AN ALGORITHM FOR SOLVING A FRACTIONAL TRANSPORTATION PROBLEM WITH SPECIFIED FLOW

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Abstract:

The present paper aimed at studying a fractional transportation problem with specified flow. If in addition to the flow constraint, the minimum requirement of each destination is also specified then the situation arises of distributing at minimum cost a certain commodity produced in a country, after keeping reserve stocks, to various states with minimum requirement of each state specified. A related transportation problem is formed in which the flow constraint is replaced by two extra destinations, one for supplementing the total flow up to the specified level, and the other for identifying the supply points preferred to keep reserves. It is shown that optimal basic feasible solution of the related transportation problem so formulated gives an optimal solution of the given problem. The algorithm is supported by a real life example of a manufacturing company.

Keywords: *fractional transportation problem, specified flow, related problem.*

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1 Introduction:

Transportation problems with fractional objective function are widely used as performance measures in many real life situations such as in the analysis of financial aspects of transportation enterprises and undertaking, and in transportation management situations, where an individual , or a group of people is confronted with the hurdle of maintaining good ratios between some important and crucial parameters concerned with the transportation of commodities from certain sources to various destinations. Fractional objective function include optimization of ratio of total actual transportation cost to total standard transportation cost, total return to total investment, ratio of risk assets to capital, total tax to total public expenditure on commodity , amount of raw material wasted to amount of raw material used, stock cutting problem, cargo loading problem etc. Gupta et al. [3] discussed a paradox in linear fractional transportation problems with mixed constraints and established a sufficient condition for the existence of a paradox. Jain and Saksena [4] studied time minimizing transportation problem with fractional bottleneck objective function which is solved by a lexicographic primal code. Dinkelbach [1] studied non- linear fractional programming in 1967. Xie et al. [7] developed a technique for duration and cost optimization for transportation problem.

In any transportation problem, the total quantity supplied by the various supply points and consequently received by the various destinations is the total flow in the system . This flow is different for different combinations of supply point and destination constraints. If in addition to the flow constraint , the minimum requirement of each destination is also specified then the situation arises of distributing at minimum cost a certain commodity produced in a country, after keeping reserve stocks, to various states with minimum requirement of each state specified. Khanna and Puri [5] in the year 1983 solved a transportation problem with mixed constraints and specified transportation flow. Khurana et .al. [6]in the year 2006 studied linear plus linear fractional transportation problem for restricted and enhanced flow. Gupta and Arora [2] developed an algorithm to find optimum time – cost tradeoff pairs in a fractional capacitated transportation problem with restricted flow.

2 Problem Formulation:

Consider a fractional transportation problem given by

$$(P1):\ \min\left[\frac{\sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij}}{\sum_{i\in I}\sum_{j\in J} d_{ij}x_{ij}}\right]$$

subject to

$$\sum_{j\in J} x_{ij} \le a_i;\ \forall i \in I$$

$$\sum_{i\in I} x_{ij} \ge r_j;\ \forall j \in J$$

$$\sum_{i\in I}\sum_{j\in J} x_{ij} = P$$

$$x_{ij} \ge 0;\ \forall\ (i,j) \in IxJ$$

where $\sum_{j\in J} r_j < P < \sum_{i\in I} a_i$, r_j being the minimum requirement specified for the j^{th} destination.

I = {1, 2, … m} is the index set of m origins.

J = {1, 2, …, n} is the index set of n destinations.

x_{ij} = number of units transported from origin i to the destination j.

c_{ij} = Total salestax paid per unit of a commodity when transported from i^{th} origin to the j^{th} destination.

d_{ij} = Total public expenditure per unit of a commoditywhen transported from i^{th} origin to the j^{th} destination.

The nature of the problem (P1) suggests that a total of $\left(P - \sum_{j\in J} r_j\right)$ is to be distributed over the various destinations, after meeting their minimum requirements. Also the flow constraint implies that a total $\left(\sum_{i\in I} a_i - P\right)$ of supply point reserves is to be retained at the various supply points. So one extra destination is introduced which receives a total of $\left(P - \sum_{j\in J} r_j\right)$ to be distributed over the various destinations. One more destination is introduced to receive all the supply point reserves. So the following related transportation problem (P2) is constructed in order to solve the problem (P1).

$$\text{(P2): } \min z = \left[\frac{\sum_{i\in I'}\sum_{j\in J'} c'_{ij} y_{ij}}{\sum_{i\in I'}\sum_{j\in J'} d'_{ij} y_{ij}} \right]$$

subject to

$$\sum_{j\in J''} y_{ij} = a_i, i \in I$$

$$\sum_{i\in I} y_{ij} = b'_j, j \in J'' = J \cup \{n+1, n+2\}$$

$$y_{ij} \geq 0; (i, j) \in IxJ''$$

where $b'_j = r_j, j \in J$

$$b'_{n+1} = P - \sum_{j\in J} r_j$$

$$b'_{n+2} - \sum_{i\in I} a_i - P$$

$$c'_{ij} = c_{ij}; \forall (i, j) \in IxJ$$

$$d'_{ij} = d_{ij}; \forall (i, j) \in IxJ$$

$$c'_{i,n+1} = c_{ip} \text{ and } d'_{i,n+1} = d_{ip} \text{ such that } c_{ip} / d_{ip} = \min_{j\in J}\left(c_{ij} / d_{ij}\right)$$

$$c'_{i,n+2} = 0 = d'_{i,n+2}; \forall i \in I$$

Transformation

Let $\left\{\overset{\circ}{y}_{ij}\right\}$ be an optimal solution of the problem (P2). Then an optimal solution $\left\{\overset{\circ}{x}_{ij}\right\}$ of the problem (P1) is given by

$$\overset{\circ}{x}_{ij} = \overset{\circ}{y}_{ij} \quad ; \forall i \in I, j \in J, j \neq p \text{ and}$$

$$\overset{\circ}{x}_{ip} = \overset{\circ}{y}_{ip} + \overset{\circ}{y}_{i,n+1} \text{ for a single p such that}$$

$$c_{ip} = c'_{i,n+1} \text{ and } d_{ip} = d'_{i,n+1} \text{ such that } c_{ip}/d_{ip} = \min_{j\in J}(c_{ij}/d_{ij})$$

when p is not uniquely determined , different choices of p will give alternate optimal solution of (P1).

3 Theoretical Development

Lemma 1: There is one to one correspondence between the basic feasible solution of problem (P1) and basic feasible solution of problem (P2).

Proof: Let $\{y_{ij}\}$ be a basic feasible solution of problem (P2)

$$\left.\begin{array}{l} \text{Define } x_{ij} = y_{ij}; \forall i \in I, j \in J, j \neq p \\ \text{and } x_{ip} = y_{ip} + y_{i,n+1} \text{ for a single p such that} \\ c_{ip} = c'_{i,n+1} \text{ and } d_{ip} = d'_{i,n+1} \text{ such that } c_{ip}/d_{ip} = \min_{j\in J}(c_{ij}/d_{ij}) \end{array}\right\} \quad (1)$$

It is clear that

$$x_{ij} \geq 0; \forall i \in I, j \in J$$

Also for $i \in I$,

$$\sum_{j\in J''} y_{ij} = a_i$$

$$\Rightarrow \sum_{j\in J} y_{ij} + y_{i,n+1} \leq a_i$$

Since $y_{i,n+2} \geq 0$ and as $b'_{n+2} > 0$ so $y_{i,n+2} > 0$ for atleast one $i \in I$.

Hence $\sum_{j\in J, j\neq p} y_{ij} + y_{ip} + y_{i,n+1} \leq a_i$

Therefore by relation (1) , $\sum_{j\in J, j\neq p} x_{ij} + x_{ip} \leq a_i$

Hence $\sum_{j\in J} x_{ij} \leq a_i; \forall i \in I$

For $j \in J, \sum_{i\in I} y_{ij} = b'_j = r_j$

So, $\sum_{i\in I} y_{ij} + \sum_{i\in L} y_{i,n+1} = r_j + \sum_{i\in L} y_{i,n+1}$

where $L = \left\{ i \middle| c_{ip} = c'_{i,n+1} \text{ and } d_{ip} = d'_{i,n+1} \text{ such that } c_{ip}/d_{ip} = \min_{j\in J}(c_{ij}/d_{ij}) \right\}$

$$\sum_{i\in I-L} y_{ij} + \sum_{i\in L} y_{ij} + \sum_{i\in L} y_{i,n+1} = r_j + \sum_{i\in L} y_{i,n+1}$$

$\therefore \sum_{i\in I-L} x_{ij} + \sum_{i\in L} x_{ij} \geq r_j$ by relation (1)

Hence $\sum_{i\in I} x_{ij} \geq r_j$

For j= n+2,

$$\sum_{i\in I} y_{i,n+2} = b'_{n+2} = \sum_{i\in I} a_i - P$$

$$\sum_{i\in I} y_{i,n+2} = \sum_{i\in I}\left(\sum_{j\in J''} y_{ij}\right) - P$$

$$= \sum_{i\in I}\sum_{j\in J} y_{ij} + \sum_{i\in I} y_{i,n+1} + \sum_{i\in I} y_{i,n+2} - P$$

Thus $\sum_{i\in I}\sum_{j\in J} y_{ij} + \sum_{i\in I} y_{i,n+1} = P$

Hence $\sum_{i\in I}\sum_{j\in J-K} y_{ij} + \sum_{i\in I}\sum_{j\in K} y_{ij} + \sum_{i\in I} y_{i,n+1} = P$

where $K = \left\{ j \middle| c_{ip} = c'_{i,n+1} \text{ and } d_{ip} = d'_{i,n+1} \text{ such that } c_{ip}/d_{ip} = \min_{j\in J}(c_{ij}/d_{ij}) \right\}$

So, $\sum_{i\in I}\sum_{j\in J-K} y_{ij} + \sum_{i\in I}\sum_{j\in K}\left(y_{ij} + y_{i,n+1}\right) = P$

Therefore by relation (1) , $\sum_{i\in I}\sum_{j\in J-K} x_{ij} + \sum_{i\in I}\sum_{j\in J} x_{ij} = P$

Hence $\sum_{i\in I}\sum_{j\in J} x_{ij} = P$

The above results shows that $\{x_{ij}\}$ is a feasible solution of problem (P1).

Let $\sum_{j\in J} x_{ij} = \overline{a_i} \le a_i; \forall i \in I$

$$\sum_{j\in J} x_{ij} = \overline{b_j} \ge r_j; \forall j \in J$$

Thus $\sum_{i\in I}\sum_{j\in J} x_{ij} = P = \sum_{i\in I} \overline{a_i} = \sum_{j\in J} \overline{b_j}$

Define $y_{i,n+2} = a_i - \overline{a_i} \ge 0$

The following transportation problem is constructed

$$(P3): \min\left\{ \frac{\sum_{i\in I}\sum_{j\in J'} c'_{ij} w_{ij}}{\sum_{i\in I}\sum_{j\in J'} d'_{ij} w_{ij}} \right\}$$

subject to

$$\sum_{j\in J'} w_{ij} = \overline{a}_i, i \in I$$

$$\sum_{i\in I} w_{ij} = b'_j, j \in J' = J \cup \{n+1\}$$

$$w_{ij} \geq 0; (i,j) \in IxJ'$$

$$\text{where } b'_j = r_j, j \in J$$

$$b'_{n+1} = P - \sum_{j\in J} r_j$$

Let $\left\{\overset{\circ}{w}_{ij}\right\}$ be an optimal solution of this problem (P3)

Then $x_{ij} = \overset{\circ}{w}_{ij}; \forall i \in I, j \in J, j \neq p$

and $x_{ip} = \overset{\circ}{w}_{ip} + \overset{\circ}{w}_{i,n+1}$ for a single p such that

$c_{ip} = c'_{i,n+1}$ and $d_{ip} = d'_{i,n+1}$ such that $c_{ip}/d_{ip} = \min_{j\in J}(c_{ij}/d_{ij})$

$$\left.\begin{array}{l}\text{Define } y_{ij} = \overset{\circ}{w}_{ij}, i \in I, j \in J' \\ \text{Also } y_{i,n+2} = a_i - \overline{a}_i, \forall\ i \in I\end{array}\right\} \quad (2)$$

Clearly, $y_{ij} \geq 0; \forall i \in I, j \in J''$

Also for $i \in I$,

$$\sum_{j\in J''} y_{ij} = \sum_{j\in J'} y_{ij} + y_{i,n+2}$$

$$\Rightarrow \sum_{j\in J'} \overset{\circ}{w}_{ij} + a_i - \overline{a}_i \quad \text{using relation (2)}$$

For j= n+2,

$$\sum_{i\in I} y_{i,n+2} = \sum_{i\in I}\left(a_i - \overline{a}_i\right) = \sum_{i\in I} a_i - P$$

$$\text{So, } \sum_{i\in I} y_{i,n+2} = b'_{n+2}$$

$$\text{Thus } \sum_{i\in I} y_{ij} = b'_j, \forall j \in J''$$

Thus it is established that $\{y_{ij}\}$ is a feasible solution of the problem (P2).

Lemma 2 : The value of the objective function of the problem (P1) at its feasible solution is equal to value of the objective function of the problem (P2) at its corresponding feasible solution.

Proof: Objective function value of the problem (P1) at its feasible solution $\{x_{ij}\}$ is

$$= \left\{ \frac{\sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij}}{\sum_{i\in I}\sum_{j\in J} d_{ij}x_{ij}} \right\}$$

$$\Rightarrow \left\{ \frac{\left(\sum_{i\in I}\sum_{\substack{j\in J\\ j\neq p}} c_{ij}x_{ij}\right) + \left(\sum_{i\in I} c_{ip}x_{ip}\right)}{\left(\sum_{i\in I}\sum_{\substack{j\in J\\ j\neq p}} d_{ij}x_{ij}\right) + \left(\sum_{i\in I} d_{ip}x_{ip}\right)} \right\}$$

$$\Rightarrow \left\{ \frac{\left(\sum_{i\in I}\sum_{\substack{j\in J\\ j\neq p}} c_{ij}\mathring{w}_{ij}\right) + \left(\sum_{i\in I} c_{ip}\left(\mathring{w}_{ip}+\mathring{w}_{i,n+1}\right)\right)}{\left(\sum_{i\in I}\sum_{\substack{j\in J\\ j\neq p}} d_{ij}\mathring{w}_{ij}\right) + \left(\sum_{i\in I} d_{ip}\left(\mathring{w}_{ip}+\mathring{w}_{i,n+1}\right)\right)} \right\}$$

$$\Rightarrow \left\{ \frac{\left(\sum_{i\in I}\sum_{\substack{j\in J\\ j\neq p}} c_{ij}y_{ij}\right) + \left(\sum_{i\in I} c_{ip}y_{ip} + \sum_{i\in I} c_{ip}y_{i,n+1}\right)}{\left(\sum_{i\in I}\sum_{\substack{j\in J\\ j\neq p}} d_{ij}y_{ij}\right) + \left(\sum_{i\in I} d_{ip}y_{ip} + \sum_{i\in I} d_{ip}y_{i,n+1}\right)} \right\} \text{using (2)}$$

$$\Rightarrow \left\{ \frac{\left(\sum_{i\in I}\sum_{j\in J} c_{ij}y_{ij}\right) + \left(\sum_{i\in I} c'_{i,n+1}y_{i,n+1}\right)}{\left(\sum_{i\in I}\sum_{j\in J} d_{ij}y_{ij}\right) + \left(\sum_{i\in I} d'_{i,n+1}y_{i,n+1}\right)} \right\} \text{by definition of } c_{ip} \text{ and } d_{ip}$$

$$\Rightarrow \left\{ \frac{\left(\sum_{i\in I}\sum_{j\in J''} c'_{ij}y_{ij}\right)}{\left(\sum_{i\in I}\sum_{j\in J''} d'_{ij}y_{ij}\right)} \right\}, \text{since} \begin{bmatrix} c_{ij}=c'_{ij}, \forall i\in I, j\in J \\ d_{ij}=d'_{ij}, \forall i\in I, j\in J \\ c'_{i,n+2}=0=d'_{i,n+2} \end{bmatrix}$$

= Objective function value of the problem (P2) at its corresponding feasible solution $\{y_{ij}\}$

Lemma 3 : An optimal solution of problem (P2) gives an optimal solution of problem (P1).

Proof : Let $\left\{\overset{\circ}{x}_{ij}\right\}$ be an optimal solution of the problem (P1) yielding value z^0 and $\left\{\overset{\circ}{y}_{ij}\right\}$ be the corresponding feasible solution of (P2) . Then objective function value yielded by $\left\{\overset{\circ}{y}_{ij}\right\}$ is also z^0 using lemma 2 . If possible, let $\left\{\overset{\circ}{y}_{ij}\right\}$ be not an optimal solution of problem (P2). So there exists a feasible solution $\left\{y'_{ij}\right\}$ of problem (P2) with value say $z^1 < z^0$. Let $\left\{x'_{ij}\right\}$ be the corresponding feasible solution of problem (P1). Then by lemma 2 $\left\{\frac{\left(\sum_{i\in I}\sum_{j\in J} c_{ij}x'_{ij}\right)}{\left(\sum_{i\in I}\sum_{j\in J} d_{ij}x'_{ij}\right)}\right\} = z^1$. Thus the assumption that z^0 is the optimal value of problem (P1) is contradicted because $z^1 < z^0$. Similarly an optimal solution of problem (P2) will give an optimal solution of problem (P1).

Theorem 1: Optimizing problem (P1) is exactly equivalent to optimizing problem (P2).

Proof: The proof follows from lemma 3

Theorem 2: A feasible solution $X^0 = \{\overset{\circ}{x}_{ij}\}_{I\times J}$ of problem (P2) with objective function value $\frac{N^\circ}{D^\circ}$ will be an optimum basic feasible solution iff the following conditions holds.

$$\delta^1_{ij} = \frac{\theta_{ij}\left[D^\circ(c_{ij} - z^1_{ij}) - N^\circ(d_{ij} - z^2_{ij})\right]}{D^\circ\left[D^\circ + \theta_{ij}(d_{ij} - z^2_{ij})\right]} \geq 0; \forall (i,j) \notin B$$

where $N^\circ = \sum_{i\in I}\sum_{j\in J} c_{ij}\overset{\circ}{x}_{ij}$, $D^\circ = \sum_{i\in I}\sum_{j\in J} d_{ij}\overset{\circ}{x}_{ij}$, B denotes the set of cells (i,j) which are basic.

$u^1_i, u^2_i, v^1_j, v^2_j; i \in I, j \in J$ are the dual variables such that $u^1_i + v^1_j = c_{ij}$, $\forall (i,j) \in B$; $u^2_i + v^2_j = d_{ij}$, $\forall (i,j) \in B$; $u^1_i + v^1_j = z^1_{ij}$, $\forall (i,j) \notin B$; $u^2_i + v^2_j = z^2_{ij}$, $\forall (i,j) \notin B$

Note: $u^1_i, v^1_j, u^2_i, v^2_j$ are the dual variables which are determined by using above equations and taking one of the u_i's or v_j's as zero.

Proof: Let $X^0 = \{\overset{\circ}{x}_{ij}\}_{I\times J}$ be a basic feasible solution of problem (P2) with equality constraints. Let z^0 be the corresponding value of objective function. Then

$$z^\circ = \left[\frac{\sum_{i\in I}\sum_{j\in J} c_{ij}x^\circ_{ij}}{\sum_{i\in I}\sum_{j\in J} d_{ij}x^\circ_{ij}}\right] = \frac{N^\circ}{D^\circ} \text{ (say)}$$

$$= \left[\frac{\sum_{i\in I}\sum_{j\in J}(c_{ij}-u^1_i-v^1_j)x^\circ_{ij} + \sum_{i\in I}\sum_{j\in J}(u^1_i+v^1_j)x^\circ_{ij}}{\sum_{i\in I}\sum_{j\in J}(d_{ij}-u^1_i-v^1_j)x^\circ_{ij} + \sum_{i\in I}\sum_{j\in J}(u^2_i+v^2_j)x^\circ_{ij}}\right]$$

$$= \left[\frac{\sum\sum_{(i,j)\notin B}(c_{ij}-u^1_i-v^1_j)x^\circ_{ij} + \sum_{i\in I}\sum_{j\in J}(u^1_i+v^1_j)x^\circ_{ij}}{\sum\sum_{(i,j)\notin B}(d_{ij}-u^2_i-v^2_j)x^\circ_{ij} + \sum_{i\in I}\sum_{j\in J}(u^2_i+v^2_j)x^\circ_{ij}}\right]$$

$$= \left[\frac{\sum\sum_{(i,j)\notin B}(c_{ij}-z^1_{ij})x^\circ_{ij} + \sum_{i\in I}a_iu^1_i + \sum_{j\in J}b_jv^1_j}{\sum\sum_{(i,j)\notin B}(d_{ij}-z^2_{ij})x^\circ_{ij} + \sum_{i\in I}a_iu^2_i + \sum_{j\in J}b_jv^2_j}\right]$$

Let some non basic variable $x_{ij} \notin B$ undergoes change by an amount of θ_{rs} θ_{rs}, then new value of the objective function $\hat{z}$ will be given by

$$\hat{z} = \frac{N^\circ + \theta_{rs}(c_{rs}-z^1_{rs})}{D^\circ + \theta_{rs}(d_{rs}-z^2_{rs})}$$

$$\hat{z} - z^\circ = \left[\frac{N^\circ + \theta_{rs}(c_{rs}-z^1_{rs})}{D^\circ + \theta_{rs}(d_{rs}-z^2_{rs})} - \frac{N^\circ}{D^\circ}\right]$$

$$= \frac{\theta_{rs}\left[D^\circ(c_{rs}-z^1_{rs}) - N^\circ(d_{rs}-z^2_{rs})\right]}{D^\circ\left[D^\circ + \theta_{rs}(d_{rs}-z^2_{rs})\right]} = \delta^1_{rs} \text{ (say)}$$

Hence X^0 will be local optimal solution iff $\delta^1_{ij} \geq 0; \forall (i,j) \notin B$ .If X^0 is a global optimal solution of (P2), then it is an optimal solution and hence the result follows.

4 Algorithm:

Step 1: Given a fractional transportation problem (P1) with specified flow, construct the related transportation problem (P2) by introducing two dummy destinations such that

$c'_{ij} = c_{ij}; \forall(i,j) \in IxJ$

$d'_{ij} = d_{ij}; \forall(i,j) \in IxJ$

$c'_{i,n+1} = c_{ip}$ and $d'_{i,n+1} = d_{ip}$ such that $c_{ip}/d_{ip} = \min_{j\in J}\left(c_{ij}/d_{ij}\right)$

$c'_{i,n+2} = 0 = d'_{i,n+2}; \forall i \in I$

$b'_j = r_j; \forall j \in J, b'_{n+1} = P - \sum_{j\in J} r_j, b'_{n+2} = \sum_{i\in I} a_i - P$

Step 2 : Find the initial basic feasible solution of problem (P2) . Let B be its corresponding basis.

Step 3: Calculate $\theta_{ij}, (c_{ij} - z^1_{ij})$, $(d_{ij} - z^2_{ij})$, N^0, D^0 for all non- basic cells such that

$u^1_i + v^1_j = c_{ij} \quad ; \quad \forall(i,j) \in B$

$u^2_i + v^2_j = d_{ij} \quad ; \quad \forall(i,j) \in B$

$u^1_i + v^1_j = z^1_{ij} \quad ; \quad \forall(i,j) \notin B$

$u^2_i + v^2_j = z^2_{ij} \quad ; \quad \forall(i,j) \notin B$

N^0 = value of $\sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij}$ at the current basic feasible solution corresponding to the basis B.

D^0 = value of $\sum_{i\in I}\sum_{j\in J} d_{ij}x_{ij}$ at the current basic feasible solution corresponding to the basis B.

θ_{ij} = level at which a non basic cell (i,j) enters the basis replacing some basic cell of B.

Note: $u^1_i, v^1_j, u^2_i, v^2_j$ are the dual variables which are determined by using above equations and taking one of the u_i ’s or v_j ’s as zero.

Step 4 : Find $\delta^1_{ij}; \forall(i,j) \notin B$ where $\delta^1_{ij} = \dfrac{\theta_{ij}\left[D^\circ(c_{ij} - z^1_{ij}) - N^\circ(d_{ij} - z^2_{ij})\right]}{D^\circ\left[D^\circ + \theta_{ij}(d_{ij} - z^2_{ij})\right]}$

If $\delta^1_{ij} \geq 0; \forall(i,j) \notin B$ then the current solution so obtained is the optimal solution to (P2).Go to step 5.Otherwise, some (i,j) $\notin B$ for which $\delta^1_{ij} < 0$ will undergo change. Go to step 3.

Step 5: $Z=\frac{N^0}{D^0}$ will be the optimal solution of (P2) yielded by the optimal solution $\left\{\overset{\circ}{y}_{ij}\right\}$.Then perform the following transformation to find the optimal solution $\left\{\overset{\circ}{x}_{ij}\right\}$ of problem (P1).

$\overset{\circ}{x}_{ij} = \overset{\circ}{y}_{ij}$; $\forall i \in I, j \in J, j \neq p$ and

$\overset{\circ}{x}_{ip} = \overset{\circ}{y}_{ip} + \overset{\circ}{y}_{i,n+1}$ for a single p such that

$c_{ip} = c'_{i,n+1}$ and $d_{ip} = d'_{i,n+1}$ such that $c_{ip}/d_{ip} = \min_{j \in J}(c_{ij}/d_{ij})$

The optimal solution to problem (P1) is $Z=\frac{N^0}{D^0}$ yielded by the optimal solution $\left\{\overset{\circ}{x}_{ij}\right\}$.

5Problem of the manager of a manufacturing company

A company produces integrated circuits which are used in LCD's. These circuits are manufactured in the factories (i) located at Haryana, Punjab and Chandigarh. After production, these circuits are transported to main distribution centres (j) at Kolkata, Chennai, Mumbai and Delhi. While transporting the goods, the company has to pay sales tax per circuit . The total sales tax paid per circuit from Haryana to Kolkata, Chennai, Mumbai and Delhi are □ 5 , 9 , 9 and 8 respectively. The tax figures when the goods are transported from Punjab to Kolkata, Chennai, Mumbai and Delhi are □ 4, 6, 2 and 5 respectively. The total tax paid per circuit from Chandigarh to Kolkata, Chennai, Mumbai and Delhi are □ 4,1,2 and 3 respectively. The total public expenditure per unit when the goods are transported from Haryana to Kolkata, Chennai , Mumbai and Delhi are □ 1,2 4 and 7 respectively while the figures forPunjab are □ 3,7,4 and 6. When the goods are transported from Chandigarh to distributioncentres at Kolkata, Chennai, Mumbai and Delhi ,the total public expenditure per unit is □ 2,9 5 and 2 respectively. Factory at Haryana Punjab and Chandigarh can produce a maximum of 10, 6 and 8 circuits respectively in a month. The minimum monthly requirement of each distribution centre at Kolkata, Chennai, Mumbai and Delhi is 2 ,3 4 and 6 circuits respectively. The company can produce a total of 20 circuits in a month. The manager after fulfilling the minimum requirement of 2,3 4 and 6 circuits at Kolkata, Chennai, Mumbai and Delhi distributes the remaining stock of 5 units tothese distribution centres. He wishes to determine the number of circuits to be distributed from each factory to different distribution centres in such a way that the ratio of total sales tax paid to the total public expenditure per circuit is minimum.

Solution: The problem of the manager can be formulated as a 3×4 fractional transportation problem (P1) with specified flow. Table 1 gives the values of c_{ij}, d_{ij}, a_i, r_j for i=1,2,3 and j=1,2,3,4

Table 1: Problem (P1)

	D_1	D_2	D_3	D_4	a_i
O_1	5 1	9 2	9 4	8 7	10
O_2	4 3	6 7	2 4	5 6	6
O_3	4 2	1 9	2 5	3 2	8
r_j	2	3	4	6	

Note: O_1 and O_2 and O_3 denotes factories at Haryana , Punjab and Chandigarh. D_1 , D_2 and D_3 and D_4 are the distribution centres at Kolkata, Chennai, Mumbai and Delhi respectively. Values in the upper left corners are c_{ij}'s which shows the total sales tax paid per circuit while transporting goods from factories (i) to distribution centres (j). Values in lower left corners are d_{ij}'s that shows the total public expenditure per circuit for i=1,2 and 3 and j=1,2,3,4. Values in the r_j row shows the minimum monthly requirement of each distribution centre (j). Values in the a_i column shows the maximum capacity of each factory (i).

Let x_{ij} be the number of circuits transported from the i^{th} factory to the j^{th} distribution centre.

Here, P = 20. Here $\sum_{j=1}^{4} r_j = 15 < P = 20 < \sum_{i=1}^{3} a_i = 24$

Introduce two dummy destinations in the above problem (P1) with

$$b_j^{'} = r_j; \forall j = 1,2,3,4; b_5^{'} = P - \sum_{j=1}^{4} r_j = 20 - 15 = 5; b_6^{'} = \sum_{i=1}^{3} a_i - P = 24 - 20 = 4$$

$$c_{ij}^{'} = c_{ij}; i = 1,2,3 \text{ and } j = 1,2,3,4$$

$$d_{ij}^{'} = d_{ij}; i = 1,2,3 \text{ and } j = 1,2,3,4$$

$$c_{i5}^{'} = c_{ip} \text{ and } d_{i5}^{'} = d_{ip} \text{ such that } c_{ip} / d_{ip} = \min_{j \in J} \left(c_{ij} / d_{ij} \right) \text{for } i = 1,2,3$$

$$c_{i6}^{'} = 0 = d_{i6}^{'}; \forall i = 1,2,3$$

In this way we form the related transportation problem (P2). An optimal solution of the problem (P2) is shown in the table 2 below.

Table 2: An optimal solution of problem (P2)

	D_1	D_2	D_3	D_4	D_5	D_6	a_i	u_i^1	u_i^2
O_1	5 **2** 1	9 2	9 4	8 7	8**4** 7	0 4 0	10	0	0
O_2	4 3	6 7	2 **4** 4	5**2** 6	2 4	0 0	6	-3	-1
O_3	4 2	1 **3** 9	2 5	3 **6** 2	1 **5** 9	0 **0** 0	8	0	0
b'_j	2	3	4	6	5	4			
v_j^1	5	1	5	8	1	0			
v_j^2	1	9	5	7	9	0			

Note: Entries in bold are basic cells.

Here $N^0 = 68$, $D^0 = 130$.

Table 3:Calculation of optimality condition

NB	O_1D_2	O_1D_3	O_1D_5	O_2D_1	O_2D_2	O_2D_5	O_2D_6	O_3D_1	O_3D_3	O_3D_4
θ_{ij}	3	4	4	2	2	2	2	0	0	0
$(c_{ij} - z_{ij}^1)$	8	4	7	2	8	4	3	-1	-3	-5
$(d_{ij} - z_{ij}^2)$	-7	-1	-2	3	-1	-4	1	1	0	-5
δ_{ij}^1	0.0316	0.335	0.2638	0.006	0.13317	0.099	0.0375	0	0	0

Since $\delta_{ij}^1 \geq 0; \forall (i, j) \notin B$, the solution in table 2 is an optimal solution of (P2) with Z = 68/130=0.5230 The required optimal solution of problem (P1) after transformation is given in table 4 below.

Table 4:Optimal solution of problem (P1)

	D_1	D_2	D_3	D_4	a_i
O_1	5 **2** 1	9 2	9 4	8**4** 7	10
O_2	4 3	6 7	2 **4** 4	5**2** 6	6
O_3	4 2	1 **8** 9	2 5	3 2	8
r_j	2	3	4	6	

Therefore the company should transport 2 circuits from Haryana to Kolkata and 4 circuits to Delhi. The factory at Punjab should transport 4 circuits to Mumbai and 2 circuits to distribution centre at Delhi. The factory at Chandigarh should transport 8 units to distribution centre at Chennai. In this way, the manager can distribute all its 20 circuits to its distribution centres satisfying the minimum requirement of each distribution centre with a minimum sales tax/total public expenditure ratio Z =68/130=0.5230 .

Conclusion: In order to solve a transportation problem with specified flow, a related transportation problem is formed. It is shown that optimal solution to the specified problem may be obtained from the optimal solution of the related transportation problem. Since optimal solution of the related transportation problem is attainable at an extreme point , therefore optimal basic feasible solution of the related problem will give an optimal solution of the given problem

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