ON THE NON-HOMOGENEOUS BINARY QUADRATIC EQUATION

$$x^2 - 3xy + y^2 + 2x = 0$$

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ABSTRACT

The Binary quadratic equation given by $x^2 - 3xy + y^2 + 2x = 0$ is analyzed for its patterns of non – zero distinct integral solutions. A few interesting relations among the solutions are exhibited.

KEY WORDS

Binary Quadratic equation, Integral solutions

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INTRODUCTION

Binary quadratic diophantine equation offers an unlimited field for research because of their variety [1-4]. In the context one may refer [5-23] .This communication concerns with yet another interesting binary quadratic equation $x^2-3xy+y^2+2x=0$ for determining its infinitely many non zero integral solutions.Also a few interesting relations among the solutions have been presented.

METHOD OF ANALYSIS

The diophantine equation representing the binary quadratic equation under consideration is

$$x^2-3xy+y^2+2x=0 \tag{1}$$

Different patterns of solutions of (1) are Presented below.

Pattern-1

The substitution of the linear transformations

$$x=\frac{2}{5}[3\alpha+5T+2],\quad y=\frac{2}{5}[2\alpha+3] \tag{2}$$

in (1) leads to $\alpha^2=5T^2+1$ (3)

whose general solution (α_n, T_n) is

$$\alpha_n=\frac{1}{2}[(9+4\sqrt{5})^{n+1}+(9-4\sqrt{5})^{n+1}]$$

$$T_n=\frac{1}{2\sqrt{5}}[(9+4\sqrt{5})^{n+1}-(9-4\sqrt{5})^{n+1}]$$

In view of (2), the corresponding non-zero integral solutions of (1) are given by

$$x_n=\frac{1}{5}[3f_n+\sqrt{5}g_n+4]$$

$$y_n=\frac{1}{5}[2f_n+6]$$

where

$$f_n=(9+4\sqrt{5})^{2n+2}+(9-4\sqrt{5})^{2n+2}$$

$$g_n=(9+4\sqrt{5})^{2n+2}-(9-4\sqrt{5})^{2n+2}$$

A few interesting properties observed are as follows:

1. For all values of n, x_n and y_n are even

2. $5[x_{n+2} - 322x_{n+1} + x_n] + 1280 = 0$

3. $5[y_{n+2} - 322y_{n+1} + y_n] + 1980 = 0$

4. $5[377x_n - x_{n+1}] \equiv 0 (\bmod 32)$

5. $1885x_n - 5x_{n+1} - 144\{1610y_{n+1} - 5y_{n+2} - 1926\} \equiv 0 (\bmod 1504)$

6. $1610y_{n+1} - 5y_{n+2} \equiv 0 (\bmod 2)$

7. $144x_n - y_{n+1} \equiv 4 (\bmod 22)$

8. $10x_{n+1} - 3770x_n + 1440y_n + 1280 = 0$

9. $720x_n - 275y_n - 240 = 5y_{n+1}$

10. Each of the following represents a nasty number
    - $15\{377x_n - x_{n+1}\} - 2784$
    - $4830y_{n+1} - 15y_{n+2} - 5766$
    - $33\{144x_n - y_{n+1} - 70\}$

11. Each of the following represents is a cubical integer.
    - $4\{5y_{3n+2} + 15y_n - 24\}$
    - $20y_{3n+2} + 19320y_{n+1} - 60y_{n+2} - 23136$

Pattern-2

Treating (1) as Quadratic in x,we have

$$x = \frac{1}{2}[3y \pm \sqrt{5y^2 - 12y + 4}] \tag{4}$$

Let $\alpha^2 = 5y^2 - 12y + 4$ (5)

which can be written as $Y^2 = 5\alpha^2 + 16$ (6)

where $Y = (5y - 6)^2$

and whose general solution (Y_n, α_n) is

$$Y_n = 3f_n + \sqrt{5}g_n$$
$$T_n = f_n + \frac{3}{5}\sqrt{5}g_n \qquad ,n=0,1,2,\ldots$$

in which

$$f_n = (9+4\sqrt{5})^{2n+1} + (9-4\sqrt{5})^{2n+1}$$
$$g_n = (9+4\sqrt{5})^{2n+1} - (9-4\sqrt{5})^{2n+1}$$

In view of (6) and (4), the corresponding non-zero integral solutions of (1) are given by

$$x_n = \frac{9}{10}f_n + \frac{3}{2\sqrt{5}}g_n \pm [\frac{f_n}{2} + \frac{3}{2\sqrt{5}}g_n] + \frac{4}{5}$$
$$y_n = \frac{1}{5}[3f_n + \sqrt{5}g_n + 6]$$

A few interesting properties observed are as follows

1. $5[x_{n+2} - 644x_{n+1} + 4x_n] + 2556 = 0$
2. $5y_{n+2} - 3220y_{n+1} + 20y_n + 3834 = 0$
3. $5[3y_n - x_n] \equiv 14(\mathrm{mod}\, 2f_n)$
4. $5x_{n+1} - 1440y_n + 550x_n + 1284 = 0$
5. $5y_{n+1} + 1440x_n + 3366 = 3770y_n$
6. $4\{15y_{3n+1} - 5x_{3n+1} + 45y_n - 15x_n - 56\}$ is a cubical integer.

CONCLUSION:

To conclude, one may search for other patterns of solutions and their corresponding properties.

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