DETERMINATION OF SOLUTIONS OF ONE-DIMENSIONAL TRANSIENT HEAT TRANSFER PROBLEM BY DUAL RECIPROCITY BOUNDARY ELEMENT METHOD

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Abstract

In this paper, we describe a dual-reciprocity boundary element method for the solution of a one-dimensional transient heat equation with a source term. A brief introduction of the development of the boundary element method and DRBEM is given. Discretization of heat equation using BEM and the solution of boundary integral equation is discussed. Due to the presence of the source term, the BEM fails, thus we use the DRBEM which is employed in the non-homogeneous equation. In DRBEM, all the source terms and time-dependent terms are converted into boundary integrals and hence the computational domain of the problem reduces to one.

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Introduction

There are numerous scientific,engineering and technological processes that can be mathematically modeled by transient diffusion equations such as smelting of metal,sintering of ceramics, and joining (welding) of polymers by microwave radiation, spontaneous ignition of reactive solid, heat transfer in nuclear reactor components, mass transport in ground water, e.tc. The study of these diffusion problems is thus of fundamental importance. In the past years, the Boundary Element Method (BEM) has become increasingly attractive to scientists and engineers as an alternative numerical method to themore established ones such as the Finite Difference (FDM) and Finite Element (FEM) Methods, for solving diffusion problems. In many aspects,the BEM proves advantageous over the FDM and FEM.Its major advantageous and attractive characteristic is its ability in reducingthe dimension of the problem by one.In other words, it provides a complete solution to the problem with the effort of solving an integral equation on the boundary of the computational domain only in obtaining a boundary-only integral equation, the BEM makes use of the fundamental solution (Green's function) of the partial differential equation (PDE), and the reciprocity theorem is then applied, in a discretized form, to a certain number of nodal points on the boundary.(e.g., by using a collocation technique), resulting in algebraic equations.Once all the unknowns on the boundary are found, the solution at any interior point can be easily obtained with high accuracy using only the computed boundary values. Generally, the treatment of transient diffusion problems with the BEM can be categorized into two main approaches. The first one solves the problem directly in the time domain. The second one solves the problem in a transformed domain. The Boundary Element Method, is now a well-established numerical technique for solving boundary-value problems that involve linear as well as certain types of non-linear partial differential equations .The basic idea of the technique is to find an integral equation equivalent to the original PDE, and then solve this integral equation using a discretization procedure as with any other numerical approach. For certain types of linear and homogeneous PDEs, only a boundary discretization is necessary; this reduction in the dimensionality of the problem permits accurate solutions to be obtained very efficiently, and is the main attraction of the BEM approach. However, for an inhomogeneous PDE, the integral equation involves a domain integral, and the reduction in dimensionality is apparently lost. Heat transfer is a discipline of thermal engineering that concerns the generation, use, conversion and thermal exchange of thermal of thermal energy and heat between physical systems. Heat transfer in solids, with the changes of temperature in time on physical boundaries of analyzed objects, occur in many engineering mechanisms (engines, compressors), heating and cooling system and hydraulic networks(Zhang et al. ,2009; LU and Viljanen, 2006).The analysis of basic mechanism of heat transfer in solids, that is heat conduction problem, is significant for process of designing and optimization of mechanical systems and devices. Accordingly, the heat conduction equations with conditions of variable temperature or heat flux on the boundaries become an important instrument for mathematical description of many engineering, geothermal and biological problems. As a result, there is a need to develop effective computational methods and the tools for solving heat conduction problem(Mansur et al.,2009; Yang Gao, 2010). Physical problems involving the heat exchange between the ends of a conductor and the surrounding environment can be formulated as a set of partial differential equations representing the heat equation and boundary conditions relating the heat fluxes at the ends of the conductor to the difference between the temperature at the ends of the rod and that of the surrounding fluid through a function f.Heat transfer is mainly concerned with temperature and the flow of heat. Temperature represents the amount of thermal energy available, whereas heat flow represents the movement of thermal energy from one region to another. On a microscopic scale, thermal energy is related to the kinetic energy of molecules. The greater a material's temperature, the greater the thermal agitation of the constituent molecules. Several material properties serve to modulate the heat transferred between two regions at differing temperatures. Examples include thermal conductivities, specific heats, material densities, fluid viscosities, fluid velocities and more. Heat transfer takes place by three fundamental modes namely conduction, convection and radiation. Conduction is heat transfer from one part to another of the same substance or from one substance to another sharing a physical contact without displacement of molecules forming the substance. Heat conduction is accomplished by two main mechanisms namelyby molecular interaction and drift of free electrons. According to Pitts and Sissom (2004), a temperature gradient within a homogeneous substance results in an energy transfer rate within the medium. This can be determined using the Fourier's law of heat conduction as;

$$q = -kA\frac{\partial T}{\partial n} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (1.1)$$

where q is the rate of heat transfer and $\frac{\partial T}{\partial n}$ is the temperature gradient.

Convection is the mode of heat transfer which occurs due to movement of fluid particles undergoing density change associated with temperature differential in the fluid. The flow of fluid may be by external process or sometimes (in gravitational fields) by buoyancy forces caused when thermal energy expands the fluid, thus influencing its own transfer.The latter processis often called natural convection.All convective processes also moveheat partly by diffusion,as well.Another form of convection is forced convection.In this case, the fluid is forced to flow by use of a pump, fan or mechanical means.When a solid is exposed to a moving fluid having a temperature different from that of the solid, energy is carried from or to the solid body by the fluid and obeying the Newton's law of cooling, defined by

$$q = hA(T_s - T_\infty) \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (1.2)$$

where T_s is the surface temperature, T_∞ is the free stream temperature, h is the constant of proportionality and A is the area. Radiation mode of heat transfer is due to electromagnetic wave propagation. Experimental evidence indicates that radiant heat transfer is proportional to the fourth power of the absolute temperature of the body according to the fundamental Stefan-Boltzmann law

$$q = \varepsilon AT^4 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (1.3)$$

where ε is the proportionality constant. Generally temperature at any point in a solid is completely defined by its numerical value because it is a scalar quantity whereas heat flow is a vector quantity defined by its value and direction. Anon-uniformtemperature distribution within a solid body shows the presence of heat flow in the solid which is always in the direction of decreasing temperature. Heat –flux vector $q(\vec{r},t)$denotes heat flow at a spatial position r in a solid body at any instant t.The magnitude of the heatflux vector is equal to the quantity of heat crossing a unit area, normal to the direction of heat flow at the position under consideration per unit time. The basic law which gives the relationship between the heat flow and the temperature gradient for a stationary homogeneous isotropicsolid is given in the form,

$$q(\vec{r},t) = -kT(\vec{r},t) \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (1.4)$$

where$k$ is called the thermal conductivity; itis ascalar quantity whose value depend on the property of the media. Based on the hypothesis that heat crosses from the inside to the outside of the isothermal surface, the unit direction vector$\hat{s}$is referred to as outward- drawn normal to the isothermal surface. Denotingdifferentiation along the$\hat{s}$ direction by$\frac{\partial}{\partial s}$, the derivative of temperature in the$\hat{s}$directionis $\frac{\partial T}{\partial s}$, which representsthe maximum rate of decrease of temperature. Since, $\frac{\partial T}{\partial s}$and ∇T areof equalmagnitudesbut point in opposite directions, the magnitude of the heat-flux vector is given by;

$$|\vec{q}| = -k\frac{\partial T}{\partial S} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (1.5)$$

Transient Conduction

Transient conduction occurs when there is a change in temperature at the boundaries of an object or may occur with temperature changes inside an object as a result of a new source or sink of heat causes the changes in temperatures. When equilibrium is reached, the heat flow into the system will equal the heat flow out of the system and temperatures at each point inside the system no longer vary. It is at this point that transient conduction ends and steady-state conduction may continue to occur if there is a continual flow of heat. If there are changes in external temperatures or the internal heat generation changes are too rapid for equilibrium of temperature in space to take place, then the system never reaches a state of unchanging temperature distribution in time, and the system remains in a transient state.

Differential equation of heat conduction

Temperature distribution is determined from the solution of the differentiation equation of heat conduction. The differential equation of heat conduction for a stationary, homogeneous, isotropic solid with heat generation within the region is derived below. A heat source with a negative sign denotes the heat sink. Consider a small control volume v.

Rate of energy storage in v equals rate of heat entering v through its boundary surfaces plusrate of heat generated in v. Rate of energy storage in the volume v is given by

$$\int_v \rho c_p \frac{\partial T(\vec{r},t)}{\partial t} dv \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (1.6)$$

Heat entering v through a small area dA on the boundary surface is -$\vec{q}(\vec{r},t).\hat{s}$,

where$\hat{s}$is the outward-drawn normal unit direction vector and $\vec{q}$ the heat-flux vector at dA.

Rate of heat entering the volume v through its bounding surfaces is

$$-\int_A \vec{q}\cdot\hat{s}dA = -\int_v \nabla\cdot\vec{q}dv \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (1.7)$$

where divergence theorem is used to convert the surface integral to volume integral.

Heat generation in the volume$v$ is

$$\int_v g(\vec{r},t)dv \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (1.8)$$

Considering the energy-balance equation and usingequations (1.6), (1.7) and (1.8)we obtain

$$\int_V \left\{\rho C_P \frac{\partial T(\vec{r},t)}{\partial t} + \nabla\vec{q}(\vec{r},t) - g(\vec{r},t)\right\} dv = 0 \dots\dots\dots\dots\dots\dots\dots (1.9)$$

When v is chosen so small as to eliminate the integral we obtain

$$\rho c_p \frac{\partial T(\vec{r},t)}{\partial t} = \nabla\cdot[k\nabla T(\vec{r},t)] + g(\vec{r},t) \dots\dots\dots\dots\dots\dots\dots\dots (1.10)$$

This is the differential equation of heat conduction for a stationary, homogeneous isotropic solid with heat generation within the solid.

Boundary condition

Boundary Value Problem (BVP) is a differential equation together with a set of additional restraints called the boundary conditions. A solution to a boundary value problem is a solution to the differential equation which also satisfies the boundary condition. Boundary value problems arise in several branches of physics as any physical problem. Problems involving the wave equation such as the determination of normal nodes are often stated as boundary value problems. A large class of important boundary value problems is Sturm-Liouville problems. The analysis of these problems involves the Eigen functions of a differential operator.

To be useful in application, a boundary value problem should be well posed; this means that given the input to the problem, there exists a unique solution which depends continuously on the input. Among the earliest boundary value problem to be studied is the Dirichlet problem of finding harmonic functions (solutions of Laplace equation); the solution was given by the Dirichlet's principle. The differential equation of heat conduction will have numerous solutions unless a set of boundary condition and an initial condition (for the time-dependent problem) are presented. However, the boundary condition that prescribes the conditions at the boundary surfaces of the region may be linear or nonlinear. Considering the linear boundary condition, we narrow into three groups, namely;

Boundary condition of the first kind

In this boundary condition temperature is prescribed along the boundary surface and for the general case it is a functional of both position and time i.e. $T = f(\vec{r}, t)$

If the temperature at the boundary vanishes we have $T = 0$ on the boundary surfaces. This special case is called the homogeneous boundary condition of the first kind.

Boundary condition of the second kind

This boundary condition is of the form of normal derivative of temperature prescribed at the boundary surface and it may be function of both position and time.

$\frac{\partial T}{\partial n_i} = f_i(\vec{r_s}, t)$on the boundary surface s_i,

where, $\frac{\partial}{\partial n_i}$denotes differentiation along the outward-drawn normal at the boundary surfaces_i.

This boundary condition is equivalent to that of prescribing the magnitude of the heat flux along the boundary surface, since the left hand side becomes the magnitude of heat flux at the surface when multiplied by thermal conductivity on both sides. If the normal derivative of temperature at the boundary surface vanishes, we have, $\frac{\partial T}{\partial n_i} = 0$ on the surface s_i. This special case is called homogenous boundary condition of the second kind. An insulated boundary condition satisfies this condition.

Boundary condition of the third kind

A linear combination of the temperature and its normal derivative is prescribed at the boundary surface. For a boundary surface that fits the coordinate surface of an orthogonal coordinate system is given as $k\frac{\partial T}{\partial n} + hT = f(\vec{r}, t)$on the boundary surface.

A special case where $k\frac{\partial T}{\partial n} + hT = 0$ is called the homogeneous boundary condition of the third kind.

Results

One of the most valuable techniques is the divergence theorem which transforms a volume into a surface integral.

$$\iiint_V (\nabla \cdot \vec{A}) dv = \iint_S (\vec{A} \cdot \hat{n}) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.0)$$

where$\vec{A}$is a vector is$\hat{n}$is the unit outward normal of s and dv stands for volume integral.

Another useful formula is the Stokes Theorem presented by George Gabriel Stokes (1819-1903), which transforms a surface integral into a contour integral.

$$\iint_s (\nabla \times \vec{A}) \cdot \hat{n} ds = \oint_c \vec{A} \cdot ds \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.1)$$

Where s is an open, two sided curved surface, c is the closed contour bonding s.

The most important work related to the boundary integral equation for solving potential problem came from George Green whose presented an identity;

If $A = \phi\nabla\psi$, where ϕ and ψ are arbitrary scalar fields, defined in an enclosed region V.

$$\nabla \cdot A = \phi\nabla^2\psi + \nabla\phi \cdot \nabla\psi \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.2)$$

$$A \cdot \hat{n} = \phi\nabla\psi \cdot \hat{n} = \phi \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.3)$$

This equation (3.2) and (3.3) when substituted into the divergence theorem, results in Green's first identity:

$$\iiint_v [\phi\nabla^2\psi + \nabla\psi \cdot \nabla\phi] dv = \iint_s \phi \frac{\partial\psi}{\partial n} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.4)$$

When, $A = \psi\nabla\phi$ and we substitute into the divergence theorem, we obtain:

$$\iiint_v [\psi\nabla^2\phi + \nabla\phi \cdot \nabla\psi] dv = \iint_s \psi \frac{\partial\phi}{\partial n} ds \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.5)$$

Subtracting equation (3.4) from (3.5) results the corollary of divergence theorem known as Green's second identity or Green's theorem:

$$\iiint_v [\phi\nabla^2\psi - \psi\nabla^2\phi] dv = \iint_s \left[\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right] ds \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.6)$$

The mathematical background of BEM is represented by the divergence theorem and Green's theorem. Also the definition of the $Kronecker$ symbol, the Heaviside function, the Dirac distribution, and the *erf* and *erfc* functions are introduced.

The $Kronecker\ delta$ symbolis given by

$$\delta_{i,j} = \begin{cases} 1 & \text{if} i = j \\ 0 & \text{if} \quad i \neq j \end{cases} \quad \text{……………………..………………..…(3.7)}$$

The $Heaviside\ function$ is given by

$$H(t) = \begin{cases} 1 & \text{if} \quad t > 0 \\ 0 & \text{if} \quad t \leq 0 \end{cases} \text{…………………………………………..……(3.8).}$$

The $Dirac\ delta\ distribution$functions are as follows:

$$\delta(x, \xi) = \delta(x - \xi) = \begin{cases} 0 & \text{if} \quad x \neq \xi \\ \infty & \text{if} \quad x = \xi \end{cases} \text{……………………………………....(3.9)}$$

The fundamental properties of the Dirac delta distribution

$$\delta(x) = H'(x), \int_\Omega f(\xi)\delta(x,\xi)d\xi = f(x), \int_\Omega f(\xi)\frac{\partial\delta}{\partial\xi}(x,\xi)d\xi = -f(x),$$

$$x \in \Omega \text{………………………………………………………………………(3.10)}$$

The erf and $erfc$ functions are given by

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x exp\,(-\sigma^2)d\sigma, \qquad \text{erfc}(x) = 1 - \text{erf}\,(x) \text{……………………………(3.11)}$$

The complex error function $erfi$ is given by

$$erfi(x) = -ierf(ix) = \frac{2}{\sqrt{\pi}}\int_0^x \exp(\sigma^2)\,d\sigma. \quad \text{……………………………………..(3.12)}$$

The Boundary Element Method (BEM)

BEM is a numerical method and hence its importance in numerical analysis community. It attempts to use the given boundary conditions to fit boundary values into the Boundary Integral Equation (BIE), rather than values throughout the space defined by a partial differentiation equation. BEM is applied to problems for which Green's function can be calculated. These usually involve fields in homogeneous media.

There are several advantages of using the BEM over the finite element (FEM) or the finite-difference (FDM) methods. First, the BEM only requires a boundary mesh to discretize the problem and as such, it reduces the dimensionality of the problem, it deals easily with both infinite and semi-infinite domains, etc., it is very flexible and applicable to complex geometries without having to resort to intricate internal mesh generation of unnecessary internal information as required by the traditional FDM or FEM. Secondly, the unknown ambient temperature, boundary temperature and heat flux are boundary quantities to be determined and the discretization of the boundary only is the essence of the BEM. Further, the heat flux is computed as part of the solution and is not a post-processing numerical differentiation. However, it is important to note that all these gains are possible only when the fundamental solution of the governing equation is available, a fact that forms the major drawback in the use of the BEM in general.

Fundamental Solution

Many practical problems in engineering and science are mathematically modeled by a partial differential equation of the form

$$LT\left(\underline{p}\right) = 0 \quad \underline{p} \in \Omega \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.13)$$

Where L the differential operator is $\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t}$, T is the temperature and Ω is the domain.

In a one-dimensional domain $\underline{p} = (x, t)$, and in order to determine the values of the temperature T at every point $\underline{x} \in \Omega$ some boundary conditions have to be imposed on the boundary of the domain $\partial\Omega$. Different boundary conditions may arise for the values of unknown function T or its normal derivative $\frac{\partial T}{\partial n}$, such as the convective boundary conditions are represented by linear relations between the temperature and the heat flux, radiative boundary conditions which is a nonlinear fourth order power in temperature in relation to the heat flux, others include mixed boundary conditions, exponential decay in temperature relation to the heat flux, complex relation between heat flux and temperature, etc. The function $G(\underline{p}; \underline{p'})$ is the fundamental solution for the heat equation (14) if

$$L^{*}G\left(\underline{p}; \underline{p'}\right) = -\delta\left(\underline{p}; \underline{p'}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.14)$$

where $L^{*} = \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial t}$ the ad joint of is L, $\underline{p} = (x, t)$ is a field point, $\underline{p'} = (\xi, \tau)$ is a source point and δ is the Dirac delta distribution function defined in (3.9).

Thus, the fundamental solution of the heat conduction equation (3.14) satisfies the equation

$$\frac{\partial G}{\partial \tau}(x, t; \xi, \tau) + \frac{\partial^2 G}{\partial \xi^2}(x, t; \xi, \tau) = -\delta(x, t; \xi, \tau) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.15)$$

If we look for a solution of equation (3.15) that depends only on the distance $r = |x - \xi|$ and time period $t' = |t - \tau|$, then the following expression is obtained,

$$G(x, t; \xi, \tau) = \frac{H(t-\tau)}{2\sqrt{\pi(t-\tau)}} \exp\left(-\frac{(x-\xi)^2}{4(t-\tau)}\right), \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.16)$$

where H is the Heaviside function given in (3.8), which is introduced to emphasize the fact that the fundamental solution is zero when $t \leq \tau$.

Boundary Integral Equation (BIE)

The governing partial differential equation (3.13) transforms into an integral equation by means of fundamental solution and Green's second formula given in (3.6).

$$\iiint_v \{\phi\nabla^2\psi - \psi\nabla^2\phi\} dv = \iint_s \left\{\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\chi}{\partial n}\right\} ds \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.17)$$

For an arbitrary function ϕ, the following equation holds

$$\int_Q \left\{T\left(\underline{p}'\right)L^*\psi\left(\underline{p}'\right) - \psi\left(\underline{p}'\right)LT(\underline{p}')\right\}dQ = \int_{\partial Q}\left\{T\left(\underline{p}'\right)\frac{\partial\psi}{\partial n}\left(\underline{p}'\right) - \psi(\underline{p}')\frac{\partial T}{\partial n}(\underline{p}')\right\}ds. \quad \ldots(3.18)$$

Next, taking $\psi = G(\underline{p};\underline{p}')$ we obtain that for every $p \in Q$.

$$\int_Q \left\{T(\underline{P}')L^*G\left(\underline{p};\underline{p}'\right) - G\left(\underline{p};\underline{p}'\right) - G\left(\underline{p};\underline{p}'\right)LT(\underline{P}')\right\}dQ = \int_{\partial Q}\left\{T\left(\underline{p}'\right)\frac{\partial G}{\partial n\left(\underline{p}'\right)}\left(\underline{p};\underline{p}'\right) - \right.$$

$Gp;p'\partial T\partial np'dS.$ ……(3.19)

If equations (3.18) and (3.19) are used then we obtain

$$-\int_Q T\left(\underline{p}'\right)\delta\left(\underline{p};\underline{p}'\right)dQ = \int_{\partial Q}\left\{T\left(\underline{p}'\right)\frac{\partial G}{\partial n\left(\underline{p}'\right)}\left(\underline{p};\underline{p}'\right) - G(\underline{p};\underline{p}')\frac{\partial T}{\partial n}(\underline{p}')\right\}dS \quad \ldots(3.20)$$

Using the property (3.10) of the Dirac delta function, we obtain

$$T\left(\underline{p}\right) = \int_{\partial Q}\left\{G\left(\underline{p};\underline{p}'\right)\frac{\partial T}{\partial n}\left(\underline{p}'\right) - T(\underline{p}')\frac{\partial G}{\partial n}(\underline{p};\underline{p}')\right\}dS, p \in Q \quad \ldots (3.21)$$

On considering $x \in \partial\Omega$, and taking the domain augumented by a small region which is centered at the point x, by a limit process commonly used in potential theory, it can be shown that if$\partial\Omega$ is smooth then, see e.g. Brebbia et al. (1984), Chen and Zhou (1992), Wrobel (2002),

$$\frac{1}{2}T\left(\underline{p}\right) = \int_{\partial Q}\left\{G\left(\underline{p};\underline{p}'\right)\frac{\partial T}{\partial n}\left(\underline{p}'\right) - T(\underline{p}')\frac{\partial G}{\partial n}(\underline{p};\underline{p}')\right\}dS, p \in \partial\Omega \times (0, t_f) \quad \ldots(3.22)$$

Or, combining (3.21) and (3.22), we obtain

$$\eta(x)T\left(\underline{p}\right) = \int_{\partial Q}\left\{G\left(\underline{p};\underline{p}'\right)\frac{\partial T}{\partial n}\left(\underline{p}'\right) - T(\underline{P}')\frac{\partial G}{\partial n}(\underline{p};\underline{p}')\right\}dS, \quad \ldots(3.23)$$

For any $\underline{p} = (x,t) \in \overline{Q}$, where $\eta(x) = 1$ if $x \in \Omega$ and $\eta(x) = \frac{1}{2}$ if $x \in \partial\Omega$ (smooth domain).

The boundary integral equation (BIE) (3.23) can then be transformed into the form

$$\eta(x)T\left(\underline{p}\right) = \int_{S_1}\left\{G\left(\underline{p};\underline{p}'\right)\frac{\partial T}{\partial n}(\underline{P}') - T\left(\underline{p}'\right)\frac{\partial G}{\partial n\left(\underline{p}'\right)}\left(\underline{p};\underline{p}'\right)\right\}dS_1 + \int_{S_2} T\left(\underline{p}'\right)G\left(\underline{p};\underline{p}'\right)dS_2$$

$$\underline{p}' = (x,t) \in \overline{Q}, \quad \ldots(3.24)$$

Where in the one-dimensional case, $S_1 = \{0,1\} \times (0, t_f]$ and $S_2 = [0,1] \times \{0\}$.The integral over S_3=$\{0,1\} \times (0, t_f)$ vanishes due to the Heaviside function in expression (3.8) and $n(0) = -1$ and $n(1) = 1$ for the boundaries $x = 0$ and $x = 1$, respectively.

In higher dimensions, if the domain Ω has corners, then the coefficient η at these corners can be evaluated by similar process using different parts of the cylindrical surface instead of the semi-circular surface and then η depends on the angle at the corners, see Brebbia et al. [4] .

However, since in numerical approach the boundary integral equation (3.23) is applied at a finite number of nodes, and assuming that the boundary $\partial\Omega$ of the domain Ω consists of several smooth segments, these nodes can be chosen such that they are not at corner points of the domain. Therefore we do not need to evaluate the coefficient η at the corner points.

We note that the boundary integral equation (BIE) (3.23) provides a linear relation between the boundary values of the solution T and those of its normal derivative $\frac{\partial T}{\partial n}$. Hence, given$T$ at each point on part of the boundary and $\frac{\partial T}{\partial n}$ on the remainder, we obtain a linear BIE, or a pair of such equations for the complementary boundary values. By solving these equations and then substituting their solution, together with boundary data, into equation (3.24), we obtain the solution at any point $\underline{p}$ in the domain Q.

REFERENCES

[1]A. J. Nowak and C. A**.** Brebbia, "Solving Helmholtz equation by Multiple reciprocity method" *in computer and experiments in fluids flow* (eds G.M Carlomagno and C.A. Brebbia), (comput.,southampton, 1989) 265-270

[2] A.C.Neves and C.A. Brebbia, "The multiple reciprocity element method inelasticity: A new approach for transforming domain integrals to boundary", *Int. J. Numer. Methods Eng.***31** (1991) 709-729.

[3] Bialecki R. A, Jurga's P. , (2002), Dual reciprocity Boundary Element Method without matrix inversion for transient heat conduction, *Engineering Analysis with Boundary Elements*, 26, 227-236.

[4] M.Itagaki and C.A.Brebbia, "Generation of higher order fundamental solutions to thetwo-dimensional modified Helmholtz equation", *Eng. Anal. Boundary Elements***11** (1993) 87-90

[5] C.A.Brebbia, J. C. F.Tells and L.C. Wrobel (ads.), *Boundary element techniques* (Springer, Berlin, 1984)

[6]D. Nardini and C. A. Brebbia, "A new approach to free vibration analysis using boundary elements"**.(***comput.Mech. ,Southampton, and Springer*, *Berlin*, 1982).

[7]P.W. Partridge and C. A.Brebbia, **"**Computer implementation of the BEM dualreciprocitymethod for the solution of Poisson-type equations*",* *Software Engrg.workstation5* (1989) 199-206

[8]Edmond Ghandour (1974). Initial Value Problem for Boundary Values of a Green's Function. *Journal of Applied Mathematics*.Vol.27, pg 650-652.

[9] L. Wrobel (2002), The Boundary Element Method; *Applications in Thermo-FluidsandAcoustics***.**Vol.1.pg 20-35.

[10] Paris F. (1997).*Boundary Element Method; Fundamentals and Applications*,

Oxford University Press.

[11] Iso Y. (1991) Convergence of Boundary Element solutions for Heat Equation**.** *Journal of Computational and Applied Mathematics*, vol. 38,pg 201-209.

[12] P.W.Partridge and L.C.Wrobel, "*The dual reciprocity method for spontaneous Ignition".Int. J.Numer.methods Eng*.30 (1990) 953-963

[13]S. P. Zhu and Y.L. Zhang, "Improvement on dual reciprocity boundary element for equations with convective terms", *Comm. Numer. methods Eng*. **10** (1994) 361-371.

[14] Y. L. Zhang and S. P Zhu , "On the choice of interpolation functions used in the dual-reciprocity boundary element method", *Eng.Anal. Boundary Elements***13** (1994)387-396

[15] S. P Zhu, H. W. Liu and X. P. Lu , "A combinationof the LTDRM and ATPs in solving linear diffusion problems", *Eng. Anal. Boundary Elements* **21** (1998) 285- 289.

[16] I. L. Bruton and A. J. Pullan, "A semi-analytic boundary element for parabolic problem", *Eng. Anal. Boundary Elements*,**18,** (1996) 253-264.

[17] Onyango T.T.M, Ingham D.B and D.Lensic, (2008) *Restoring Boundary ConditionsIn Heat Transfer, Journal of engineering mathematics*, vol. 62.No1 pg85-101.Universty of Leeds, U.K.

[18] Mushtag, Shak and Muhammed, (2010) Comparison of direct and indirect boundary element methods for the flow past a sphere. *Journal of American Science 2010, **61**, pg166-17*

[19]MohammadiaM ,Hematiyan M.R, Marin L. (2010), Boundary element analysis of nonlinear transient heat conduction problems involving non-homogeneous and nonlinear heat sources using time-dependent fundamental solutions, *Engineering Analysis with Boundary Elements*,**34**,655-665

[20] Ochiai Y., Kitayama Y. (2009), Three-dimensional unsteady heat conduction analysis of triple-reciprocity boundary element method, *Engineering Analysis withBoundary Elements*, **33,** 789-795.

[21] Yang K., Gao X. W. (2010), Radial integration of Boundary Element Method for transient heat conduction problems, Engineering Analysis with Boundary Elements, **34**, 557-563.

[22]Sutradhar A. ,Paulino G.H. (2004), The simple boundary element method for transient heat conduction in functionally graded materials, *Computer methods in Applied Mechanics and Engineering*, **193**, 4511-4539.