# ON TERNARY BI-QUADRATIC EQUATION

$$2x^2 - 3xy + 2y^2 = 23z^4$$

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**Abstract:**

We obtain non-trivial integral solutions for the Ternary Bi-quadratic Equation $2x^2 - 3xy + 2y^2 = 23z^4$ . A few interesting relations for each pattern among the solutions are presented.

**Keywords:** Ternary Bi-quadratic, Integral solutions.

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## INTRODUCTION:

Diophantine equations have an unlimited field for research by reason of their variety. In particular, the Bi-quadratic Diophantine equations, Homogenous and Non Homogenous have aroused the interest of numerous mathematicians since antiquity [1-5]. However, often we come across homogenous bi-quadratic equations and as such one may require its integral solutions in its most general form. In this context one may refer [6-13] for problem on ternary bi-quadratic equations. This paper concerns with the problem of determining non-trivial integral solutions of the bi-quadratic equation with three unknowns given by $2x^2 - 3xy + 2y^2 = 23z^4$. Explicit integral solutions of the above equations are presented. A few interesting relations among the solutions are obtained.

**Notations:**

$P_n^m$ - Pyramidal number of rank n with size m.

$T_{m,n}$ - Polygonal number of rank n with size m.

$CP_{m,n}$ - Centered Polygonal number of rank n with size m.

$F_{4,n,6}$ - Four Dimensional figurate number of rank n with size 6.

$Pt_n$ - Pentalope number of rank n.

## METHOD OF ANALYSIS:

The Ternary Bi-quadratic Diophantine Equation to be solved is given by

$$2x^2 - 3xy + 2y^2 = 23z^4 \qquad (1)$$

The substitution of the linear transformations

$$x = u + v, \; y = u - v \qquad (2)$$

in (1) leads to

$$u^2 + 7v^2 = 23z^4 \qquad (3)$$

(3) is solved through different approaches and the different patterns of solutions of (1) obtained are presented below:

**PATTERN: 1**

Consider (3) as

$$u^2 + 7v^2 = 16z^4 + 7z^4$$

and write it in the form of ratio as

$$\frac{(u + 4z^2)}{7(z^2 + v)} = \frac{(z^2 - v)}{(u - 4z^2)} = \frac{a}{b}, b \neq 0 \qquad (4)$$

The above equation is equivalent to the system of equations

$$bu - 7av + (4b - 7a)z^2 = 0 \qquad (5)$$

$$-au - bv + (4a + b)z^2 = 0 \qquad (6)$$

Solving (4) & (5) by the method of cross multiplication, we have

$$\left.\begin{array}{l} u = 28a^2 - 4b^2 + 14ab \\ v = -7^2 + b^2 + 8ab \\ z^2 = 7a^2 + b^2 \end{array}\right\} \qquad (7)$$

Substituting $a = 2pq, b = 7p^2 - q^2$ in (7) and using (2), we get

$$x = x(p,q) = -147p^4 - 3q^4 + 126p^2q^2 + 308p^3q - 44pq^3$$
$$y = y(p,q) = -245p^4 - 5q^4 + 210p^2q^2 + 84p^3q - 12pq^3$$
$$z = z(p,q) = 7p^2 + q^2$$

which represent the non-zero distinct integer solutions to (1).

**PROPERTIES:**

- $x(1,q) + y(1,q) + t_{4,q^2} + 42F_{4,q,6} + 22CP_{9,q} + 6P_q^4 + t_{640,q} \equiv -8 \pmod{64}$
- $x(1,q) - y(1,q) - t_{4,q^2} - 12F_{4,q,6} + 22CP_{9,q} + 6P_q^4 + t_{168,q} \equiv 98 \pmod{131}$
- $2\{z(p,p)\} = (4p)^2$ is a Perfect Square.

**PATTERN: 2**

Instead of (4), consider the form of ratio as,

$$\frac{(u+4z^2)}{(z^2-v)}=\frac{7(z^2+v)}{(u-4z^2)}=\frac{a}{b},b\neq 0 \qquad (8)$$

(10)

Following the analysis similar to pattern-1, the corresponding integer solutions to (1) are given by

$$x=x(p,q)=245p^4+5q^4-210p^2q^2+84p^3q-12pq^3$$
$$y=y(p,q)=147p^4+3q^4-126p^2q^2+308p^3q-44pq^3$$
$$z=z(p,q)=7p^2+q^2$$

**PROPERTIES:**

- $x(1,q)+y(1,q)-2t_{4,q^2}-6F_{4,q,6}+222P_q^4+t_{476,q}\equiv 2\pmod{195}$
- $x(1,q)-y(1,q)-t_{4,q^2}-6F_{4,q,6}+2CP_{9,q}+96P_q^4+t_{78,q}\equiv 98\pmod{246}$
- $12\,z(p,p)$ is a Nasty Number.

**PATTERN: 3**

Assume

$$23=(4+i\sqrt{7})(4-i\sqrt{7}) \qquad (9)$$

Write z as

$$z=z(a,b)=a^2+7b^2 \qquad (10)$$

Substituting (9) and (10) in (3) and employing the method of factorization, define

$$(u+i\sqrt{7}v)=(4+i\sqrt{7})(a+i\sqrt{7}b)^4$$

Equating real and imaginary parts in the above equation, we get

$$\left.\begin{aligned} u&=4a^4+196b^4-168a^2b^2-28a^3b+196ab^3\\ v&=a^4+49b^4-42a^2b^2+16a^3b-112ab^3\end{aligned}\right\} \qquad (11)$$

Substituting (11) in (2), the corresponding integer solutions to (1) are given by

$$x(a,b) = 5a^4 + 245b^4 - 210a^2b^2 - 12a^3b + 84ab^3$$

$$y(a,b) = 3a^4 + 147b^4 - 126a^2b^2 - 44a^3b + 308ab^3$$

$$z(a,b) = a^2 + 7b^2$$

**PROPERTIES:**

- $x(A,A) + y(A,A) - 391t_{4,A^2} - 6F_{4,A,6} - 130CP_{9,A} - 6P_A^4 + t_{342,A} + t_{344,A} \equiv 8 \pmod{267}$
- $12\{x(B,B) - y(B,B) - 107t_{4,B^2} - 6F_{4,B,6} - 150CP_{9,B} + 6P_B^4 + 84t_{3,B} + 41t_{4,B}\} - 24$ is a Nasty Number
- $48\, z(A,A)$ is a Nasty Number.

**PATTERN: 4**

Instead of (9), consider 23 as

$$23 = \frac{(9 + i\sqrt{7})(9 - i\sqrt{7})}{4^2} \qquad (12)$$

Following the analysis similar to pattern-3, the corresponding integer solutions to (1) are given by

$$x(A,B) = 80A^4 + 3920B^4 - 3360A^2B^2 + 19272A^3B - 1344AB^3$$

$$y(A,B) = 72A^4 + 3528B^4 - 3024A^2B^2 - 416A^3B + 2912AB^3$$

$$z(A,B) = 4A^2 + 28B^2$$

**PROPERTIES:**

- $x(B,B) + y(B,B) - 6926t_{4,B^2} - 3132F_{4,B,6} - 6P_B^4 + t_{14864,B} \equiv 152 \pmod{11425}$
- $x(B,B) - y(B,B) - 392t_{4,B^2} + 3236CP_{9,B} + 6P_B^4 + t_{6720,B} \equiv 8 \pmod{14713}$
- $z(A,A) - t_{58,B} \equiv 4 \pmod{27}$.

**PATTERN: 5**

Consider (3) as

$$u^2 + 7v^2 = 23z^4 * 1 \qquad (13)$$

Write 1 as

$$1 = \frac{(3 + i\sqrt{7})(3 - i\sqrt{7})}{4^2} \qquad (14)$$

Substituting (9), (10) and (14) in (13) and employing the method of factorization, define

$$u + i\sqrt{7}v = (1 + i\sqrt{7})(a + i\sqrt{7}b)^4 \left[\frac{3 + i\sqrt{7}}{4}\right]$$

Equating the real and imaginary parts in the above equation, we get

$$\left.\begin{aligned} u &= \frac{1}{4}\left[2a^4 + 98b^4 - 42a^2b^2 + 196a^3b - 1372ab^3\right] \\ v &= \frac{1}{4}\left[7a^4 + 343^4 - 294a^2b^2 + 8a^3b - 56ab^3\right] \end{aligned}\right\} \qquad (15)$$

Substituting (15) in (2), we've

$$x = \frac{1}{4}\left[9a^4 + 441b^4 - 336a^2b^2 + 204a^3b - 1428ab^3\right]$$

$$y = \frac{1}{4}\left[-5a^4 - 245b^4 + 252a^2b^2 + 188a^3b + 1316ab^3\right]$$

Replacing 'a' by "4A" and 'b' by "4B" in the above equations and (10), we have

$$x(A,B) = 576A^4 + 28224B^4 - 21504A^2B^2 + 13056A^3B - 91392AB^3$$

$$y(A,B) = -320A^4 - 15680B^4 + 16128A^2B^2 + 12032A^3B + 84224AB^3$$

$$z(A,B) = 16A^2 + 112B^2$$

which represent the non- zero distinct integer solutions to (1).

**PROPERTIES:**

- $x(1,B)+y(1,B)-12543t_{4,B^2}-6F_{4,B,6}+5782CP_{9,B}-6P_B^4+t_{10764,B}\equiv 256(\bmod 17316)$
- $y(1,B)+z(1,B)+15681t_{4,B^2}-6F_{4,B,6}-32472P_B^4-53330CP_{9,B}-t_{6,B}\equiv -304(\bmod 33286)$
- $z(A,1)-t_{34,A}\equiv 7(\bmod 15)$

**PATTERN: 6**

Substituting (10), (12) and (14) in (13) and following the procedure as in pattern-5, the corresponding integer solutions to (1) are given by,

$$x(A,B)=1152A^4+56448B^4-16128A^2B^2-2016AB-6656A^3B+46592AB^3$$

$$y(A,B)=448A^4+21592B^4-6272A^2B^2-784AB-13056A^3B+92032AB^3$$

$$z(A,B)=16A^2+112B^2$$

**PROPERTIES:**

- $x(A,1)+y(A,1)-100t_{4,A^2}-9000F_{4,A,6}+16140CP_{9,A}+6P_A^4+t_{227898,A}+t_{227900,A}\equiv 78400(\bmod 100140)$
- $x(A,1)-y(A,1)-4t_{4,A^2}-4200F_{4,A,6}+300P_A^4-2800CP_{9,A}-2t_{22814,A}\equiv -21952(\bmod 160126)$
- $2z(B,B)$ is a Nasty Number

**CONCLUSION:**

In this paper, six different patterns of integer solutions to (1) are presented. To conclude, one may search for other patterns of non-zero distinct integer solutions to the considered Bi-quadratic with three unknowns and their corresponding properties.

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