A COMMON FIXED POINT THEOREM OF COMPATIBLE MAPPING OF TYPE (A-1) IN COMPLETE FUZZY METRIC SPACE

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ABSTRACT

In this paper we prove some common fixed point theorems for compatible mappings of type (A-1) in fuzzy metric spac .our result modify and extends the result of Jungck G.[7].

Key words : Compatible mappings, Compatible mappings of type(A), Compatible mappings of type(A-1), Common fixed point , Complete Fuzzy metric space, Fuzzy metric space.

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1. INTRODUCTION

The first important result in the theory of fixed point of compatible mapping was obtained by Gerald Jungck in 1986[6] as a generalization of commuting mapping. In 1993 Jungck and Cho [7] introduced the concept of , Compatible mappings of type (A) by generalizing the definition of weakly uniformly contraction maps. Pathak and Khan [12] introduced the concept of type A-compatible and S-compatible by splitting the definition of compatible mapping of type(A).Pathak et.al. [8] renamed A-compatible and S-compatible as compatible mappings of and type(A-1) and compatible mappings of type(A-2) respectively and introduced it in fuzzy metric space.

Zadeh [16] introduced the concept of fuzzy sets. The idea of fuzzy metric space was introduced by Kramosil and Michalek [11] which was modified by George and Veernmani [2,3]. Singh B. and M.S. Chauhan [14] introduced the concept of compatibility in fuzzy metic space and proved some common fixed point theorems in fuzzy metric spaces in the sense of George and Veermani with continuous t-norm $*$ defined by $a*b = \min\{a, b\}$ for all $a, b \in [0,1]$.

The aim of the paper is to prove some common fixed point theorems of compatible mappings of type(A-1) and type(A-2).These results modify and extend the result in [8,12,15].

2. PRELIMINARIES

DEFINITION 2.1 ([13]) A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is called a continuous t-norm if, it satisfies the following conditions:

(i) $*$ is associative and commutative.

(ii) $*$ is continuous.

(iii) $a * 1 = a$,for all $a \in [0,1]$.

(iv) $a * b \le c * d$ whenever $a \le c$ and $b \le d$, for all a, b, c, d in [0, 1]

DEFINITION 2.2 [2] 3-tuple (X, M ,*) is called a fuzzy metric space if X is an arbitrary (non-empty) , * is continuous t-norm , and M is a Fuzzy set on $X^2 \times (0,\infty)$ satisfying the following conditions :

(i) $M(x, y, t) > 0$.

(ii) $M(x, y, t) = 1$ if and only if $x = y$.

(iii) $M(x, y, t) = M(y, x, t)$.

(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$

(v) $M(x, y, .) : (0,\infty) \rightarrow [0,1]$ is continuous.

For all $x, y, z \in X$ and $s, t > 0$

.Let (X, d) be a metric space, and let $a*b = \min\{a, b\}$ .Let $M(x, y, t) = \frac{t}{t + d(x, y)}$ for all $x, y \in X$ and $t > 0$.

Then (X, M ,*) is a fuzzy metric M induced by d is called standard fuzzy metric space [3].

DEFINITION 2.3 [4] A sequence $[x_n]$ in a fuzzy metric space (X, M ,*) is said to be convergent to a point x in X (denoted by $\lim_{n\to\infty} x_n = x$), if for each $\varepsilon > 0$ and each $t > 0$,there exists $n_0 \in N$ such that $M(x_n, x, t) > 1-\varepsilon$ for all $n \geq n_0$.

The completeness and non completeness of fuzzy metric space was discussed in George and Veeramani [3] and . Grabiec[5].

DEFINITION 2.4. [2] A sequence$\{x_n\}$ in a fuzzy metric space (X, M, *) is called Cauchy sequence if for each $\varepsilon>0$ and each $t > 0$, there exists $n_0 \in N$ such that $M(x_n, x_m, t) > 1-\varepsilon$ for all $n, m \geq n_0$.

DEFINITION 2.5[8] Two self mapping A and S of a fuzzy metric space (X,M,*) are said to be compatible ,if $\lim_{n\to\infty} M(ASx_n, SAx_n, t) = 1$ whenever $\{x_n\}$ is a sequence such that

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$,for some z in X.

DEFINITION 2.6[7]Self mappings A and S of a fuzzy metric space (X,M,*) are said to be compatible of type(A) if $\lim_{n\to\infty} M(ASx_n, SSx_n, t) = \lim_{n\to\infty} M(SAx_n, AAx_n, t) = 1$ for all $t > 0$,whenever $\{x_n\}$is a sequence in X such that

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$,for some $z \in X$.

DEFINITION 2.7[8] Self mappings A and S of a fuzzy metric space (X, M ,*) are said to be compatible of type(A-1) if $\lim_{n\to\infty} M(SAx_n, AAx_n, t) = 1$ for all t > 0 whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$, for some z ∈ X.

LEMMA 2.8[4] Let (*X*, *M*, *) be a fuzzy metric space. Then for all *x*, *y* in *X*, *M* (*x*, *y*, *) is non-decreasing.

LEMMA 2.9 [4] Let (*X*, *M*, *) be a fuzzy metric space. If there exists $q \in (0, 1)$ such that

$M(x, y, qt)$ $M(x, y, t/q)$ for positive integer n. Taking limit as $n\to\infty$ $M(x, y, t) \geq 1$ and hence $x = y$.

LEMMA 2.10. [10] The only *t*-norm * satisfying $s*s \geq s$ for all $s \in [0,1]$,is the minimum *t*-norm, that is,

$a * b = \min\{a, b\}$ for all *a* ,*b* [0,1].

PROPOSITION 2.11. [7] Let (*X*, *M*, *) be a fuzzy metric space and let *A* and *S* be continuous mappings of *X* then *A* and *S* are compatible if and only if they are compatible of type (A).

PROPOSITION 2.12. [8] Let (*X*, *M*, *) be a fuzzy metric space and let *A* and *S* be compatible mappings of type (A-1) and $Az = Sz$ for some $z \in X$, then $SAz = AAz$.

PROPOSITION 2.13. [8] Let (*X*, *M*, *) be a fuzzy metric space and let *A* and *S* be compatible mappings of type (A-1) and $Az = Sz$ for some $z \in X$, then $ASz = SSz$.

PROPOSITION 2.14. [8] Let (*X*, *M*, *) be a fuzzy metric space and let *A* and *S* be compatible mappings of type (A-1) and let Ax_n, $Sx_n \to z$ as $n\to\infty$ for some x ∈ X then $AAx_n \to Sz$ if S is continuous at z.

3. MAIN RESULTS

We prove the following theorem.

THEOREM 3.1. Let (X, M, *) be a complete fuzzy metric space and let P,Q, S and T be a self mappings of X satisfying the following conditions:

(i) $P(X) \subset T(X)$, $Q(X) \subset S(X)$,

(ii) S and T are continuous.

(iii) the pairs{P ,S} and {Q, T} are compatible mapping of type (A-1) on X.

(iv) there exists $k \in (0, 1)$ such that for every $x, y \in X$ and $t > 0$,

$$M(Px, Qy, kt) \geq M(Sx, Ty, t)*M(Px, Sx, t)*M(Qy, Ty, t)*M(Px, Ty, t)$$

Then P , Q , S and T have a unique common fixed point in X.

PROOF : Since $P(X) \subset T(X)$ and $Q(X) \subset S(X)$ for any $x_0 \in X$, there exists $x_1 \in X$ such that

$P x_0 = T x_1$ and for this $x_1 \in X$, $y_{2n-1} = Tx_{2n-1} = Ax_{2n-2}$ and $y_{2n} = Sx_{2n} = Bx_{2n-1}$, for all n = 0, 1, 2,…

From (iv), $M (y_{2n+1}, y_{2n+2}, kt) = M (Px_{2n}, Qx_{2n+1}, kt)$.

$$\geq M (Sx2n, Tx_{2n+1}, t)*M (Px_{2n}, Sx_{2n}, t)*M (Qx_{2n+1}, Tx_{2n+1}, t) *M (Px_{2n},Tx_{2n+1}, t)$$

$$= M (y_{2n}, y_{2n+1}, t)*M (y_{2n+1}, y_{2n}, t)*M (y_{2n+2}, y_{2n+1}, t) * M (y_{2n+1},y_{2n+1}, t)$$

$$\geq M (y_{2n}, y_{2n+1}, t)*M (y_{2n+1}, y_{2n+2}, t)$$

From lemma 2.9 and 2.10, We have

$$M (y_{2n+1}, y_{2n+2}, kt) \geq M (y_{2n}, y_{2n+1}, t) \quad (3.1.1)$$

Similarly, we have

$$M (y_{2n+2}, y_{2n+3}, kt) \geq M (y_{2n+1}, y_{2n+2}, t) \quad (3.1.2)$$

From (3.1.1) and (3.1.2), we have

$$M (y_{n+1}, y_{n+2}, kt) \geq M (y_n, y_{n+1}, t) \quad (3.1.3)$$

$$M (y_n, y_{n+1}, t) \geq M (y_n, y_{n-1},t/k)$$

$$\geq M(y_{n+2}, y_{n-1},t/k^2)$$

$$\geq \ldots\ldots\ldots\ldots\ldots\ldots \geq M(y_1,y_2,t/k^n) \rightarrow 1 \text{ as } n\rightarrow\infty.$$

So $M(y_n, y_{n+1}, t) \to 1$ as $n\to\infty$ for any $t > 0$.For each $\varepsilon > 0$ and each $t > 0$,we can choose $n_0 \in N$ such that $M(y_n, y_{n+1}, t) > 1-\varepsilon$ for all $n > n_0$.

For $m, n \in N$ we suppose $m \geq n$. Then we have that

$$M(y_n, y_m, t) \geq M(y_n, y_{n+1}, \frac{t}{m-n})* M(y_{n+1}, y_{n+2}, \frac{t}{m-n})*\ldots\ldots M(y_{m-1}, y_m, \frac{t}{m-n})$$

$$\geq (1-\varepsilon)* (1-\varepsilon)*\ldots\ldots\ldots\ldots\ldots\ldots.(m-n)\text{times}.$$

$$\geq (1-\varepsilon)$$

And hence $\{y_n\}$ is a Cauchy sequence in X.

Since (X, M, *) is complete, $\{y_n\}$ converges to some point $z \in X$, and so

$\{Px_{2n-2}\}$, $\{Sx_{2n}\}$, $\{Qx_{2n-1}\}$ and $\{Tx_{2n-1}\}$ also converges to z.

From proposition 2.15 and (iii), we have

$$PPx_{2n-2} \to Sz \qquad (3.1.4)$$

and

$$QQx_{2n-1} \to Tz \qquad (3.1.5)$$

Now, from (iv),we get

$$M(PPx_{2n-2}, QQx_{2n-1}, kt) \geq M(SPx_{2n-2}, TQx_{2n-1}, t)*M(PPx_{2n-2}, SPx_{2n-2}, t)* M(QQx_{2n-1}, TQx_{2n-1}, t)$$

$$*M(PPx_{2n-2}, TQx_{2n-1}, t)$$

Taking limit as $n\to\infty$ and using (3.1.4) and (3.1.5) we have

$$M(Sz, Tz, kt) \geq M(Sz, Tz, t)*M(Sz, Sz, t)*M(Tz, Tz, t)*M(Sz, Tz, t)$$

$$\geq M(Sz, Tz, t) * 1 * M(Sz,Tz,t)$$

$$\geq M(Sz, Tz, t)$$

It follows that Sz = Tz (3.1.6)

Now from (iv)

M(Pz, QQ_{2n-1}, kt) ≥ M(Sz, TQx_{2n-1}, t)*M(Pz, Sz, t)*M(QQx_{2n-1}, TQx_{2n-1}, t)*M(PPx_{2n-2}, TQx_{2n-1}, t)

Again taking limit n → ∞ and using (3.1.5) and (3.1.6),we have

M(Pz, Tz, kt) ≥ M(Sz, Sz, t)*M(Pz, Tz, t)*M(Pz, Tz, t)*M(Pz, Tz, t)

≥ M(Pz, Tz, t)

and hence Pz =Tz. (3.1.7)

From (iv), (3.1.6) and (3.1.7)

M(Pz, Qz, kt) ≥ M(Sz, Tz, t)*M(Pz, Sz, t)*M(Qz, Tz, t)*M(Pz, Tz, t)

= M(Pz, Pz, t)*M(Pz, Pz, t)*M(Qz, Pz, t)*M(Pz, Pz, t)

≥M(Pz,Qz,t)

hencePz=Qz. (3.1.8)

From(3.1.6),(3.1.7)and(3.1.8),

we have Pz=Qz=Tz=Sz (3.1.9)

Now, we show that Qz = z.

From (iv),

M (Px_{2n}, Qz, kt) ≥ M (Sx_{2n}, Tz, t)*M (Px_{2n}, Sx_{2n}, t)*M (Qz, Tz, t)*M (Px_{2n}, Tz, t)

And, taking limit as n →∞ and using (3.1.6) and (3.1.7), we have

M(z, Qz, kt) ≥ M(z, Tz, t)*M(z, z, t)*M(Qz, Tz, t)*M(z, Tz, t)

= M (z, Qz, t)*1*M(Qz, Qz, t)*M(z, Qz, t)

≥ M (z, Bz, t).

And hence Qz = z. Thus from (3.1.9), z = Pz = Q z = Tz = Sz and z is a common fixed point of P, Q, S and T.

In order to prove the uniqueness of fixed point, let w be another common fixed point of P, Q, S and T. Then

M(z, w, kt) = M(Pz, Qw, kt)

≥ M (Sz, Tw, t)*M (Pz, Sz, t)*M (Qw, Tw, t)*M (Pz, Tw, t)

≥ M (z, w, t).

From lemma 2.10, z = w. This completes the proof of theorem .

COROLLARY 3.2. Let (X, M , *) be a complete fuzzy metric space and let P, Q, S and T be a self mappings of X satisfying (i) - (iii) of theorem 3.1 and there exists k ∈ (0,1) such that

M (Px, Qy, kt) ≥ M (Sx, Ty, t)*M (Px, Sx, t)*M (Qy, Ty, t)*M (Qy, Sx, 2t)*M (Px, Ty, t)

for every x, y ∈ X and t >0 .Then P,Q,S and T have a unique common point in X.

COROLLARY 3.3. Let (X, M , *) be a complete fuzzy metric space and let P, Q, S and T be a self mappings of X satisfying (i) - (iii) of theorem 3.1 and there exists k ∈ (0,1) such thatM (Px, Qy, kt) ≥ M (Sx, Ty, t) for every x, y ∈ X and t >0 .Then P,Q,S and T have a unique common point in X.

CORROLARY 3.4 . Let (X, M , *) be a complete fuzzy metric space and let A, B, S and T be a self mappings of X satisfying (i) - (iii) of theorem 3.1 and there exists k ∈ (0,1) such that

M (Px, Qy, kt) ≥ M (Sx, Ty, t)*M (Sx, Px, t)*M (Px, Ty, t) ,

for every x, y ∈ X and t >0 .Then P,Q,S and T have a unique common point in X.

COROLLARY 3.5. Let (X, M, *) be a complete fuzzy metric space. Then continuous self mappings S and T of X have a common fixed point in X if and only if there exists a self mapping P of X such that the following condition are satisfied :

(i) P(X) $\subset$ T(X) $\cap$ S(X),

(ii) The pair {P,S} and {P,T} are compatible mapping of type(A-1) on X.

(iii) There exists k $\in$ (0,1) such that for every x, y $\in$ X and t >0

M (Px, Py, kt) $\geq$ M (Sx, Ty, t)*M (Px, Sx, t)*M (Qy, Ty, t)*M (Px, Ty, t).

In fact, P,S and T have a unique common fixed point in X.

Proof : We shown that the necessity of the conditions (i)-(iii). Suppose that S and T have a common fixed point in X, say z. Then Sz = z = Tz.

Let Px = z for all x $\in$ X .Then we have P(X) $\subset$ T(X) $\cap$ S(X), and we know that [P, S] and [P, T] are compatible mapping of type (A-1), in fact PoS = S oP and PoT = ToP, and hence the conditions (i) and (ii) are satisfied.

For some k $\in$ (0,1),we get M (Px, Py, kt) = 1 $\geq$ M (Sx, Ty, t)*M (Px, Sx, t)

*M (Py, Ty, t)*M (Px, Ty, t).

for every x, y $\in$ X and t >0 and hence the condition (iii) is satisfied.

Now, for the sufficiency of the conditions, let $P = Q$ in theorem 3.1. Then P, S and T have a unique common fixed point in X.

In fact ,P,S and T have a unique common fixed point in X.

COROLLARY 3.6. Let (X, M,*) be a complete fuzzy metric space. Then continuous self mappings S and T of X have a common fixed point in X if and only if there exists a self mapping P of X satisfying (i) – (ii) of theorem 3.5 and there exists a self mapping of X satisfying (i)- (iii) of theorem 3.5 and there exists k $\in$ (0,1) such that for every x, y $\in$ X and t >0

M (Px, Py, kt) $\geq$ M (Sx, Ty, t)*M (Px, Sx, t)*M (Py, Ty, t)*M (Px, Sx, t)* M (Px, Ty, t).

COROLLARY 3.7. Let (X, M,*) be a complete fuzzy metric space. Then continuous self mappings S and T of X have a common fixed point in X if and only if there exists a self mapping P of X satisfying (i) – (ii) of theorem 3.5 and there exists a self mapping of X satisfying (i)- (iii) of theorem 3.5 and there exists k ∈ (0,1) such that for every x, y ∈ X and t >0

M (Px, Py, kt) ≥ M (Sx, Ty, t).

In fact ,P,S and T have a unique common fixed point in X.

COROLLARY 3.8. Let (X, M,*) be a complete fuzzy metric space. Then continuous self mappings S and T of X have a common fixed point in X if and only if there exists a self mapping P of X satisfying (i) – (ii) of theorem 3.5 and there exists a self mapping of X satisfying (i)- (iii) of theorem 3.5 and there exists k ∈ (0,1) such that for every x, y ∈ X and t >0

M (Px, Py, kt) ≥ M (Sx, Ty, t)*M (Sx ,Px, t)* M (Px, Ty, t).

In fact ,P,S and T have a unique common fixed point in X.

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