On a New Summation Formula for 2ψ2

Basic Bilateral Hypergeometric Series and Its Applications

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Abstract:

We have obtained a new summation formula for 2ψ2 bilateral basic hypergeometric series by the method of parameter augmentation and demonstrated its various uses leading to some development of eta functions, q-gamma, and q-beta function identities.

Keywords: Bilateral Hypergeometric Series, Summation Formula, 2ψ2,

Introduction

The summation formulae for hypergeometric series form a very interesting and useful component of the theory of basic hypergeometric series. The q-binomial theorem of Cauchy [1]is perhaps the first identity in the class of the summation formulae, which can be stated as

$$\sum_{k=0}^{\infty}\frac{(a)_k}{(q)_k}z^k=\frac{(az)_\infty}{(z)_\infty},\quad |z|<1,|q|<1, \tag{1.1}$$

Where

$$(a)_\infty := (a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k), \tag{1.2}$$

$$(a)_\infty := (a;q)_\infty := \frac{(a)_\infty}{(aq^k)_\infty}\text{, k is an integer.}$$

For more details about the q-binomial theorem and about the identities which fall in this sequel, one may refer to the book by Gasper and Rahman [2]. Another famous identity in the sequel is the Ramanujan's 1ψ1 summation formula [3]

$$\sum_{k=0}^{\infty}\frac{(a)_k}{(b)_k}z^k=\frac{(az)_\infty(q)_\infty(q/az)_\infty(b/a)_\infty}{(z)_\infty(b)_\infty(b/az)_\infty(q/a)_\infty},\quad \left|\frac{b}{a}\right|<|z|<1,|q|<1, \tag{1.3}$$

There are a number of proofs of the 1ψ1 summation formula 1.3 in the literature. For more details, one refers to the book by Berndt [4]and a recent paper of Johnson [5].

In this paper, we derive a new summation formula for 2ψ2 basic bilateral hypergeometric series using the 1ψ1 summation formula 1.3 by the method of parameter augmentation. We then use the formula to derive the q-analogue of Gauss summation formula and to obtain a number of etafunction, q-gamma, and q-beta function identities, which complement the works of Bhargava and Somashekara [6], Bhargava et al. [7], Somashekara and Mamta [8], Srivastava [9], and Bhargava and Adiga [10].

First, we recall that q-difference operator and the q-shift operator are defined by

$$D_q f(a) = \frac{f(a) - f(aq)}{a}, \quad \zeta\big(f(a)\big) = f(aq), \tag{1.4}$$

respectively. In [11], Chen and Liu have constructed an operator θ as

$$\theta = \zeta^{-1} D_q, \tag{1.5}$$

and thereby they defined the operator $E(b\theta)$ as

$$E(b\theta) = \sum_{k=0}^{\infty} \frac{(b\theta)^k q^{k(k-1)/2}}{(q;q)_k}. \tag{1.6}$$

Then, we have the following operator identities [12, Theorem 1]:

$$E(b\theta)\{(at;q)_\infty\} = (at, bt; q)_\infty,$$

$$E(b\theta)\{(as, bt; q)_\infty\} = \frac{(as, at, bs, bt; q)_\infty}{(abst/q; q)_\infty}, \qquad \left|\frac{abst}{q}\right| < 1. \tag{1.7}$$

Further, the Dedekind etafunction is defined by

$$\eta(\tau) \coloneqq e^{\pi i\tau/12} \prod_{k=1}^{\infty} \left(1 - e^{2\pi i k\tau}\right) \coloneqq q^{1/24} (q;q)_\infty,$$

where $q = e^{2\pi i\tau}$, and Im $(\tau) > 0$.

Jackson [13] defined the q-analogue of the gamma function by

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty} (1-q)^{1-x}, \quad 0 < q < 1.$$

In his paper on the q-gamma and q-beta function, Askey [14]has obtained q-analogues of several classical results about the gamma function. Further, he has given the definition for q-beta function as

$$\mathrm{B}_q(x,y) = (1-q)\sum_{k=0}^{\infty} q^{nx}\frac{(q^{n+1})_\infty}{(q^{x+y})_\infty} \tag{1.10}$$

In fact, he has shown that

$$\mathrm{B}_q(x,y) = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}. \tag{1.11}$$

In Section 2, we prove our main result. In Section 3, we deduce the well-known qanalogue of the Gauss summation formula and some etafunction, q-gamma, and q-beta function identities.

Result and Discussion:

Theorem 2.1.$If\ 0 < |z| < 1, |q| < 1$, then

$$\sum_{k=-\infty}^{\infty}\frac{(a)_k(bc/azq)_k}{(b)_k(c)_k}z^k = \frac{(az)_\infty(q)_\infty(q/az)_\infty(b/a)_\infty(c/a)_\infty(bc/azq)_\infty}{(z)_\infty(b)_\infty(c)_\infty(b/az)_\infty(c/az)_\infty(q/a)_\infty} \tag{2.1}$$

Proof.Ramanujan's 1ψ1 summation formula 1.3 can be written as

$$\sum_{k=0}^{\infty}\frac{(a)_k}{(b)_k}z^k + \sum_{k=1}^{\infty}\frac{(q/b)_k}{(q/a)_k}\left(\frac{b}{az}\right)^k = \frac{(az)_\infty(q)_\infty(q/az)_\infty(b/a)_\infty}{(z)_\infty(b)_\infty(b/az)_\infty(q/a)_\infty} \tag{2.2}$$

This is the same as

$$\sum_{k=0}^{\infty}(a)_k z^k\left\{(bq^k)_\infty\left(\frac{b}{az}\right)_\infty\right\} + \sum_{k=1}^{\infty}\frac{(-1)^k q^{k(k+1)/2}}{(q/a)_k}\left(\frac{1}{az}\right)^k\left\{(bq^{-k})_\infty\left(\frac{b}{az}\right)_\infty\right\} \tag{2.3}$$

$$= \frac{(az)_\infty(q)_\infty(q/az)_\infty}{(z)_\infty(q/a)_\infty}\left\{\left(\frac{b}{a}\right)_\infty\right\}.$$

On applying $E(c\theta)$ to both sides with respect to b, we obtain

$$\sum_{k=0}^{\infty}(a)_k z^k\left\{\frac{(bq^k)_\infty(b/az)_\infty(cq^k)_\infty(c/az)_\infty}{(bcq^k/azq)_\infty}\right\}$$
$$+\sum_{k=1}^{\infty}\frac{(-1)^k q^{k(k+1)/2}}{(q/a)_k}\left(\frac{1}{az}\right)^k\left\{\frac{(bq^{-k})_\infty(b/az)_\infty(cq^{-k})_\infty(c/az)_\infty}{(bcq^{-k}/azq)_\infty}\right\} \quad (2.4)$$

$$=\frac{(az)_\infty(q)_\infty(q/az)_\infty}{(z)_\infty(q/a)_\infty}\left\{\left(\frac{b}{a}\right)_\infty\left(\frac{c}{a}\right)_\infty\right\}.$$

Multiplying 2.4 throughout by $\{(bc/azq)_\infty/(b)_\infty(c)_\infty\}$,we obtain

$$\sum_{k=0}^{\infty}\frac{(a)_k(bc/azq)_k}{(b)_k(c)_k}z^k\sum_{k=1}^{\infty}\frac{(a)_{-k}(bc/azq)_{-k}}{(b)_{-k}(c)_{-k}}z^{-k}$$
$$=\frac{(az)_\infty(q)_\infty(q/az)_\infty(b/a)_\infty(c/a)_\infty(bc/azq)_\infty}{(z)_\infty(b)_\infty(c)_\infty(b/az)_\infty(c/az)_\infty(q/a)_\infty}, \quad (2.5)$$

which yields 2.1.

Some Applications of the Main Identity

The following identity is the well-known q-analogue of the Gauss summation formula

Corollary 3.1 (see [15]). $If\ |q| < 1, |\gamma/\alpha\beta| < 1$, then

$$\sum_{k=0}^{\infty}\frac{(\alpha)_k(\beta)_k}{(q)_k(\gamma)_k}\left(\frac{\gamma}{\alpha\beta}\right)^k=\frac{(\gamma/\alpha)_\infty(\gamma/\beta)_\infty}{(\gamma)_\infty(\gamma/\alpha\beta)_\infty} \quad (3.1)$$

Proof. Putting $a = \alpha, b = q, c = \gamma, and\ z = \gamma/\alpha\beta$ in 2.1, we obtain 3.1.

Corollary 3.2.$If\ |q| < 1$, then

$$\sum_{k=-\infty}^{\infty}\frac{(q^2;q^4)_k}{(1-q^{4k+1})(q^3;q^4)_k}q^k = q^{-1/8}\frac{\eta^7(2\tau)}{\eta^3(\tau)\eta^2(4\tau)}, \quad (3.2)$$

$$\sum_{k=-\infty}^{\infty}\frac{(-q^2;q^2)_k}{(1-q^{2k+1})^2(-q;q^2)_k}q^k = 2q^{-3/4}\frac{\eta^6(4\tau)}{\eta^3(2\tau)}, \quad (3.3)$$

$$\sum_{k=-\infty}^{\infty}\frac{(-q;q^2)_k(-q^2;q^2)_k}{(1-q^{2k+1})^2(q;q^2)_n}q^k=2q^{-3/4}\frac{\eta^8(4\tau)}{\eta^7(2\tau)}, \tag{3.4}$$

$$\sum_{k=-\infty}^{\infty}\frac{(1+q)(-q;q^2)_k(-q^2;q^2)_k}{(1-q^{2k+1})^2(q^2;q^2)_k}q^k=\frac{\eta(2\tau)}{\eta(\tau)}, \tag{3.5}$$

$$\sum_{k=-\infty}^{\infty}\frac{(q^2;q^6)_k}{(1-q^{6k+1})^2(q^5;q^6)_k}q^{3k}=q^{-1/4}\frac{\eta^4(2\tau)}{\eta^2(\tau)}, \tag{3.6}$$

$$\sum_{k=-\infty}^{\infty}\frac{(1+q^2)(-q^2;q^2)_{k-1}(-q^2;q^2)_{k+1}}{(1-q^{2k+2})^2(q^2;q^2)_k}q^k=\frac{\eta^2(4\tau)}{\eta^4(2\tau)}, \tag{3.7}$$

Proof.Putting $a = z = q^{1/4}, b = q^{3/4}, c = q^{5/4}$, and then changing $q\ to\ q^4$ in 2.1, we obtain

$$\sum_{k=-\infty}^{\infty}\frac{(q;q^4)_k(q^2;q^4)_k}{(q;q^4)_{k+1}(q^3;q^4)_k}q^k=\frac{(q^2;q^4)_{\infty}^4\ (q^4;q^4)_{\infty}^2}{(q;q^4)_{\infty}^3\ (q^3;q^4)_{\infty}^3}, \tag{3.8}$$

Simplifying the right hand side and then using 1.8, we obtain 3.2

Similarly, putting $a = -q^{1/2}, z = q^{1/2}, b = -q^{3/2}, c = -q^{3/2}$, and then changing $q\ to\ q^2$, we obtain 3.3. Putting $a = -q^{1/2}, z = q^{1/2}, b = c = q^{3/2}$, and then changing $q\ to\ q^2$, we obtain 3.4. Putting $a = -q^{1/2}, z = q^{1/2}, b = q, c = q^2$, and then changing $q\ to\ q^2$, we obtain 3.5. Putting $a = q^{1/6}, z = q^{1/2}, b = q^{7/6}, c = q^{5/6}$, and then changing $q\ to\ q^6$, we obtain 3.6. Finally, putting $a = -1, z = q, b = c = q^2$, and then changing $q\ to\ q^2$, we obtain 3.7.

Corollary 3.3.$If\ 0 < q < 1, 0 < x, y < 1, and\ 0 < x + y < 1$, then

$$B_q^3(x,y)=\frac{\Gamma_q(1-x+y)\Gamma_q(x-y)(1-q)^2}{(1-q^y)^2}\sum_{k=-\infty}^{\infty}\frac{(q^{1-x})_k(q^{x+y})_k}{(q^{1+y})_k^2}q^{ky}. \tag{3.9}$$

Proof.Putting $a = q^{1-x}, z = q^y$, and $b = c = q^{1+y}$ in 2.1, we get

$$\sum_{k=-\infty}^{\infty}\frac{(q^{1-x})_k(q^{x+y})_k}{(q^{1+y})_k^2}q^{ky}=\frac{(q^{1-x+y})_\infty(q)_\infty(q^{x-y})_\infty}{(q^y)_\infty(q^{1+y})_\infty(q^{1+y})_\infty}\left(\frac{(q^{x+y})_\infty}{(q^x)_\infty}\right)^3 \tag{3.10}$$

On using (1.9), (1.10), and (1.11), we obtain (3.9).

Corollary 3.4.$If\ 0 < q < 1, 0 < x, y < 1, and\ 1 < x + y < 2$, then

$$B_q^2(x,y) = \frac{\Gamma_q(y-x)\Gamma_q(x-y+1)\Gamma_q(x+y-1)}{\Gamma_q(y)\Gamma_q(1+x)} \sum_{k=-\infty}^{\infty} \frac{(q^{-x})_k (q^{x+y-1})_k}{(q^y)_k^2} q^{ky}. \qquad (3.11)$$

Proof.Putting $a = q^{-x}, z = b = c = q^y$, in (2.1), and then using (1.9), (1.10), and (1.11), we obtain 3.11.

Corollary 3.5.$If\ 0 < x, y < 1, and\ 0 < x + y < 1$, then

$$B^3(x,y) = \frac{\Gamma(1-x+y)\Gamma(x-y)}{y^2} \left[\sum_{k=0}^{\infty} \frac{(1-x)_k (x+y)_k}{(1+y)_k^2} + \sum_{k=1}^{\infty} \frac{(-y)_k^2}{(x)_k (1-x-y)_k} \right] \qquad (3.12)$$

Proof. Letting $q \to 1$ in 3.9, we obtain 3.12.

Corollary 3.6.$If\ 0 < x, y < 1, and\ 1 < x + y < 2$, then

$$B^2(x,y) = \frac{\Gamma(y-x)\Gamma(x-y+1)\Gamma(x+y-1)}{\Gamma(y)\Gamma(1+x)} \times \left[\sum_{k=0}^{\infty} \frac{(-x)_k (x+y-1)_k}{(y)_k^2} + \sum_{k=1}^{\infty} \frac{(1-y)_k^2}{(1+x)_k (2-x-y)_k} \right] \qquad (3.13)$$

Proof. Letting $q \to 1$ in 3.11, we obtain 3.13.

Corollary 3.7.$If\ 0 < q < 1, 0 < x, y < 1, and\ 0 < x + y < 1$, then

$$B_q(x,y) = \Gamma_q(y)\Gamma_q(1-y) \sum_{k=0}^{\infty} \frac{(q^{1-x-y})_k (q^y)_k}{(q)_k^2} q^{kx} \qquad (3.14)$$

Proof. Putting $a = q^{1-x-y}, z = q^x\ and\ b = c = q$ in 2.1, we obtain 3.14.

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