COMPARABILITY OF THE NEOCLASSICAL ECONOMIC GROWTH MODELS: THE RAMSEY MODELS IN PERSPECTIVE

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Abstract

The paper outlined two systems of differential equations: the Ramsey-Cass-Koopmans(RCK) growth model with exponential growth of labour (population) and Ramsey-Cass-Koopmans model with logistic growth rate of labour. The stability properties of these models are outlined in the neighbourhood of the steady state with a set of bench-mark parameters. We compare the performance ability of these models in the case of developed and developing countries.

Keywords: Per worker capital, Ramsey-Cass-Koopmans (RCK) model, Stability

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1 Introduction

Over the years, economists from different schools of thought have had series of exchanges on the various models of economic growth. In 1928 a sophisticated model of a society's optimal saving was published by Frank P. Ramsey[1]. Ramsey's contribution was mathematically involving and experienced no strong response at the time. After three decades his contribution was taken up seriously by Robert M. Solow in 1956 [2]. The model was unified with Solow's simpler growth model and became the foundation in neoclassical growth theory in 1960. Cass [3] and Koopmans [4] both provided extensions to the Ramsey model. They used discounting unlike in the Ramsey model and extended the Phelps Golden Rule formulation. They argued that, the objective function should maximize utility and not directly on consumption stream. They explained society is not indifferent to the timing of utility receipts, and attaches greater value to utility today than they will attach to utility some years from now. The model assumes the Malthusian population growth, thus called the Ramsey-Cass-Koopmans(RCK) model with exponential growth of labour. Brida and Accinelli [5] explored the unrealistic nature of the growth of labour (population) in the model. They described population growth to be logistic other than exponential, and illustrated it with a set of three differential equations.

These models form the basic models of macroeconomics for modeling growth rate of an economy. The current study investigates the strength of the models for developed and developing countries. We assess the effectiveness of these models with a set of benchmark parameters.

2 The Models

2.1 Exponential RCK model

The exponential RCK assumes that, there exist a large number of identical firms with production function $Y = F(K, AL)$, where Y is the total output, K is capital, L is labour and A is technology. Households grow at rate $n : L(t) = L(0)e^{nt}$. Technological progress grows at rate $g : A(t) = A(0)e^{gt}$. Capital Stock depreciates at a constant rate: $\delta \geq 0$. Output is either consumed or invested: $Y(t) = C(t) + I(t)$. The intensive form of the production function is given as $y = f(k)$.

Where $y = \dfrac{Y}{L}, k = \dfrac{K}{L}, f(0) = 0; f'(k) > 0, \forall k \in \Re^+; \lim_{k \to +\infty} f'(k) = \infty \; ; f''(k) < 0, \forall k \in \Re^+$.

The production function results in the following differential equation [6]:

$$k'(t) = f(k(t)) - c(t) - (\delta + n + g)k(t) \qquad (1)$$

The utility functional of the household is of the form:

$$U = \int_{t=0}^{\infty} e^{-\rho t} u(C(t)) \frac{L(t)}{H_d} dt \qquad (2)$$

Where $u(\bullet)$ is the utility function at any given date, $C(t)$ is household consumption, H_d is the number of households, $L(t)/H_d$ is the total number of members in the household and $u(C(t))L(t)/H_d$ is the household's total instantaneous utility at a given time t. Discount rate is denoted by ρ. The higher the value of ρ, then future consumption is less preferred to current consumption. When the utility functional is maximised subject to equation (1), the result is the capital stock equation [7] which is given as:

$$\frac{c'(t)}{c(t)} = \left\{ \frac{f'(k(t)) - \delta - \rho - \theta g}{\theta} \right\} \qquad (3)$$

Where θ is the coefficient of relative risk aversion. This determines a consumer's willingness to shift consumption between different periods. Equations (1) and (3) form the system of differential equations of the RCK model with exponential population growth law.
We equate the differential equations to zero to find the equilibrium values.

$$c'(t) = 0 \Rightarrow \left\{ \frac{f'(k(t)) - \delta - \rho - \theta g}{\theta} \right\} c(t) = 0$$

$$k'(t) = 0 \Rightarrow f(k(t)) - c(t) - (\delta + n + g)k(t) = 0$$

The steady state values are given as:

$$c_e = Ak_e^{\alpha} - (\delta + n + g)k_e \text{ and } k_e = \left\{ \frac{A\alpha}{\delta + \rho + \theta g} \right\}^{\frac{1}{1-\alpha}}$$

The Jacobian matrix is given by:

$$J(c,k) = \begin{bmatrix} \frac{f'(k(t)) - \delta - \rho - \theta g}{\theta} & \frac{f''(k)c}{\theta} \\ -1 & f'(k) - (\delta + n + g) \end{bmatrix}$$

The eigenvalues are given as by the values of λ that solve the following quadratic form $(A - \lambda I) = 0$

$$\det \begin{bmatrix} -\lambda & \frac{f''(k)c}{\theta} \\ -1 & (\rho - n - (1-\theta)g) - \lambda \end{bmatrix} = 0$$

The roots of the equations are given as:

$$\lambda_{1,2} = \frac{1}{2}\left\{\rho - n - (1-\theta)g \pm \sqrt{(\rho - n - (1-\theta)g)^2 - 4\frac{f''(k_e)c_e}{\theta}}\right\} \qquad (4)$$

2.2 Logistic RCK model

This RCK model assumes that, labour force follows a logistic law [8]. This is given as:

$$L(t) = \frac{aL(0)e^{at}}{a + bL(0)(e^{at} - 1)} \qquad (5)$$

where a is the logistic population growth rate, $L'(t) > 0 \forall t \geq 0$ and $\lim_{t\to\infty} L(t) = a/b$, which is the environmental carrying capacity.

The following equation is obtained [5]:

$$c'(t) = \left\{\frac{f'(k(t)) - (\delta + a - bL(t) + g + \rho)}{\theta}\right\}c(t) \qquad (6)$$

when the utility functional is maximised subject to the modified form of Equation (1) and the logistic growth law(given in equation (7) below).

$$k'(t) = f(k(t)) - c(t) - \{\delta + a - b(t) + g\}k(t) \qquad (7)$$

$$L'(t) = aL(t) - bL(t)^2$$

The differential equations that forms the logistics RCK model are given by equations (6) and (7). The values at the equilibrium are:

$$c_e = f(k_e) - \{\delta + g\}k_e,\ k_e = \left\{\frac{A\alpha}{\delta + \rho + g}\right\}^{\frac{1}{1-\alpha}} \text{ and } L_e = \frac{a}{b},$$

The Jacobian matrix is given by:

$$J(c,k,L) = \begin{bmatrix} \frac{f'(k(t)) - \{\delta + a - bL(t) + g + \rho\}}{\theta} & \frac{f''(k(t))}{\theta}c(t) & \frac{b}{\theta}c(t) \\ -1 & f'(k(t)) - \{\delta + a - bL(t) + g\} & bk(t) \\ 0 & 0 & a - 2bL(t) \end{bmatrix}$$

The eigenvalues are given by the values of λ that solve the following quadratic form $(A - \lambda I) = 0$

$$\det\begin{bmatrix} \lambda & \frac{f''(k_e)}{\theta}c_e & \frac{bc_e}{\theta} \\ -1 & \rho-\lambda & bk_e \\ 0 & 0 & -\alpha-\lambda \end{bmatrix} = 0$$

The Characteristic roots are given as:

$$\lambda_{1,2} = \frac{1}{2}\left\{\rho \pm \sqrt{\rho^2 - 4\frac{f''(k_e)c_e}{\theta}}\right\} \qquad (8)$$

$$\lambda_3 = -a \qquad (9)$$

3 Analysis and Discussion

A set of benchmark parameters are chosen for the comparative analysis. The share of capital in the production process α is assumed to be between 0.4 and 0.6, range of depreciation of capital is 0.02 and 0.10, the coefficient of relative risk aversion θ is also between 0.5 and 2 and discount rate $\rho = 0.05$ [9]. We set the population growth of the developed countries 0.0056 and that of the developing countries to 0.022 which is the average growth of data for some chosen countries, World Bank Population growth (annual %). We set growth rate of technological progress g to 0.02 [6].

Table 1: Parameter Values

Parameter	Description	Developed	Developing
δ	Depreciation	0.08	0.02
n	Growth rate of labour	0.0056	0.022
ρ	discount rate	0.05	0.05
θ	coefficient of relative risk aversion	2	0.5
α	Elasticity of capital in production	0.6	0.4
A	Level of technology	1	1
g	Growth of technology	0.02	0.02

3.1 Comparability of the Exponential RCK model

The equilibrium values and the eigenvalues for the developed world are given as:

$$k_e = 23.4022 \qquad c_e = 4.1593$$

$$\lambda_1 = -0.0519 \quad \text{and} \quad \lambda_2 = 0.1163$$

The corresponding eigenvectors are given by

$$\begin{bmatrix} -0.1156 \\ -0.9933 \end{bmatrix} \text{and} \begin{bmatrix} 0.0519 \\ -0.9987 \end{bmatrix}$$

respectively, so that the general solution is given by

$$\begin{bmatrix} p \\ q \end{bmatrix} = \gamma \begin{bmatrix} -0.1156 \\ -0.9933 \end{bmatrix} e^{-0.0519t} + \phi \begin{bmatrix} 0.0519 \\ -0.9987 \end{bmatrix} e^{0.1163t} \qquad (10)$$

The equilibrium values and the eigenvalues for the developing countries are given as:

$$k_e = 14.6201 \qquad c_e = 2.0176$$

$$\lambda_1 = -0.1065 \text{ and } \lambda_2 = 0.1245$$

The corresponding eigenvectors are given by

$$\begin{bmatrix} -0.1235 \\ -0.9923 \end{bmatrix} \text{and} \begin{bmatrix} 0.1059 \\ -0.9944 \end{bmatrix}$$

respectively, so that the general solution is given by

$$\begin{bmatrix} p \\ q \end{bmatrix} = \gamma_* \begin{bmatrix} -0.1`235 \\ -0.9923 \end{bmatrix} e^{-0.1065t} + \phi_* \begin{bmatrix} 0.0519 \\ -0.9987 \end{bmatrix} e^{0.1245t} \qquad (11)$$

where, γ, ϕ, γ_* , and ϕ_* are some constants. Hence the behaviour of the original non linear system is a saddle. The per worker level of consumption and the per worker level of capital at the equilibrium state for the developed world are much higher than that of the developing world. As we assumed that output follow the Cobb-Douglas production, the above data shows that output of the developed countries are greater than that of the developing world. The eigenvalues and the eigenvectors are not much different from each other for these economies but there are distinct set of values for the equilibrium values of per worker level of consumption and per worker level of capital.

3.2 Comparability of the Logistic RCK model

At the steady state total labour force equals the economy carrying capacity. A total labour force of 100 is chosen as the carrying capacity for the sake of our calculation. Values in the open neighbourhood of this does not have any serious appal on our calculation, [10].

The equilibrium points for the logistic model of the developed world is given as $(k_e, c_e, L_e) = (32.000, 4.800, 100.00)$.

$\lambda_1 = -0.0466,\ \lambda_2 = 0.0966$ and $\lambda_3 = -0.0056$

The matrix representing this linear system has one positive eigenvalue and two negative eigenvalues. Then, there is a straight line of solutions that tends to move away from the equilibrium and a plane of solutions that turns toward the equilibrium as time increases. All other solutions will eventually move away from the equilibrium. whiles the corresponding eigenvectors yielded

$$\begin{bmatrix} -0.0961 \\ -0.9954 \\ 0.0000 \end{bmatrix}, \begin{bmatrix} -0.0465 \\ -0.9989 \\ 0.0000 \end{bmatrix} \text{ and } \begin{bmatrix} 0.0000 \\ 0.0000 \\ 0.9994 \end{bmatrix}$$

respectively, so that the general solution is given by

$$\begin{bmatrix} p \\ q \\ h \end{bmatrix} = \gamma \begin{bmatrix} -0.0961 \\ -0.9954 \\ 0.0000 \end{bmatrix} e^{-0.0466t} + \phi \begin{bmatrix} -0.0465 \\ -0.9989 \\ 0.0000 \end{bmatrix} e^{0.0966t} + \varphi \begin{bmatrix} 0.0000 \\ 0.0000 \\ 0.9994 \end{bmatrix} e^{-0.0056t} \quad (12)$$

The steady state values for the logistic model from the data of the developing countries is given as $(k_e, c_e, L_e) = (12.0142, 2.226, 100.00)$.

$$\lambda_1 = -0.1185,\ \lambda_2 = 0.1685 \text{ and } \lambda_3 = -0.0220$$

The corresponding eigenvectors yielded:

$$\begin{bmatrix} -0.1662 \\ -0.9861 \\ 0.0000 \end{bmatrix}, \begin{bmatrix} 0.1177 \\ -0.9930 \\ 0.0000 \end{bmatrix} \text{ and } \begin{bmatrix} 0.0000 \\ 0.0000 \\ 0.9984 \end{bmatrix}$$

respectively, so that the general solution is given by

$$\begin{bmatrix} p \\ q \\ h \end{bmatrix} = \gamma^* \begin{bmatrix} -0.1662 \\ -0.9861 \\ 0.0000 \end{bmatrix} e^{-0.1185t} + \phi^* \begin{bmatrix} 0.1177 \\ -0.9930 \\ 0.0000 \end{bmatrix} e^{0.1685t} + \varphi^* \begin{bmatrix} 0.0000 \\ 0.0000 \\ 0.9984 \end{bmatrix} e^{0.0220t} \quad (13)$$

where, $\gamma, \phi, \varphi, \gamma^*, \phi^*$, and φ^* are some constants. This gives the local dynamics of system near the equilibrium k_e, c_e, L_e :ie the model exhibits saddle-path stability. Total output and the per worker levels of capital stock and consumption for the developed countries were much higher than that of the developing countries. Comparing these economic variables, the values of the exponential model shows an appreciable increment for both developed and developing countries. The steady state values of capital per worker of the developed world increased from 23.4022 to 32.000. The steady state values of the logistic RCK are greater than that of the exponential RCK.

4 Conclusion

In this study, we compared the performance ability of the exponential RCK and the logistic RCK model with a set of parameters for the developing and developed countries. The values of per capita output, per capita capital and per capita consumption at the steady state for the logistic RCK exceed that of the exponential RCK. For both models, the per worker levels for the developed countries are greater than that of the developing countries. But the gap between these values for the RCK model with logistic growth exceed that of the model with the exponential growth of labour. Given these benchmark parameters, the equilibrium values for consumption per capita for the developing countries increased from 2.0176 to 2.2226. However, when the axiom of exponential growth of labour is substituted with the logistic growth, capital per worker decreased from 14.6201 to 12.0142. There is a fall in output since output per worker is a function of capital stock per worker, $y = Ak^{\alpha}$. These results indicates that, the logistic population growth rate is not a characteristic of the developing countries but of the developed countries. Though developed countries can be modeled with RCK with exponential growth, economic output will be higher for logistic RCK model and vice versa for developing countries.

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