GENERALIZATION OF FUZZY B-BOUNDARY

K.Bageerathi*

Abstract

The aim of this paper is to present the notion of fuzzy ℭ-b- boundary through the arbitrary complement function ℭ. Further we develop the concept by fuzzy ℭ-b-closure and fuzzy ℭ-b-interior using this fuzzy ℭ-closure operator. Topologically, fuzzy boundary was defined by Warren[13] in 1977. Later Pu and Liu[12], gave another definition of Fuzzy boundary. Later Cuchillo- Inbanez and Tarres[6], provided a new definition for boundary. So, finally a relative study of these three fuzzy b-boundaries in different ways is also established.

Mathematics Subjcet Classification: 54A40, 3E72

Key words: Fuzzy complement function, Fuzzy ℭ-b-boundary, Fuzzy w-ℭ-b-boundary, Fuzzy c- ℭ- b-boundary and fuzzy topologically

---

* Assistant Professor of Mathematics, Aditanar College of Arts and Science, Tiruchendur-628216, India.

1. Introduction.

Chang introduced the concept of fuzzy topological space[5] which is a natural generalization of topological space in 1968. In [5], Chang's study is restricted to the concepts of such as open set, closed set, neighborhood, interior, continuity and compactness. For a fuzzy set A in a fuzzy topological space X, we denote the Fuzzy closure of A by Cl A that equals to the infimum of fuzzy closed sets containing A. The connection between the fuzzy open sets and the fuzzy closed sets is given by the standard complement $\mathfrak{C}A$ of A with the membership function $\mathfrak{C}(x)=1-x$.

The standard complement is obtained by using the function $\mathfrak{C}: [0,1]\longrightarrow[0,1]$ defined by $\mathfrak{C}(x)=1-x$ for all $x\in[0,1]$.

Boundary generally marks the division of two contiguous properties. In topology boundary of a set is defined as the set of points that belongs to both closure of the set and closure of the complement of the set. Topologically, fuzzy boundary was defined by Warren[13] in 1977. Later Pu and Liu[12], gave another definition of Fuzzy boundary. Later Cuchillo- Inbanez and Tarres[6], provided a new definition for boundary. Athar and Ahmad studied the properties of fuzzy boundary recently. K. Bageerathi et.al[4] extended this standard complement function to the analog concepts with respect to an arbitrary complement function $\mathfrak{C}:[0,1]\longrightarrow[0,1]$.

In this paper the author using the complement function $\mathfrak{C}:[0,1]\longrightarrow[0,1]$, introduce the concept of fuzzy $\mathfrak{C}$-b-closed set and fuzzy $\mathfrak{C}$-b-closure operator, fuzzy $\mathfrak{C}$-b-open set and fuzzy $\mathfrak{C}$-b-interior operator in a fuzzy topological space. In this paper, first we introduce the concept of fuzzy b-boundary and generalize the concept of fuzzy b-boundary using the arbitrary complement function $\mathfrak{C}$.

For the basic concepts and notations, one can refer Chang[8]. The concepts that are needed in this paper are discussed in the second section. The section three is dealt with the concept of fuzzy $\mathfrak{C}$ -b-boundary .

2. PRELIMINARIES

Throughout this paper (X,τ) denotes a fuzzy topological space in the sense of Chang. Let $\mathfrak{C}$: [0, 1]→[0, 1] be a complement function. If λ is a fuzzy subset of (X,τ) then the complement $\mathfrak{C}\lambda$ of a fuzzy subset λ is defined by $\mathfrak{C}\lambda(x) = \mathfrak{C}(\lambda(x))$ for all $x\in X$. A complement function $\mathfrak{C}$ is said to satisfy

(i) the boundary condition if $\mathfrak{C}(0) = 1$ and $\mathfrak{C}(1) = 0$,

(ii) monotonic condition if $x \le y \Rightarrow \mathfrak{C}(x) \ge \mathfrak{C}(y)$, for all $x, y \in [0, 1]$,

(iii) involutive condition if $\mathfrak{C}(\mathfrak{C}(x)) = x$, for all $x \in [0, 1]$.

The properties of fuzzy complement function $\mathfrak{C}$ and $\mathfrak{C}\lambda$ are given in George Klir[8] and Bageerathi et al[2]. The following lemma will be useful in sequel.

Lemma 2.1 [2]

Let $\mathfrak{C} : [0, 1] \to [0, 1]$ be a complement function that satisfies the monotonic and involutive conditions. Then for any family $\{\lambda_\alpha: \alpha\in\Delta\}$ of fuzzy subsets of X, we have

(i) $\mathfrak{C}(\sup\{\lambda_\alpha(x): \alpha\in\Delta\}) = \inf\{\mathfrak{C}(\lambda_\alpha(x)): \alpha\in\Delta\}= \inf\{(\mathfrak{C}\lambda_\alpha(x)): \alpha\in\Delta\}$ and

(ii) $\mathfrak{C}(\inf\{\lambda_\alpha(x): \alpha\in\Delta\}) = \sup\{\mathfrak{C}(\lambda_\alpha(x)): \alpha\in\Delta\}= \sup\{(\mathfrak{C}\lambda_\alpha(x)): \alpha\in\Delta\}$ for $x\in X$.

Definition 2.2 [Definition 2.15, [3]]

A fuzzy topological space (X, τ) is $\mathfrak{C}$ -product related to another fuzzy topological space (Y, σ) if for any fuzzy subset ν of X and ζ of Y, whenever $\mathfrak{C}\lambda \not\ge \nu$ and $\mathfrak{C}\mu \not\ge \zeta$ imply $\mathfrak{C}\lambda \times 1 \vee 1 \times \mathfrak{C}\mu \ge \nu \times \zeta$, where $\lambda \in \tau$ and $\mu \in \sigma$, there exist $\lambda_1 \in \tau$ and $\mu_1 \in \sigma$ such that $\mathfrak{C}\lambda_1 \ge \nu$ or $\mathfrak{C}\mu_1 \ge \zeta$ and $\mathfrak{C}\lambda_1 \times 1 \vee 1 \times \mathfrak{C}\mu_1 = \mathfrak{C}\lambda \times 1 \vee 1 \times \mathfrak{C}\mu$.

Lemma 2.3 [Theorem 2.19, [3]]

Let (X, τ) and (Y, σ) be $\mathfrak{C}$ -product related fuzzy topological spaces. Then for a fuzzy subset λ of X and a fuzzy subset μ of Y, $bcl_{\mathfrak{C}}(\lambda \times \mu) = bcl_{\mathfrak{C}}\lambda \times bcl_{\mathfrak{C}}\mu$.

Lemma 2.4 [3]

Let (X, τ) be a fuzzy topological space and let $\mathfrak{C}$ be a complement function that satisfies the monotonic and involutive properties. Then a fuzzy set λ of a fuzzy topological space (X,τ) is

(i) fuzzy $\mathfrak{C}$ - b-open if and only if $\lambda \le int\, cl_{\mathfrak{C}}\lambda \vee cl_{\mathfrak{C}}\, int\, \lambda$.

(ii) fuzzy $\mathfrak{C}$ -b-closed if and only if $\mathfrak{C}\lambda$ is fuzzy $\mathfrak{C}$ -b-open.

(iii) The arbitrary union of fuzzy $\mathfrak{C}$-b-open sets is fuzzy $\mathfrak{C}$ -b-open.

Definition 2.5[11]

Let (X,τ) be a fuzzy topological space and $\mathfrak{C}$ be a complement function. Then for a fuzzy subset λ of X, the fuzzy $\mathfrak{C}$ - b-interior of λ (briefly $bint_{\mathfrak{C}}\,\lambda$), is the union of all fuzzy C-b-open sets of X contained in λ. That is,

$bint_{\mathfrak{C}}(\lambda) = \vee\{\mu: \mu \le \lambda, \mu$ is fuzzy $\mathfrak{C}$-b-open$\}$.

Proposition 2.6 [11]

Let (X, τ) be a fuzzy topological space and let $\mathfrak{C}$ be a complement function that satisfies the monotonic and involvtive conditions. Then for any fuzzy subsets λ and μ of a fuzzy topological space X, we have

(i) $bint_{\mathfrak{C}}\,\lambda \le \lambda$,

(ii) λ is fuzzy $\mathfrak{C}$ - b-open $\Leftrightarrow bint_{\mathfrak{C}}\,\lambda = \lambda$,

(iii) $bint_{\mathfrak{C}}(bint_{\mathfrak{C}}\,\lambda) = bint_{\mathfrak{C}}\,\lambda$,

(iv) If $\lambda \le \mu$ then $bint_{\mathfrak{C}}\,\lambda \le bint_{\mathfrak{C}}\,\mu$.

Proposition 2.7 [11]

Let (X, τ) be a fuzzy topological space and let $\mathfrak{C}$ be a complement function that satisfies the monotonic and involutive conditions. Then for any two fuzzy subsets λ and μ of a fuzzy topological space, we have (i) $bint_{\mathfrak{C}}(\lambda\vee\mu) \ge bint_{\mathfrak{C}}\,\lambda\vee bint_{\mathfrak{C}}\,\mu$ and (ii) $bint_{\mathfrak{C}}(\lambda\wedge\mu) \le bint_{\mathfrak{C}}\,\lambda\wedge bint_{\mathfrak{C}}\,\mu$.

Definition 2.8 [11]

Let (X,τ) be a fuzzy topological space. Then for a fuzzy subset λ of X, the fuzzy $\mathfrak{C}$-b-closure of λ (briefly $bcl_{\mathfrak{C}}\,\lambda$), is the intersection of all fuzzy $\mathfrak{C}$-b-closed sets containing λ. That is $bcl_{\mathfrak{C}}\,\lambda = \wedge\{\mu: \mu \ge \lambda, \mu$ is fuzzy $\mathfrak{C}$-b-closed$\}$.

The concepts of "fuzzy $\mathfrak{C}$ - b-closure" and "fuzzy b- closure" are identical if $\mathfrak{C}$ is the standard complement function.

Proposition 2.9 [11]

If the complement functions $\mathfrak{C}$ satisfies the monotonic and involutive conditions. Then for any fuzzy subset λ of X, (i) $\mathfrak{C}(bint_{\mathfrak{C}}\,\lambda) = bcl_{\mathfrak{C}}(\mathfrak{C}\,\lambda)$ and

(ii) $\mathfrak{C}(bcl_{\mathfrak{C}}\,\lambda) = bint_{\mathfrak{C}}(\mathfrak{C}\,\lambda)$, where $bint_{\mathfrak{C}}\,\lambda$ is the union of all fuzzy $\mathfrak{C}$-b-open sets contained in λ.

Proposition 2.10 [11]

Let (X, τ) be a fuzzy topological space and let $\mathfrak{C}$ be a complement function that satisfies the monotonic and involutive conditions. Then for the fuzzy subsets λ and μ of a fuzzy topological space X, we have

(i) $\lambda \leq bcl_{\mathfrak{C}}\,\lambda$,

(ii) λ is fuzzy $\mathfrak{C}$-b-closed $\Leftrightarrow bcl_{\mathfrak{C}}\,\lambda = \lambda$,

(iii) $bcl_{\mathfrak{C}}\,(bcl_{\mathfrak{C}}\,\lambda) = bcl_{\mathfrak{C}}\,\lambda$,

(iv) If $\lambda \leq \mu$ then $bcl_{\mathfrak{C}}\,\lambda \leq bcl_{\mathfrak{C}}\,\mu$.

Proposition 2.11 [11]

Let (X, τ) be a fuzzy topological space and let $\mathfrak{C}$ be a complement function that satisfies the monotonic and involutive conditions. Then for any two fuzzy subsets λ and μ of a fuzzy topological space, we have (i) $bcl_{\mathfrak{C}}\,(\lambda\vee\mu) = bcl_{\mathfrak{C}}\,\lambda \vee bcl_{\mathfrak{C}}\,\mu$ and (ii) $bcl_{\mathfrak{C}}\,(\lambda\wedge\mu) \leq bcl_{\mathfrak{C}}\,\lambda \wedge bcl_{\mathfrak{C}}\,\mu$.

3. Fuzzy $\mathfrak{C}$ -b-boundary

The concept of fuzzy b-boundary is defined as $bBd\lambda = bcl\,\lambda\wedge bcl\,(\lambda^{\mathfrak{C}})$. In this section, the concept of fuzzy $\mathfrak{C}$-b-boundary is introduced and its properties are discussed.

Definition 3.1

Let λ be a fuzzy subset of a fuzzy topological space X and let $\mathfrak{C}$ be a complement function. Then the fuzzy $\mathfrak{C}$-b-boundary of λ is defined as $Bd_{\mathfrak{C}b}\,\lambda = bcl_{\mathfrak{C}}\,\lambda\wedge bcl_{\mathfrak{C}}\,(\mathfrak{C}\,\lambda)$.

Since the arbitrary intersection of fuzzy $\mathfrak{C}$-b-closed sets is fuzzy $\mathfrak{C}$-b-closed, $Bd_{\mathfrak{C}b}\,\lambda$ is fuzzy $\mathfrak{C}$-b-closed.

Proposition 3.2

Let (X,τ) be a fuzzy topological space and $\mathfrak{C}$ be a complement function that satisfies the involutive condition. Then for any fuzzy subset λ of X, $Bd_{\mathfrak{C}b}\lambda=Bd_{\mathfrak{C}b}(\mathfrak{C}\,\lambda)$.

Proof.

By using Definition 3.1, $Bd_{\mathfrak{C}b}\,\lambda = bcl_{\mathfrak{C}}\,\lambda\wedge bcl_{\mathfrak{C}}\,(\mathfrak{C}\,\lambda)$. Since $\mathfrak{C}$ satisfies the involutive condition $\mathfrak{C}(\mathfrak{C}\lambda) = \lambda$, that implies $Bd_{\mathfrak{C}b}\,\lambda = bcl_{\mathfrak{C}}\,(\mathfrak{C}\lambda) \wedge bcl_{\mathfrak{C}}\,\mathfrak{C}\,(\mathfrak{C}\lambda)$. Again by using Definition 3.1, $Bd_{\mathfrak{C}b}\,\lambda = Bd_{\mathfrak{C}b}\,(\mathfrak{C}\,\lambda)$.

The following example shows that, the word "involutive" can not be dropped from the hypothesis of Proposition 3.2.

Example 3.3

Let X = {a, b, c} and τ = **{**0, {a.6, b.7, c.8}, {a.8, b1, c.7}, {a.8, b.6, c.7}, {a.8, b1, c.8}, {a.6, b.7, c.7}, {a.8, b.7, c.8}, {a.6, b.6, c.7}, {a.2, b0, c.1}1**}**. Let $\mathfrak{C}\,(x) = \frac{1-x}{1+x^2}$, $0 \le x \le 1$, be the complement function. We note that the complement function $\mathfrak{C}$ does not satisfy the involutive condition. The family of all fuzzy $\mathfrak{C}$ -closed sets is $\mathfrak{C}\,(\tau)$=**{**0,{a.294, b.201, c.122}, {a.122,b0, c.201},{a.122, b.294, c.201},{a.122, b0,c.122}, {a.294, b.201, c.201}, {a.122, b.201, c.122}, {a.294, b.294, c.201}, {a.769, b1, c.891}, 1**}**. Let λ = {a.2, b0, c.1}. Then $Int\,\lambda$ = {a.2, c.1}, $Cl_{\mathfrak{C}}\,Int\,\lambda$ = {a.294, b.201, c.122} and $Int\,Cl_{\mathfrak{C}}\,\lambda$= {a.2, c.1}. This implies that $\lambda \ge Cl_{\mathfrak{C}}\,Int\,\lambda \wedge Int\,Cl_{\mathfrak{C}}\,\lambda$ = {a.2, c.1}. By using Proposition 4.2, λ is fuzzy $\mathfrak{C}$-b-closed. Then it can be verified that $bcl_{\mathfrak{C}}\,\lambda$ = {a.2, b0, c.1}. Now $\mathfrak{C}\,\lambda$ = {a.769, b1, c.891} and the value of $bcl_{\mathfrak{C}}\,\mathfrak{C}\,\lambda$ = {a.769, b1, c.891}. Hence $Bd_{\mathfrak{C}b}\,\lambda = bcl_{\mathfrak{C}}\,\lambda \wedge bcl_{\mathfrak{C}}\,(\mathfrak{C}\,\lambda)$ = {a.2, b0, c.1}. Also $\mathfrak{C}\,(\mathfrak{C}\,\lambda)$ = {a.12, b0, c.0607}, $bcl_{\mathfrak{C}}\,\mathfrak{C}\,(\mathfrak{C}\,\lambda)$ = {a.12, b0, c.0607}. $Bd_{\mathfrak{C}b}\,\mathfrak{C}\lambda = bcl_{\mathfrak{C}}\,\mathfrak{C}\,\lambda \wedge bcl_{\mathfrak{C}}\,\mathfrak{C}\,(\mathfrak{C}\,\lambda)$ = {a.12, b0, c.0607}. This implies that $Bd_{\mathfrak{C}b}\,\lambda \neq Bd_{\mathfrak{C}b}\,\mathfrak{C}\,\lambda$.

Proposition 3.4

Let (X,τ) be a fuzzy topological space and $\mathfrak{C}$ be a complement function that satisfies the monotonic and involutive conditions. If λ is fuzzy $\mathfrak{C}$-b-closed, then $Bd_{\mathfrak{C}b}\,\lambda \le \lambda$.

Proof.

Let λ be fuzzy $\mathfrak{C}$-b-closed. By using Definition 3.1, $Bd_{\mathfrak{C}b}\,\lambda = bcl_{\mathfrak{C}}\,\lambda \wedge bcl_{\mathfrak{C}}\,(C\,\lambda)$. Since $\mathfrak{C}$ satisfies the monotonic and involutive conditions, by using Proposition 2.10(ii), we have $bcl_{\mathfrak{C}}\,\lambda = \lambda$. Hence $Bd_{\mathfrak{C}b}\,\lambda \le bcl_{\mathfrak{C}}\,\lambda = \lambda$.

The following example shows that if the complement function $\mathfrak{C}$ does not satisfy the monotonic and involutive conditions, then the conclusion of Proposition 3.4 is false.

Example 3.5

Let X = {a, b} and τ = {0, {a.5, b.6}, {a.75, b.2}, {a.5, b.2}, {a.75, b.6}, 1}. Let $\mathfrak{C}\,(x) = \frac{2x}{1+x}$, $0 \le x \le 1$, be a complement function. From this, we see that the complement function $\mathfrak{C}$ does not satisfy the monotonic and involutive conditions. The family of all fuzzy $\mathfrak{C}$ -closed sets is given by $\mathfrak{C}$ (τ) = {0, {a.667, b.75}, {a.857, b.333}, {a.667, b.333}, {a.857, b.75}, 1}. Let λ = {a.665, b.462}, it can be found that $cl_{\mathfrak{C}}$ λ = {a.667, b.33} and $Int\ cl_{\mathfrak{C}}$ λ = {a.5, b.2} and $cl_{\mathfrak{C}}\ Int$ λ = {a.667, b.3}. This shows that λ is fuzzy $\mathfrak{C}$ -b-closed. Further it can be calculated that $bcl_{\mathfrak{C}}$ λ = {a.667, b.75}. Now $\mathfrak{C}$ λ = {a.8, b.857} and $bcl_{\mathfrak{C}}$ $\mathfrak{C}$λ = {1}. Hence $Bd_{\mathfrak{C}b}$ λ = $bcl_{\mathfrak{C}}$ λ∧ $bcl_{\mathfrak{C}}$ ($\mathfrak{C}$ λ) = {a.667, b.75}. This implies that $Bd_{\mathfrak{C}b}$ λ $\not\leq$ λ. This shows that the conclusion of Proposition 3.4 is false.

Proposition 3.6

Let (X,τ) be a fuzzy topological space and $\mathfrak{C}$ be a complement function that satisfies the monotonic and involutive conditions. If λ is fuzzy $\mathfrak{C}$-b-open then $Bd_{\mathfrak{C}b}$λ ≤ $\mathfrak{C}$ λ.

Proof.

Let λ be fuzzy $\mathfrak{C}$-b-open. Since $\mathfrak{C}$ satisfies the involutive condition, this implies that $\mathfrak{C}$ ($\mathfrak{C}$ λ) is fuzzy $\mathfrak{C}$-b-open. By using Lemma 2.4, $\mathfrak{C}$ λ is fuzzy $\mathfrak{C}$-b-closed. Since $\mathfrak{C}$ satisfies the monotonic and the involutive conditions, by using Proposition 3.4, $Bd_{\mathfrak{C}b}$ ($\mathfrak{C}$λ) ≤ $\mathfrak{C}$λ. Also by using Proposition 3.2, we get $Bd_{\mathfrak{C}b}$ (λ) ≤ $\mathfrak{C}$ λ. This completes the proof.

Example 3.7

Let X = {a, b, c} and τ = {0, {a.3, b.5}, {a.5, b.2, c.15}, {a.5, b.5, c.15}, {a.3, b.2}, 1}.

Let $\mathfrak{C}$ (x) = $\frac{1-x^2}{(1+x)^3}$, 0 ≤ x ≤ 1, be the complement function. We note that the complement function $\mathfrak{C}$ does not satisfy the involutive condition. The family of all fuzzy $\mathfrak{C}$ -closed sets is $\mathfrak{C}$ (τ)={0,{a.414, b.222, c1}, {a.222, b.556, c.642}, {a.222, b.222, c.156},{a.414, b.642, c1}, 1}. Let λ = {a.4, b.122, c.57}, the value of $bcl_{\mathfrak{C}}$ λ = {a.414, b.222, c.174} and $\mathfrak{C}$λ = {a.306, b.701, c.174}, it follows that $Bd_{\mathfrak{C}b}$ λ = $bcl_{\mathfrak{C}}$ λ∧ $bcl_{\mathfrak{C}}$ ($\mathfrak{C}$ λ) = {a.306, b.222, c.642}. This shows that $Bd_{\mathfrak{C}b}$ λ $\not\leq$ $\mathfrak{C}$ λ. Therefore the conclusion of Proposition 4.6 is false.

Proposition 3.8

Let (X, τ) be a fuzzy topological space and $\mathfrak{C}$ be a complement function that satisfies the monotonic and involutive conditions. If λ ≤ μ and μ is fuzzy $\mathfrak{C}$-b-closed then $Bd_{\mathfrak{C}b}$ λ ≤ μ.

Proof.

Let λ ≤ μ and μ be fuzzy $\mathfrak{C}$-b-closed. Since $\mathfrak{C}$ satisfies the monotonic and involutive conditions, by using Proposition 2.10(iv), we have λ ≤ μ implies $bcl_{\mathfrak{C}}$ λ ≤ $bcl_{\mathfrak{C}}$ μ. By using

Definition 3.1, $Bd_{\mathfrak{C}f}\,\lambda = bcl_{\mathfrak{C}}\,\lambda \wedge bcl_{\mathfrak{C}}\,(\mathfrak{C}\,\lambda)$. Since $bcl_{\mathfrak{C}}\,\lambda \leq bcl_{\mathfrak{C}}\,\mu$, we have $Bd_{\mathfrak{C}b}\,\lambda \leq bcl_{\mathfrak{C}}\,\mu \wedge bcl_{\mathfrak{C}}\,(C\lambda) \leq bcl_{\mathfrak{C}}\,\mu$. Again by using Proposition 2.10 (ii), we have $bcl_{\mathfrak{C}}\,\mu = \mu$. This implies that $Bd_{\mathfrak{C}b}\,\lambda \leq \mu$.

The following example shows that if the complement function $\mathfrak{C}$ does not satisfy the monotonic and involutive conditions, then the conclusion of Proposition 3.8 is false.

Example 3.9

Let X = {a, b} and τ = **{**0, {a.6, b.9}, {a.7, b.3}, {a.6, b.3}, {a.7, b.9}, 1**}**. Let $\mathfrak{C}\,(x) = \frac{2x}{1+x}$, $0 \leq x \leq 1$, be a complement function. From this, we see that the complement function $\mathfrak{C}$ does not satisfy the monotonic and involutive conditions. The family of all fuzzy $\mathfrak{C}$-closed sets is given by $\mathfrak{C}\,(\tau)$={0, {a.75, b.947}, {a.8235, b.462}, {a.75, b.462},{a.8235, b.947}, 1}. Let λ = {a.7, b.45} and μ = {a.76, b.5}. Then it can be found that $Int\ cl_{\mathfrak{C}}\,\mu$ = {a.7, b.3}, $Int\,\mu$ = {a.6, b.3} and $cl_{\mathfrak{C}}\ Int\,\mu$ = {a.75, b.462} . This implies that $\mu \geq Int\ cl_{\mathfrak{C}}\,\mu \wedge cl_{\mathfrak{C}}\ Int\,\mu$ = {a.7, b.3}. This show that μ is fuzzy $\mathfrak{C}$ - b-closed. It can be computed that $bcl_{\mathfrak{C}}\,\lambda$ = {a.8, b.47}. Now $\mathfrak{C}\,\lambda$ = {a.824, b.62} and $bcl_{\mathfrak{C}}\,\mathfrak{C}\,\lambda$ = {a.824, b.47}. $Bd_{\mathfrak{C}b}\,\lambda = bcl_{\mathfrak{C}}\,\lambda \wedge bcl_{\mathfrak{C}}\,(\mathfrak{C}\lambda)$ = { a.8, b.47}. This shows that $Bd_{\mathfrak{C}b}\,\lambda \not\leq \mu$. Therefore the conclusion of Proposition 3.8 is false.

Proposition 3.10

Let (X,τ) be a fuzzy topological space and $\mathfrak{C}$ be a complement function that satisfies the monotonic and involutive conditions. If $\lambda \leq \mu$ and μ is fuzzy $\mathfrak{C}$ - b-open then $Bd_{\mathfrak{C}b}\,\lambda \leq \mathfrak{C}\,\mu$.

Proof.

Let $\lambda \leq \mu$ and μ is fuzzy $\mathfrak{C}$-b-open. Since $\mathfrak{C}$ satisfies the monotonic condition, by using Proposition 2.10(iv), we have $\mathfrak{C}\,\mu \leq \mathfrak{C}\,\lambda$ that implies $bcl_{\mathfrak{C}}\,C\,\mu \leq bcl_{\mathfrak{C}}\,\mathfrak{C}\,\lambda$. By using Definition 3.1, $Bd_{\mathfrak{C}b}\,\lambda = bcl_{\mathfrak{C}}\,\lambda \wedge bcl_{\mathfrak{C}}\,\mathfrak{C}\,\lambda$. Taking complement on both sides, we get $\mathfrak{C}\,(Bd_{\mathfrak{C}b}\,\lambda)=C\,(bcl_{\mathfrak{C}}\,\lambda \wedge bcl_{\mathfrak{C}}\,(\mathfrak{C}\,\lambda))$. Since C satisfies the monotonic and involutive conditions, by using Lemma 2.1, we have $\mathfrak{C}\,(Bd_{\mathfrak{C}b}\,\lambda) = \mathfrak{C}\,(bcl_{\mathfrak{C}}\,\lambda) \vee \mathfrak{C}(bcl_{\mathfrak{C}}\,(\mathfrak{C}\,\lambda))$. Since $bcl_{\mathfrak{C}}\,\mathfrak{C}\,\mu \leq bcl_{\mathfrak{C}}\,\mathfrak{C}\,\lambda$, $\mathfrak{C}\,(Bd_{\mathfrak{C}b}\,\lambda) \geq \mathfrak{C}\,(bcl_{\mathfrak{C}}\,C\,\mu) \vee \mathfrak{C}\,(bcl_{\mathfrak{C}}\,\lambda)$, by using Proposition 2.9(ii), $\mathfrak{C}\,(Bd_{\mathfrak{C}f}\,\lambda) \geq bint_{\mathfrak{C}}\,\mu \vee bint_{\mathfrak{C}}\,\mathfrak{C}\,\lambda \geq bint_{\mathfrak{C}}\,\mu$. Since μ is fuzzy $\mathfrak{C}$-b-open, $\mathfrak{C}\,(Bd_{\mathfrak{C}b}\lambda) \geq \mu$. Since $\mathfrak{C}$ satisfies the monotonic conditions, $Bd_{\mathfrak{C}b}\,\lambda \leq \mathfrak{C}\,\mu$.

The following example shows that if the complement function $\mathfrak{C}$ does not satisfy the monotonic and involutive conditions, then the conclusion of Proposition 3.10 is false.

Example 3.11

Let X = {a, b} and τ = {0, {a.6, b.9}, {a.7, b.3}, {a.6, b.3}, {a.7, b.9}, 1}. Let $\mathfrak{C}(x) = \frac{2x}{1+x}$, 0≤x≤ 1, be a complement function. From this, we see that the complement function $\mathfrak{C}$ does not satisfy the monotonic and involutive conditions. The family of all fuzzy $\mathfrak{C}$ -closed sets is given by $\mathfrak{C}$ (τ)={0, {a.75, b.947}, {a.8235, b.462}, {a.75, b.462},{a.8235, b.947}, 1}. Let λ = {a.6, b.3} and μ = {a.65, b.4}. Then it can be evaluated that $Int\,\lambda$ = {a.6, b.3}, and $cl_{\mathfrak{C}}\, Int\,\lambda$ = {a.75, b.462}, $Int\, cl_{\mathfrak{C}}\,\lambda$ = {a.6, b.3}. Thus we see that $\lambda \le Int\, cl_{\mathfrak{C}}\,\lambda \vee cl_{\mathfrak{C}}\, Int\,\lambda$ = {a.75, b.462}. By using Lemma 2.4, λ is fuzzy $\mathfrak{C}$ -b-open. It can be computed that $bcl_{\mathfrak{C}}\,\lambda$= {a.85, b.632}. Now $\mathfrak{C}\lambda$ = {a.75, b.462} and $bcl_{\mathfrak{C}}\,\mathfrak{C}\,\lambda$ = {a.85, b.632}. $Bd_{\mathfrak{C}b}\,\lambda = bcl_{\mathfrak{C}}\,\lambda \wedge bcl_{\mathfrak{C}}\,(\mathfrak{C}\lambda)$ = {a.85, b.632}. This shows that $Bd_{\mathfrak{C}b}\,\lambda \not\le \mathfrak{C}\,\mu$.

Proposition 3.12

Let (X,τ) be a fuzzy topological space. Let $\mathfrak{C}$ be a complement function that satisfies the monotonic and involutive conditions. Then for any fuzzy subset λ of X, we have $\mathfrak{C}(Bd_{\mathfrak{C}b}\lambda) = bint_{\mathfrak{C}}\,\lambda \vee bint_{\mathfrak{C}}\,(\mathfrak{C}\,\lambda)$.

Proof.

By using Definition 3.1, $Bd_{\mathfrak{C}b}\,\lambda = bcl_{\mathfrak{C}}\,\lambda \wedge bcl_{\mathfrak{C}}\,(\mathfrak{C}\lambda)$. Taking complement on both sides, we get $\mathfrak{C}\,(Bd_{\mathfrak{C}b}\,\lambda) = \mathfrak{C}\,(bcl_{\mathfrak{C}}\,\lambda \wedge bcl_{\mathfrak{C}}\,(\mathfrak{C}\lambda))$. Since $\mathfrak{C}$ satisfies the monotonic and involutive conditions, by using Lemma 2.4(ii), $\mathfrak{C}\,(Bd_{\mathfrak{C}b}\lambda) = \mathfrak{C}\,(bcl_{\mathfrak{C}}\,\lambda) \vee \mathfrak{C}\,(bcl_{\mathfrak{C}}\,(\mathfrak{C}\,\lambda))$. Also by using Proposition 2.17(ii), that implies $\mathfrak{C}\,(Bd_{\mathfrak{C}b}\,\lambda) = bint_{\mathfrak{C}}\,(\mathfrak{C}\,\lambda) \vee bint_{\mathfrak{C}}\,(\mathfrak{C}(\mathfrak{C}\lambda))$. Since $\mathfrak{C}$ satisfies the involutive condition, $\mathfrak{C}\,(Bd_{\mathfrak{C}b}\,\lambda) = bint_{\mathfrak{C}}\,\lambda \vee bint_{\mathfrak{C}}\,(\mathfrak{C}\,\lambda)$.

The following example shows that if the monotonic and involutive conditions of the complement function $\mathfrak{C}$ can be dropped, then the conclusion of Proposition 3.12 is false.

Example 3.13

Let X = {a, b} and τ = {0 , {a.3, b.8}, {a.2, b.5}, {a.7, b.1}, {a.3, b.5}, {a.3, b.1}, {a.2, b.1}, {a.7, b.8}, {a.7, b.5}, 1}. Let $\mathfrak{C}(x) = \sqrt{x}$, 0 ≤ x ≤ 1 be the complement function. From this example, we see that $\mathfrak{C}$ does not satisfy the monotonic and involutive conditions. The family of all fuzzy $\mathfrak{C}$-closed sets is $\mathfrak{C}(\tau) = \{0, \{a_{.548}, b_{.894}\}, \{a_{.447}, b_{.707}\},\{a_{.837}, b_{.316}\}, \{_{.548}, b_{.707}\},\{a_{.548}, b_{.316}\}, \{a_{.447}, b_{.316}\},\{a_{.837}, b_{.894}\}, \{a_{.837}, b_{.707}\},1\}$. Let $\lambda = \{a_{.6}, b_{.3}\}$. Then it can be evaluated that $bint_{\mathfrak{C}}\,\lambda = \{a_{.3}, b_{.1}\}$, $\mathfrak{C}\lambda = \{a_{.775}, b_{.548}\}$ and $bint_{\mathfrak{C}}\mathfrak{C}\,\lambda = \{a_{.7}, b_{.5}\}$. Thus we see that $bint_{\mathfrak{C}}\lambda \vee bint_{\mathfrak{C}}\mathfrak{C}\lambda = \{a_{.775}, b_{.548}\}$. It can be computed that $bcl_{\mathfrak{C}}\,\lambda = \{a_{.5}, b_{.8}\}$. Now $\mathfrak{C}\lambda = \{a_{.775}, b_{.548}\}$, $bcl_{\mathfrak{C}}\,\mathfrak{C}\,\lambda = \{a_{.837}, b_{.707}\}$ and $Bd_{\mathfrak{C}b}\,\lambda = bcl_{\mathfrak{C}}\,\lambda \wedge bcl_{\mathfrak{C}}\,(\mathfrak{C}\lambda) = \{a_{.5}, b_{.707}\}$. Also $\mathfrak{C}\,(Bd_{\mathfrak{C}b}\,\lambda) = \{a_{.707}, b_{.840}\}$. Thus we see that $\mathfrak{C}\,(Bd_{\mathfrak{C}b}\,\lambda) \neq bint_{\mathfrak{C}}\,\lambda \vee bint_{\mathfrak{C}}\,\mathfrak{C}\,\lambda$. Therefore the conclusion of Proposition 3.12 is false.

Proposition 3.14

Let (X,τ) be a fuzzy topological space. Let $\mathfrak{C}$ be a complement function that satisfies the monotonic and involutive conditions. Then for any fuzzy subset λ of X, we have $Bd_{\mathfrak{C}b}\,(\lambda) = bcl_{\mathfrak{C}}\,(\lambda) \wedge \mathfrak{C}\,(bint_{\mathfrak{C}}\,(\lambda))$.

Proof.

By using Definition 3.1, we have $Bd_{\mathfrak{C}b}\,(\lambda) = bcl_{\mathfrak{C}}\,(\lambda) \wedge bcl_{\mathfrak{C}}\,(\mathfrak{C}\lambda)$. Since $\mathfrak{C}$ satisfies the monotonic and involutive conditions, by using Proposition 2.9(ii), we have $Bd_{\mathfrak{C}b}(\lambda) = bcl_{\mathfrak{C}}\,(\lambda) \wedge C\,(bint_{\mathfrak{C}}\,(\lambda))$.

The next example shows that if the complement function $\mathfrak{C}$ does not satisfy the monotonic and involutive conditions, then the conclusion of Proposition 3.14 is false.

Example 3.15

Let $X = \{a, b, c\}$ and $\tau = \{0, \{a_{.2}, b_{.6}, c_{.2}\}, \{a_{.7}, b_{.3}, c_{.7}\}, \{a_{.2}, b_{.3}, c_{.2}\}, \{a_{.7}, b_{.6}, c_{.7}\}, 1\}$. Let $\mathfrak{C}(x) = \frac{1-x^3}{(1+x)^2}$, $0 \leq x \leq 1$, be the complement function. We note that the complement function $\mathfrak{C}$ does not satisfy the involutive condition. The family of all fuzzy $\mathfrak{C}$-closed sets is $\mathfrak{C}(\tau)=\{0,\{a_{.689}, b_{.3062}, c_{.689}\},\{a_{.227}, b_{.576}, c_{.227}\}, \{a_{.689}, b_{.576}, c_{.682}\}, \{a_{.227}, b_{.3062}, c_{.227}\}, 1\}$. Let $\lambda = \{a_{.5}, b_{.3062}, c_{.689}\}$, the value of $bcl_{\mathfrak{C}}\,\lambda = \{a_{.5}, b_{.3062}, c_{.689}\}$ and $\mathfrak{C}\,\lambda = \{a_{.389}, b_{.569}, c_{.478}\}$, it follows that $Bd_{\mathfrak{C}b}\,\lambda = bcl_{\mathfrak{C}}\,\lambda \wedge bcl_{\mathfrak{C}}\,(\mathfrak{C}\,\lambda) = \{a_{.389}, b_{.3062}, c_{.4}\}$. Also $\mathfrak{C}\,(bint_{\mathfrak{C}}\lambda) = \{a_{.689}, b_{.576}, c_{.689}\}$. It follows that $bcl_{\mathfrak{C}}\,\lambda \wedge \mathfrak{C}\,(bint_{\mathfrak{C}}\,\lambda) = \{a_{.227}, b_{.3062}, c_{.227}\}$. This shows that $Bd_{\mathfrak{C}b}\,\lambda \neq bcl_{\mathfrak{C}}\,\lambda \wedge \mathfrak{C}(bint_{\mathfrak{C}}\,\lambda)$. Therefore the conclusion of Proposition 3.14 is false.

Proposition 3.16

Let (X,τ) be a fuzzy topological space. Let $\mathfrak{C}$ be a complement function that satisfies the monotonic and involutive conditions. Then for any subset λ of X, $Bd_{\mathfrak{C}b}(b\ int_{\mathfrak{C}}(\lambda)) \leq Bd_{\mathfrak{C}b}(\lambda)$.

Proof.

Since the complement function $\mathfrak{C}$ satisfies the monotonic and involutive conditions, by using Proposition 3.14, we have $Bd_{\mathfrak{C}b}(bint_{\mathfrak{C}}(\lambda)) = bcl_{\mathfrak{C}}(bint_{\mathfrak{C}}(\lambda)) \wedge \mathfrak{C}(bint_{\mathfrak{C}}(bint_{\mathfrak{C}}(\lambda)))$. Since $bint_{\mathfrak{C}}(\lambda)$ is fuzzy $\mathfrak{C}$ -b-open, $Bd_{\mathfrak{C}b}(bint_{\mathfrak{C}}(\lambda)) = bcl_{\mathfrak{C}}(bint_{\mathfrak{C}}(\lambda)) \wedge C(bint_{\mathfrak{C}}(\lambda))$. Since $bint_{\mathfrak{C}}(\lambda) \leq \lambda$, by using Proposition 2.10(ii), $bcl_{\mathfrak{C}}(bint_{\mathfrak{C}}(\lambda)) \leq bcl_{\mathfrak{C}}(\lambda)$. Thus $Bd_{\mathfrak{C}b}(bint_{\mathfrak{C}}(\lambda)) \leq bcl_{\mathfrak{C}}(\lambda) \wedge \mathfrak{C}(bint_{\mathfrak{C}}(\lambda))$. Since $\mathfrak{C}$ satisfies the monotonic and involutive conditions, by using Proposition 2.9, $Bd_{\mathfrak{C}b}(bint_{\mathfrak{C}}(\lambda)) \leq bcl_{\mathfrak{C}}(\lambda) \wedge bcl_{\mathfrak{C}}(\mathfrak{C}\lambda)$. By using Definition 3.1, we have $Bd_{\mathfrak{C}b}(bint_{\mathfrak{C}}(\lambda)) \leq Bd_{\mathfrak{C}b}(\lambda)$.

Proposition 3.17

Let (X,τ) be a fuzzy topological space. Let $\mathfrak{C}$ be a complement function that satisfies the monotonic and involutive conditions. Then $Bd_{\mathfrak{C}b}(bcl_{\mathfrak{C}}(\lambda)) \leq Bd_{\mathfrak{C}b}(\lambda)$.

Proof.

Since $\mathfrak{C}$ satisfies the monotonic and involutive conditions, by using Proposition 3.14, $Bd_{\mathfrak{C}b}(bcl_{\mathfrak{C}}(\lambda)) = bcl_{\mathfrak{C}}(bcl_{\mathfrak{C}}(\lambda)) \wedge \mathfrak{C}(bint_{\mathfrak{C}}(bcl_{\mathfrak{C}}(\lambda)))$. By using Proposition 2.10(iii), we have $bcl_{\mathfrak{C}}(bcl_{\mathfrak{C}}(\lambda)) = bcl_{\mathfrak{C}}(\lambda)$, that implies $Bd_{\mathfrak{C}b}(bcl_{\mathfrak{C}}(\lambda)) = bcl_{\mathfrak{C}}(\lambda) \wedge \mathfrak{C}(bint_{\mathfrak{C}}(bcl_{\mathfrak{C}}(\lambda)))$. Since $\lambda \leq bcl_{\mathfrak{C}}(\lambda)$, that implies $bint_{\mathfrak{C}}(\lambda) \leq bint_{\mathfrak{C}}(cl_{\mathfrak{C}}(\lambda))$. Therefore, $Bd_{\mathfrak{C}b}(bcl_{\mathfrak{C}}(\lambda)) \leq bcl_{\mathfrak{C}}(\lambda) \wedge \mathfrak{C}(bint_{\mathfrak{C}}(\lambda))$. By using Proposition 2.9 (ii), and by using Definition 3.1, we get $Bd_{\mathfrak{C}b}(bcl_{\mathfrak{C}}(\lambda)) \leq Bd_{\mathfrak{C}b}(\lambda)$.

Theorem 3.18

Let (X,τ) be a fuzzy topological space. Let $\mathfrak{C}$ be a complement function that satisfies the monotonic and involutive conditions. Then $Bd_{\mathfrak{C}b}(\lambda \vee \mu) \leq Bd_{\mathfrak{C}b}\lambda \vee Bd_{\mathfrak{C}b}\mu$.

Proof.

By using Definition 3.1, $Bd_{\mathfrak{C}b}(\lambda \vee \mu) = bcl_{\mathfrak{C}}(\lambda \vee \mu) \wedge bcl_{\mathfrak{C}}(\mathfrak{C}(\lambda \vee \mu))$. Since $\mathfrak{C}$ satisfies the monotonic and involutive conditions, by using Proposition 2.11(i), that implies $Bd_{\mathfrak{C}b}(\lambda \vee \mu) = (bcl_{\mathfrak{C}}(\lambda) \vee bcl_{\mathfrak{C}}(\mu)) \wedge bcl_{\mathfrak{C}}(\mathfrak{C}(\lambda \vee \mu))$. By using Lemma 2.4 and Proposition 2.11(ii), $Bd_{\mathfrak{C}b}(\lambda \vee \mu) \leq (bcl_{\mathfrak{C}}(\lambda) \vee bcl_{\mathfrak{C}}(\mu)) \wedge (bcl_{\mathfrak{C}}(\mathfrak{C}\lambda) \wedge bcl_{\mathfrak{C}}(\mathfrak{C}\mu))$. That is, $Bd_{\mathfrak{C}b}(\lambda \vee \mu) \leq$

$(bcl_{\mathfrak{C}}(\lambda) \wedge bcl_{\mathfrak{C}}(\mathfrak{C}\lambda)) \vee (bcl_{\mathfrak{C}}(\mu)) \wedge bcl_{\mathfrak{C}}(\mathfrak{C}\mu))$. Again by using Definition 3.1, $Bd_{\mathfrak{C}b}(\lambda \vee \mu) \leq Bd_{\mathfrak{C}b}(\lambda) \vee Bd_{\mathfrak{C}b}(\mu)$.

Theorem 3.19

Let (X,τ) be a fuzzy topological space. Suppose the complement function $\mathfrak{C}$ satisfies the monotonic and involutive conditions. Then for any two fuzzy subsets λ and μ of a fuzzy topological space X, we have $Bd_{\mathfrak{C}b}(\lambda \wedge \mu) \leq (Bd_{\mathfrak{C}b}(\lambda) \wedge bcl_{\mathfrak{C}}(\mu)) \vee (Bd_{\mathfrak{C}b}(\mu) \wedge bcl_{\mathfrak{C}}(\lambda))$.

Proof.

By using Definition 3.1, we have $Bd_{\mathfrak{C}b}(\lambda \wedge \mu) = bcl_{\mathfrak{C}}(\lambda \wedge \mu) \wedge bcl_{\mathfrak{C}}(C(\lambda \wedge \mu))$. Since $\mathfrak{C}$ satisfies the monotonic and involutive conditions, by using Proposition 2.11(i), Proposition 2.11 (ii) and by using Lemma 2.4(iv), we get $Bd_{\mathfrak{C}b}(\lambda \wedge \mu) \leq (bcl_{\mathfrak{C}}(\lambda) \wedge bcl_{\mathfrak{C}}(\mu)) \wedge (bcl_{\mathfrak{C}}(\mathfrak{C}\lambda) \vee bcl_{\mathfrak{C}}(\mathfrak{C}\mu))$ is equal to $[bcl_{\mathfrak{C}}(\lambda) \wedge bcl_{\mathfrak{C}}(\mathfrak{C}\lambda)] \wedge (bcl_{\mathfrak{C}}(\mu)) \vee [bcl_{\mathfrak{C}}(\mu) \wedge bcl_{\mathfrak{C}}(\mathfrak{C}\mu)] \wedge bcl_{\mathfrak{C}}(\lambda)$. Again by Definition 3.1, we get $Bd_{\mathfrak{C}b}(\lambda \wedge \mu) \leq (Bd_{\mathfrak{C}b}(\lambda) \wedge bcl_{\mathfrak{C}}(\mu)) \vee (Bd_{\mathfrak{C}b}(\mu) \wedge bcl_{\mathfrak{C}}(\lambda))$.

Proposition 3.20

Let (X, τ) be a fuzzy topological space. Suppose the complement function $\mathfrak{C}$ satisfies the monotonic and involutive conditions. Then for any fuzzy subset λ of a fuzzy topological space X, we have (i) $Bd_{\mathfrak{C}b}(Bd_{\mathfrak{C}b}(\lambda)) \leq Bd_{\mathfrak{C}b}(\lambda)$

(ii) $Bd_{\mathfrak{C}b}\, Bd_{\mathfrak{C}b}\, Bd_{\mathfrak{C}b}\, \lambda \leq Bd_{\mathfrak{C}b}\, Bd_{\mathfrak{C}b}\, \lambda$.

Proof.

By using Definition 3.1, $Bd_{\mathfrak{C}b}\,\lambda = bcl_{\mathfrak{C}}\,\lambda \wedge bcl_{\mathfrak{C}}(\mathfrak{C}\lambda)$. We have $Bd_{\mathfrak{C}b}\, Bd_{\mathfrak{C}b}\lambda = bcl_{\mathfrak{C}}(Bd_{\mathfrak{C}b}\,\lambda) \wedge bcl_{\mathfrak{C}}[C(Bd_{\mathfrak{C}b}\lambda)] \leq bcl_{\mathfrak{C}}(Bd_{\mathfrak{C}b}\,\lambda)$. Since $\mathfrak{C}$ satisfies the monotonic and involutive conditions, by using Proposition 2.17(ii), $bcl_{\mathfrak{C}}\,\lambda = \lambda$, where λ is fuzzy $\mathfrak{C}$ -b-closed. Here $Bd_{\mathfrak{C}b}$ is fuzzy $\mathfrak{C}$ - b-closed. So, $bcl_{\mathfrak{C}}(Bd_{\mathfrak{C}b}\,\lambda) = Bd_{\mathfrak{C}b}\,\lambda$. This implies that $Bd_{\mathfrak{C}b}\, Bd_{\mathfrak{C}b}\,\lambda \leq Bd_{\mathfrak{C}b}\,\lambda$. This proves (i).

(ii) Follows from (i).

Proposition 3.21

Let λ be a fuzzy $\mathfrak{C}$-b-closed subset of a fuzzy topological space X and μ be a fuzzy $\mathfrak{C}$-b-closed subset of a fuzzy topological space Y, then $\lambda \times \mu$ is a fuzzy $\mathfrak{C}$-b- closed set of the fuzzy product space X × Y where the complement function $\mathfrak{C}$ satisfies the monotonic and involutive conditions.

Proof.

Let λ be a fuzzy ℭ-b-closed subset of a fuzzy topological space X. Then by applying Lemma 2.4, ℭ λ is fuzzy ℭ -b-open set in X. Also if ℭ λ is fuzzy ℭ - b-open set in X, then ℭ λ × 1 is fuzzy ℭ -b-open in X × Y. Similarly let μ be a fuzzy ℭ -b-closed subset of a fuzzy topological space X. Then by using Lemma 2.4, ℭ μ is fuzzy ℭ-b-open set in Y. Also if ℭ μ is fuzzy ℭ -b-open set in Y then ℭ μ × 1 is fuzzy ℭ-b-open in X × Y. Since the arbitrary union of fuzzy ℭ - b-open sets is fuzzy ℭ- b-open. So, ℭλ × 1∨ 1 × ℭ μ is fuzzy ℭ-b-open in X × Y. We now that ℭ(λ × μ) = ℭλ × 1 ∨ 1× ℭμ, hence ℭ (λ × μ) is fuzzy ℭ- b-open. By using Lemma 2.4, λ × μ is fuzzy ℭ - b- closed of the fuzzy product space X × Y.

Theorem 3.22

Let f: X→Y be a fuzzy continuous function. Suppose the complement function ℭ satisfies the monotonic and involutive conditions. Then $Bd_{\mathfrak{C}b}$ (f^{-1}(μ)) ≤ f^{-1} ($Bd_{\mathfrak{C}b}$ (μ)), for any fuzzy subset μ in Y.

Proof.

Let f be a fuzzy continuous function and μ be a fuzzy subset in Y. By using Definition 3.1, we have $Bd_{\mathfrak{C}f}$ (f^{-1}(μ)) = $bcl_{\mathfrak{C}}$ (f^{-1}(μ)) ∧ $bcl_{\mathfrak{C}}$ (ℭ(f^{-1}(μ)). By using f^{-1} (ℭ μ) = ℭ(f^{-1}(μ)), $Bd_{\mathfrak{C}b}$ (f^{-1} (μ)) = $bcl_{\mathfrak{C}}$ (f^{-1}(μ)) ∧ $bcl_{\mathfrak{C}}$ (f^{-1}(ℭμ)). Since f is fuzzy continuous and f^{-1}(μ) ≤ f^{-1}($cl_{\mathfrak{C}f}$ (μ)), it follows that $fcl_{\mathfrak{C}}$ (f^{-1}(μ)) ≤ f^{-1}($fcl_{\mathfrak{C}}$ (μ)). This together with the above imply that $Bd_{\mathfrak{C}f}$ (f^{-1}(μ)) ≤ f^{-1}($fcl_{\mathfrak{C}}$ (μ))∧f^{-1}($fcl_{\mathfrak{C}}$ (ℭμ)). By using Lemma 2.11, $Bd_{\mathfrak{C}f}$ (f^{-1}(μ)) ≤ f^{-1}($fcl_{\mathfrak{C}}$ μ)∧ $fcl_{\mathfrak{C}}$ (ℭμ)). That is $Bd_{\mathfrak{C}f}$(f^{-1} (μ)) ≤ f^{-1}($Bd_{\mathfrak{C}f}$(μ)).

4. Comparative study on Fuzzy boundaries

Here we introduce the other two Fuzzy boundaries and do some comparative study. For simplicity let us denote Fuzzy ℭ-b- boundary as $\mathfrak{C}b\ \partial_2\lambda$ and is defined as, $\mathfrak{C}b\ \partial_2\lambda$ = $bcl\mathfrak{C}$ λ∧ $bcl\mathfrak{C}$(ℭλ).

Definition 4.1

The fuzzy w- ℭ b- boundary of a fuzzy set λ in a fuzzy topological space X is denoted by $\mathfrak{C}b\partial_1\lambda$ and is defined as the infimum of all fuzzy ℭ-b-closed sets D in X with the property D(x) ≥ $bcl\mathfrak{C}\lambda(x)$ for all $x \in X$ for which [$bcl\mathfrak{C}$ λ∧$bcl\mathfrak{C}$(ℭλ)](x) > 0.

Definition 4.2

The fuzzy C $\mathfrak{C}$-b- boundary of fuzzy set λ in a fuzzy topological space (X,τ) is defined as the infimum of all $\mathfrak{C}$-b- closed fuzzy sets D in X with the property D(x) ≥ $[bcl\mathfrak{C}\ \lambda](x)$ for all $x\in X$ for which $[bcl\mathfrak{C}\ \lambda \wedge bcl\mathfrak{C}(\mathfrak{C}\lambda)]$ (x) > 0. We shall represent it by $\mathfrak{C}b\partial_3\lambda$.

Interrelationship among generalized boundaries:

1. $bcl\mathfrak{C}\ \lambda \geq \mathfrak{C}b\partial_1\lambda \geq \mathfrak{C}b\partial_3\lambda$.

2. $\mathfrak{C}b\partial_1\lambda \geq \mathfrak{C}b\partial_2\lambda$.

That is $\mathfrak{C}b\partial_1\lambda$ contains the other two $\mathfrak{C}$-b-boundaries, where all are contained in the $\mathfrak{C}$-b-closure of the set.

Remark 4.3

Fuzzy $\mathfrak{C}$-b- boundary and fuzzy $C\mathfrak{C}$-b- boundary are independent of each other.

Example 4.4

Let X = { a, b} be a set with the Fuzzy topology , $\mathfrak{I}$= { 0, {a.4 , b.8}, {a.6, b.9}, {a.5,b.7}, {a.8, b.7}, {a.3, b.2}, {a.4, b.2}, {a.5, b.2}, {a.6, b.7}, {a.5, b.8}, {a.8, b.8}, {a.4, b.7}, {a.6, b.8}, {a.8 ,b.9}, 1}. Let λ = {a. 4,b.8}, μ= {a.6, b.8}. Then $\mathfrak{C}b\ \partial_1\lambda$ = {a.5, b.8} $\mathfrak{C}b\partial_1\mu$ = {a.6, b.8} . $\mathfrak{C}b\partial_2\lambda$ = {a.5, b.2} $\mathfrak{C}b\partial_2\mu$ = {a.4, b.2} . $\mathfrak{C}b\ \partial_3\lambda$ = {a.5, b.8} $\mathfrak{C}b\partial_3\mu$ = {0} . Where our $\mathfrak{C}$- complement function is just (1-x). Hence $\mathfrak{C}b\partial_3\lambda \subsetneqq \mathfrak{C}b\partial_2 A$ and $\mathfrak{C}b\partial_2 A \subsetneqq \mathfrak{C}b\partial_3 A$.

Properties 4.5

i. $bcl\mathfrak{C}\ \lambda = \lambda \vee \mathfrak{C}b\partial_i\lambda = bint\ \lambda \vee \mathfrak{C}b\partial_i\lambda$ where i=1,3

ii. $bcl\mathfrak{C}\ \lambda \geq bint\ \lambda \vee \mathfrak{C}b\ \partial_2\lambda$

Proof.

(i)This part of proof has two cases.

Case (a): When i=1, If $[bcl\mathfrak{C}\ \lambda \wedge bcl\mathfrak{C}(\mathfrak{C}\lambda)] > 0$ then $\mathfrak{C}b\partial_1\lambda = bcl\mathfrak{C}\ \lambda$. If $bcl\mathfrak{C}\ \lambda \wedge bcl\mathfrak{C}(\mathfrak{C}A) = 0$ then $bcl\mathfrak{C}(\mathfrak{C}\lambda)=0$ then $\mathfrak{C}\lambda=0 \Rightarrow \lambda=1$ that is $bint\lambda = bcl\mathfrak{C}\ \lambda = 1 \Longrightarrow bint\ \lambda \vee \mathfrak{C}b\ \partial_1\lambda = bcl\mathfrak{C}\ \lambda$. $bint\ \lambda \leq \lambda \leq bcl\mathfrak{C}\ \lambda$. $\Longrightarrow \lambda \vee \mathfrak{C}b\partial_1\lambda = bcl_{\mathfrak{C}}\ \lambda$.

Case (b): When i=3, We know that $\mathfrak{C}b\partial_3\lambda \leq bcl_{\mathfrak{C}}\lambda$ and $bint\ \lambda \leq bcl\mathfrak{C}\ \lambda$. So, $int\ \lambda \vee \mathfrak{C}b\partial_3\lambda \leq bcl\mathfrak{C}\ \lambda$. But if $bcl\mathfrak{C}\ \lambda - bint\ \lambda > 0 \Rightarrow bcl\mathfrak{C}\ \lambda = \mathfrak{C}b\partial_3\lambda$. Also if $bcl\mathfrak{C}\lambda - bint\ \lambda = 0$ then $bcl\mathfrak{C}\ \lambda = bint\ \lambda$ .Thus $bcl\mathfrak{C}\ \lambda = bint\ \lambda \vee \mathfrak{C}b\partial_3\lambda \leq bcl\mathfrak{C}\lambda$.

(ii)We have, Since by definition $bint\ \lambda \vee \mathfrak{C}b\partial_2\lambda = \lambda \vee [\ bcl\mathfrak{C}\ \lambda \wedge bcl\mathfrak{C}\ (\mathfrak{C}\lambda)\ ] = (\lambda \vee bcl\mathfrak{C}\ \lambda\ ) \wedge bcl\mathfrak{C}\ (\mathfrak{C}\lambda)$. Also $bcl\mathfrak{C}\ \lambda \wedge bcl\mathfrak{C}\ (\mathfrak{C}\lambda) \leq bcl\mathfrak{C}\lambda$. Similarly, $\lambda \vee \mathfrak{C}b\ \partial_2\lambda \leq bcl\mathfrak{C}\ \lambda$.

Remark 4.6

1. If λ is fuzzy ℭ -b-closed set if and only if $\mathfrak{C}b\partial_2\lambda \leq \mathfrak{C}\lambda$.

2. If λ is fuzzy ℭ-b-closed then $\mathfrak{C}b\,\partial_2\lambda \leq \lambda$.

Theorem 4.7

Let λ be a fuzzy ℭ- b-closed set if and only if $\mathfrak{C}b\partial_i\lambda \leq \lambda$, i=1, 3

Proof.

(1) We have $\mathfrak{C}b\,\partial_1\lambda \leq bcl\mathfrak{C}\,\lambda = \lambda$. That is $\mathfrak{C}b\,\partial_1\lambda \leq \lambda$. Then $bcl\mathfrak{C}\,\lambda = \lambda \vee \mathfrak{C}b\,\partial_1\lambda \Rightarrow bcl\mathfrak{C}\,\lambda = \lambda \Longrightarrow \lambda$ is Fuzzy ℭ -b-closed. $\mathfrak{C}b\partial_3\lambda \leq bcl\mathfrak{C}\,(\lambda) = \lambda$ then if $\mathfrak{C}b\partial_3\lambda \leq \lambda \Rightarrow bcl\mathfrak{C}\,\lambda = \lambda \vee \mathfrak{C}b\partial_3\lambda = \lambda$. Therefore λ is fuzzy ℭ -b-closed.

Remark 4.8

If $\{bcl\mathfrak{C}\,\lambda \wedge bcl\mathfrak{C}\,(\mathfrak{C}\lambda)\} > 0$ then (i) $\mathfrak{C}b\partial_1\lambda \geq \lambda$ and (ii) $\mathfrak{C}b\partial_2\lambda \leq bcl\mathfrak{C}\,\lambda$.

Theorem 4.9

For any fuzzy set λ in the fuzzy topological space, (i) $\mathfrak{C}b\partial_i[\mathfrak{C}b\partial_i\lambda] \leq \mathfrak{C}b\,\partial_i\lambda$, ($i$=1,3) (ii) $\mathfrak{C}b\partial_i[\mathfrak{C}b\,\partial_i[\mathfrak{C}p\partial_i\lambda]] \leq \mathfrak{C}b\,\partial_i[\mathfrak{C}b\partial_i\lambda]$, ($i$=1,3)

Proof.

(i) Since $\mathfrak{C}b\partial_i\lambda$, i=1, 3 is fuzzy ℭ-b-closed, $\mathfrak{C}b\,\partial_i[\mathfrak{C}b\,\partial_i\lambda] \leq \mathfrak{C}b\,\partial_i\lambda$, ($i$=1,3) .

(ii) Proof is straight forward.

Remark 4.10

If the intersection of the ℭ-b-closure of the set and the ℭ-complement of the set is empty then value of all the three forms of ℭ-b-boundaries are equal.

REFERENCES

[1] B.Ahmad and Ahar Kharal, *Fuzzy sets, fuzzy s-open and s-closed mappings*,Advances in Fuzzy system, Article Id 303042(2009).

[2] M.Athar and B.Ahmad, *Fuzzy boundary and Fuzzy semiboundary*, Advances in Fuzzy systems(2008), Article ID 586893, 9 pages.

[3] K.K.Azad, *On Fuzzy semi-continuity, fuzzy almost continuity and fuzzy weakly continuity weakly continuity*, J.Math.Anal.Appl.82 (1) (1981), 14-32.

[4] K.Bageerathi, G.Sudha, P.Thangavelu, *A generalization of fuzzy closed sets*, International journal of fuzzy systems and rough systems4(1),(2011), 1-5.

[5] C.L.Chang, *Fuzzy Topological Space*, J.Math.Anal.Appl., 24(1968), 182-190.

[6] E.Cuchillo-Ibanez and J.Tarres, *On the boundary of fuzzy sets*, Fuzzy sets and systems 89(1997), 113-119.

[7] George J. Klir and Bo Yuan, *Fuzzy Sets and Fuzzy Logic Theory and Applications*, prentice-Hall, Inc., 2005.

[8] D. Hazarika and D.Hazarika, *On Certain Generalizations of Fuzzy Boundary*, Int.Math.Forum, vol 6, 46(2011), 2293-2303.

[9] D. Hazarika and D. Hazarika, *A comparative study of fuzzy boundaries and their applications*, 19th International Conference of the Forum for interdisciplinary Mathematics, Patna University, India, Dec 18 - 20, 2010.

[10] A.K. Katsaras and D.B.Liu, *Fuzzy vector spaces and fuzzy topological vector spaces*, J. Math. Anal. Appl. 8(3)(1978), 459-470.

[11] N. Palaniappan, *Fuzzy Topology*, Narosa, 2005.

[12] P.M. Pu and Y.M.Liu, *Fuzzy topology-I: neighborhood structure of a fuzzy point and Moore-Smith convergence*, J.Math.Anal.Appl. 76(2)(1980), 571-599.

[13] K.Bageerathi, Generalization of fuzzy b-open sets & fuzzly b-closed sets in fuzzy topological spaces, *International Journal of ultra Scientist of Physical Sciences*, (communicated).

[14] K.Bageerathi, On fuzzy ℭ-b- irresolute and fuzzy ℭ-b- continuous functions, *International Journal of Applied Mathematics and Applications*, (communicated).

[15] R. H. Warren, *Boundary of a fuzzy set*, Indiana Univ. Math. J. **26** (1977), 191–197.

[16] T. H. Yalvac, *Semi-interior and semi-closure of a fuzzy set*, J. Math. Anal. Appl. 132 (1988), 356–364.

[17] L.A. Zadeh, *Fuzzy Set*", Inform. And Control, 8(1965), 338-353

[18] H. J. Zimmermann, *Fuzzy set theory and its applications*, Kluwer Academic Publishers, 1991.