{1}{\ell}-1/n\right)|\sigma|\right),$$

where $\sigma \in \mathbb{R}$, $x \in \mathbb{C}$ and $\ell, n \in \mathbb{N}$, $n > l > 1$.

We assume now that $R(x) > 0$ and we choose $\tau > 0$ such that

$$\Omega'(b(1+1/m)\tau) = \frac{R(x)}{b(1+1/m)}\ .$$

Then again by Lemma 2.4 of [2],

$$\tau R(x) = \Omega(b(1+1/m)\tau) + \Omega^X\left(\frac{R(x)}{b(1+1/m)}\right).$$

Hence, Since $(2\beta - 1) \leq 0$ and $1 < p < \infty$, we obtain for every $|x| \geq 1$ and $R(x) > 0$

$$\int_{|x(\sigma+i\tau)|>1}\left|\left(x(\sigma+i\tau)\right)^{-(\alpha-\beta)} H_{\alpha,\beta}^{(1)}\left(x(\sigma+i\tau)\right)\phi(\sigma+i\tau)\,(\sigma+i\tau)^{4\alpha}\right| d\sigma$$

$$\leq C\, e^{M^X\left(\frac{|I(x)|}{a(1-1/\ell)}\right)-\Omega^X\left(\frac{R(x)}{b(1+1/m)}\right)}$$

$$\times\left(\int_{-\infty}^{\infty}\left(e^{-M(a(1/\ell-1/n)|\sigma|)}\,|\sigma+i\tau|^{2\alpha}\right)^{p'} d\sigma\right)^{1/p'} \qquad (2.7)$$

$$\leq C\, e^{M^X\left(\frac{|I(x)|}{a(1-1/\ell)}\right)-\Omega^X\left(\frac{R(x)}{b(1+1/m)}\right)},\quad n,m,\ell \in \mathbb{N}-\{0\},\ \ 1<\ell<n$$

because

$$\int_{-\infty}^{\infty}\left(e^{-M(a(1/\ell-1/n)|\sigma|)}\,|\sigma+i\tau|^{2\alpha}\right)^{p'} d\sigma < \infty\ .$$

If $p = 1$ or $p = \infty$, we can proceed in a similar way.

On the other hand, by (5.3c) of [2]

$$\int_{|x(\sigma+i\tau)|\leq 1}\left|\left(x(\sigma+i\tau)\right)^{-(\alpha-\beta)} H_{\alpha,\beta}^{(1)}\left(x(\sigma+i\tau)\right)\phi(\sigma+i\tau)\,(\sigma+i\tau)^{4\alpha}\right| d\sigma \qquad (2.8)$$

$$\leq C\,|x|^{-2(\alpha-\beta)} \int_{-\infty}^{\infty} e^{-(R(x))\tau+|I(x)||\sigma|}\,|\phi\,(\sigma+i\tau)(\sigma+i\tau)|\,d\sigma$$

$$\leq C\,e^{M^X\left(\frac{|I(x)|}{a\left(1-\frac{1}{\ell}\right)}\right)-\Omega^X\left(\frac{R(x)}{b(1+1/m)}\right)}, \qquad |x|\geq 1$$

and

$R\,(x)\;>0,$ for $m,\ell\;\in\;\mathbb{N}-\{0\},\;\ell>1.$

Hence from (2.7) and (2.8) we conclude that

$$\left|h_{\alpha,\beta}\,\phi(x)\right|\;\leq C\,e^{M^X\left(\frac{1}{a}\left(1+\frac{1}{\ell-1}\right)|I(x)|\right)-\,\Omega^X\left(\frac{1}{b}\left[1-\frac{1}{m+1}\right]R(x)\right)} \tag{2.9}$$

for every $|x|\geq 1$ and $R(x)>0\,,m,\ell\;\in\;\mathbb{N}$, where $\ell>1$.

Since $h_{\alpha,\beta}\,\phi$ is even, the corresponding inequality (2.9) also holds when $R(x)\;<0$. Now let $|x|<1$. By using (5.3b) of [2] we deduce that

$$\left|h_{\alpha,\beta}\,\phi(x)\right|\;\leq C\int_{0}^{\infty} e^{t|I(x)|}\,|\phi(t)|\,t^{4\alpha}\,dt.$$

Proceeding as in the above case, we conclude that $h_{\alpha,\beta}\,\phi\;\;\in W\,e^{M^X,\;1/a}_{\Omega^X,\;1/b}$.

Thus proof is completed.

Now we can prove the following result easily.

**Lemma 2.6:** Let $1\leq p\leq\infty$. Then $W\,e^{\Omega,b}_{M,a}$ is contained in $W\,e^{p,\Omega,b}_{M,a}$.

From Lemma 2.5 and 2.6 we obtain

**Theorem 2.3:** Let $1\leq p\leq\infty$. Then $W\,e^{p,\Omega,b}_{M,a}\;=W\,e^{\Omega,b}_{M,a}$.

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