NIJENHUIS TENSOR IN AN L-CONTACT MANIFOLD

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ABSTRACT

In 1989, K. Matsumoto [2] introduced the notion of manifolds with Lorentzian paracontact metric structure similar to the almost paracontact metric structure which is defined by I. Sato [3], [4]. The purpose of this paper is to study the Nijenhuis tensor in various forms in an L-Contact manifold.

Keywords: L-Contact manifold, Nijenhuis tensor.

1. Introduction

An n-dimensional differentiable manifold M_n, on which there are defined a tensor field $F$of type (1, 1), a vector field T, a 1-form A and a Lorentzian metric g, satisfying for arbitrary vector fields $X, Y, Z, \ldots$

(1.1) $\overline{\overline{X}} = -X - A(X)T,\ \overline{T} = 0,\ A(T) = -1,\ \overline{X} \overset{\text{def}}{=} FX,\ A(\overline{X}) = 0,$ rank $F = n - 1$

(1.2) $g\ (\overline{X}, \overline{Y}) = g\ (X, Y) + A(X)A(Y),$ where $A(X) \overset{\text{def}}{=} g(X, T),$

$`F(X, Y) \overset{\text{def}}{=} g\left(\overline{X},\ Y\right) = -g\left(\overline{Y},\ X\right) = -`F(Y, X),$

Then M_n is called a Lorentzian contact manifold (an L-Contact manifold) and the structure (F, T, A, g) is known as Lorentzian contact structure (an L-Contact structure).

2. Nijenhuis Tensor

Nijenhuis tensor [1] is a vector valued bilinear function given by

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(2.1) $N(X,Y) = [\bar{X},\bar{Y}] + \overline{\overline{[X,Y]}} - \overline{[\bar{X},Y]} - \overline{[X,\bar{Y}]}$

Therefore, Nijenhuis tensor in L-Contact manifold is given by

$$N(X,Y) = [\bar{X},\bar{Y}] - [X,Y] - \overline{[\bar{X},Y]} - \overline{[X,\bar{Y}]} - A([X,Y])T$$

Theorem 2.1 Let

(2.2) $B(X,Y) \overset{\text{def}}{=} \overline{[\bar{X},Y]} + \overline{[X,\bar{Y}]}$

Then

(2.3) (a) $B(\bar{X},\bar{Y}) + B(X,Y) = -A(X)\overline{[T,\bar{Y}]} - A(Y)\overline{[\bar{X},T]}$

(b) $B(\bar{X},Y) - B(X,\bar{Y}) = -A(X)\overline{[T,Y]} + A(Y)\overline{[X,T]}$

Proof. Barring X and Y in (2.2) and using (1.1), then adding to (2.2), we get (2.3) (a). Again barring X and Y in (2.2) separately and using (1.1), we obtain (2.3) (b).

Theorem 2.2 Let

(2.4) $C(X,Y) \overset{\text{def}}{=} [\bar{X},\bar{Y}] - [X,Y] - A([X,Y])T$

Then

(2.5) (a) $C(\bar{X},Y) = -[X,\bar{Y}] - [\bar{X},Y] - A(X)[T,\bar{Y}] - A([\bar{X},Y])T$

(b) $C(X,\bar{Y}) = -[X,\bar{Y}] - [\bar{X},Y] - A(Y)[\bar{X},T] - A([X,\bar{Y}])T$

(c) $C(\bar{X},\bar{Y}) = -[\bar{X},\bar{Y}] + [X,Y] + A(X)[T,Y] + A(Y)[X,T] - A([\bar{X},\bar{Y}])T$

(d) $C(\bar{X},Y) - C(X,\bar{Y}) = -A(X)[T,\bar{Y}] - A([\bar{X},Y])T + A(Y)[\bar{X},T] + A([X,\bar{Y}])T$

(e) $C(\bar{X},\bar{Y}) + C(X,Y) = A(X)[T,Y] + A(Y)[X,T] - A([\bar{X},\bar{Y}])T - A([X,Y])T$

(f) $\overline{C(\bar{X},Y)} - \overline{C(X,\bar{Y})} = -A(X)\overline{[T,\bar{Y}]} + A(Y)\overline{[\bar{X},T]}$

(g) $\overline{C(\overline{X},\overline{Y})} + \overline{C(X,Y)} = A(X)\overline{[T,Y]} + A(Y)\,\overline{[X,T]}$

Proof. Barring X and Y separately in (2.4) and using (1.1), we get (2.5) (a) and (2.5) (b) respectively. Barring X and Y both in (2.4), then using (1.1), we get (2.5) (c). Remaining equations follows from (2.4) and (2.5) (a), (2.5) (b), (2.5) (c).

Theorem 2.3 We have

(2.6) (a) $N(X,Y) = C(X,Y) + \overline{C(\overline{X},Y)} + A(X)\overline{[T,\overline{Y}]}$

(b) $N(\overline{X},Y) = C(\overline{X},Y) - \overline{C(X,Y)} + A(X)\overline{[T,Y]}$

Proof. Barring (2.5) (a) and adding with (2.4), then using (2.1), we get (2.6) (a). (2.6) (b) follows from (2.5) (a), (2.4), (2.1) and (1.1).

Corollary 2.1 We have

(2.7) (a) $A(C(X,Y)) = A([\overline{X},\overline{Y}])$

(b) $A(C(\overline{X},Y)) = -A([X,\overline{Y}]) - A(X)A([T,\overline{Y}])$

(c) $A\left(C(X,\overline{Y})\right) = -A([\overline{X},Y]) - A(Y)A([\overline{X},T])$

(d) $A\left(C(\overline{X},\overline{Y})\right) = A([X,Y]) + A(X)A([T,Y]) + A(Y)A([X,T])$

Proof. Applying the 1-form A in the equations (2.4), (2.5) (a), (2.5) (b), (2.5) (c) and using (1.1), we obtain (2.7) (a), (2.7) (b), (2.7) (c), (2.7) (d) respectively.

Corollary 2.2 We have

(2.8) (a) $C(X,T) = -[X,T] - A([X,T])T$

(b) $A\left(C(X,T)\right) = 0$

Corollary 2.3 We have

(2.9) (a) $A(N(X,Y)) = A(C(X,Y))$

(b) $A(N(\overline{X},Y)) = A(C(\overline{X},Y))$

Corollary 2.4 We have

(2.10) (a) $N(X,T) = C(X,T) + \overline{C(\overline{X},T)}$

(b) $N(\overline{X},T) = C(\overline{X},T) - \overline{C(X,T)}$

Corollary 2.5 We have

(2.11) (a) $B(X,Y) + \overline{C(\overline{X},Y)} = -A(X)\overline{[T,\overline{Y}]}$

(b) $B(\overline{X},Y) + \overline{C(\overline{X},\overline{Y})} = A(Y)\overline{[X,T]}$

(c) $B(\overline{X},\overline{Y}) - \overline{C(X,\overline{Y})} = -A(X)\overline{[T,\overline{Y}]}$

Theorem 2.4 Let

(2.12) $E(X,Y) \stackrel{\text{def}}{=} [\overline{X},\overline{Y}] - [X,Y],$

Then

(2.13) $E(X,Y) + \overline{E(\overline{X},Y)} = N(X,Y) + A([X,Y])T - A(X)\overline{[T,\overline{Y}]}$

Proof. Barring X in (2.12) and using (1.1), we get

$$E(\overline{X},Y) = -[X,\overline{Y}] - A(X)[T,\overline{Y}] - [\overline{X},Y]$$

Barring the whole equation and using (1.1), we obtain

$$\overline{E(\overline{X},Y)} = -\overline{[X,\overline{Y}]} - A(X)\overline{[T,\overline{Y}]} - \overline{[\overline{X},Y]}$$

Adding this equation to (2.12) and using (2.1), we get (2.13).

Corollary 2.6 The equation (2.13) is equivalent to

(2.14) $E(T,Y) = N(T,Y) + A([T,Y])T + \overline{[T,\overline{Y}]}$

Proof. Putting T for X in (2.13) and using (1.1), we obtain (2.14).

Theorem 2.5 Let

(2.15) $H(X,Y) \overset{\text{def}}{=} [\bar{X},\bar{Y}] - \overline{[\bar{X},Y]},$

Then

(2.16) $H(X,Y) + \overline{H(\bar{X},Y)} = N(X,Y) - A(X)\overline{[T,\bar{Y}]} - A(X)[T,Y] - A(X)A([T,Y])T$

Proof. Barring X in (2.15) and using (1.1), we get

$$H(\bar{X},Y) = -[X,\bar{Y}] - A(X)[T,\bar{Y}] + \overline{[X,Y]} + A(X)\overline{[T,Y]}$$

Barring the whole equation and using (1.1), we obtain

$$\overline{H(\bar{X},Y)} = -\overline{[X,\bar{Y}]} - A(X)\overline{[T,\bar{Y}]} - [X,Y] - A([X,Y])T - A(X)[T,Y] - A(X)A([T,Y])T$$

Adding this to (2.15) and using (2.1), we get (2.16).

Corollary 2.7 The equation (2.16) is equivalent to

(2.17) $H(T,Y) = N(T,Y) + \overline{[T,\bar{Y}]} + [T,Y] + A([T,Y])T$

Proof. Putting T for X in (2.16) and using (1.1), we obtain (2.17).

Theorem 2.6 Let

(2.18) $P(X,Y) \overset{\text{def}}{=} [\bar{X},\bar{Y}] - \overline{[X,\bar{Y}]},$

Then

(2.19) $P(X,Y) + \overline{P(X,\bar{Y})} = N(X,Y) - A(Y)\overline{[\bar{X},T]} - A(Y)[X,T] - A(Y)A([X,T])T$

Proof. Barring Y in (2.18) and using (1.1), we get

$$P(X,\bar{Y}) = -[\bar{X},Y] - A(Y)[\bar{X},T] + \overline{[X,Y]} + A(Y)\overline{[X,T]})$$

Barring the whole equation and using (1.1), we obtain

$$\overline{P(X,\overline{Y})} = -[\overline{\overline{X},Y}] - A(Y)[\overline{\overline{X},T}] - [X,Y] - A([X,Y])T - A(Y)[X,T] - A(Y)A([X,T])T$$

Adding this to (2.18) and using (2.1), we get (2.19).

Corollary 2.8 The equation (2.19) is equivalent to

(2.20) $P(X,T) = N(X,T) + \overline{[\overline{X},T]} + [X,T] + A([X,T])T$

Proof. Putting T for Y in (2.19) and using (1.1), we obtain (2.20).

Theorem 2.7 In an L-Contact manifold, we have

(2.21) $E(T,Y) - H(T,Y) = -[T,Y]$

Proof. The result follows from (2.14) and (2.17).

Theorem 2.8 In an L-Contact manifold, we have

(2.22) $H(X,Y) + \overline{H(\overline{X},Y)} = E(X,Y) + \overline{E(\overline{X},Y)} - A([X,Y])T - A(X)[T,Y] - A(X)A([T,Y])T$

Proof. The result follows from (2.13) and (2.16).

References:

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