EFFECT OF THERMAL RADIATION ON INTEGRAL BOUNDARY LAYER HEAT TRANSFER

Promise Mebine *

Abstract

The paper examines the contributions of thermal radiation to integral boundary layer heat transfer prediction. The von Karman Integral Method is applied to converting the governingequations of momentum and energy with radiative heat flux to a non-linear first-order equation of the ratio of thermal-momentum boundary layer thickness. The resulting non-linear first-order equation is solved by the use of singular perturbation. Three representative Prandtl numbers: $Pr = 0.71$, which corresponds to Air at 20 degrees Celsius, $Pr = 7.2$, which corresponds to Seawater at 20 degrees Celsius, and$Pr = 13.4$, which corresponds to Seawater at 0 degrees Celsius are used such that increase in Prandtl number reduces theratio of thermal-momentum boundary layer thickness without thermal radiation. On the hand other, the inclusion of thermal radiation increases the ratio of thermal-momentum boundary layer thickness. Physically,thermal radiation reduces the magnitude of the internal friction in the fluid thereby increasing the rate of diffusion, and hence increases the ratio of thermal-momentum boundary layer thickness. Also, other important features of the problem are discussed quantitatively.

Key Words: Boundary layer, Heat transfer, Perturbation, Thermal radiation

*Department of Mathematics/Computer Science, Niger Delta University, Wilberforce Island, Bayelsa State, NIGERIA

1. Introduction

Boundary layer flows, which are external flows around streamlined bodies, where viscous (shear and no-slip) effects occur, are confined close to the body surfaces and its wake. Rienstraand Darau(2011) modelled and clarified why a seemingly very thin mean flow boundary layer cannot be neglected, and stated that the physical insight can help to interpret experimental results.The applications of such flows are many in nature and industry. Examples are aerodynamics(airplanes, rockets,projectiles), hydrodynamics(ships, submarines, torpedoes),transportation(automobiles, trucks, cycles), windengineering(buildings, bridges, water towers), and oceanengineering(buoys,breakwaters, cables).The study of heat transfer in these and other engineering processes has become a centre point. Heat transfer is simply that science which seeks to predict the energy transfer which may take place between material bodies as a result of a temperature difference. Thermodynamics teaches that this energy is defined as heat. Other than how heat energy may be transferred, the study of heat transfer predicts the rate at which the exchange will take place under certain specified conditions.

Hydrodynamic and thermal boundary layers are pertinent in the study of heat transfer because, while hydrodynamic boundary layer is defined as that region of the flow where viscous forces are felt, a thermal boundary layer, on the hand, may be defined as that region where temperature gradients are present in the flow. These temperature gradients would result from a heat-exchange process between the fluid and the wall. The study of such processes is useful for improvingtechnological and industrial advancements becausesome of the physical properties are functions of temperature. For example, most industrial and technological metal works usually experience large differences in temperature (Attia, 2005). The large differences in temperature generate significant heat or radiative heat transfer that occurs due to the induction of eddy currents in most of these engineering applications. Other examples, where high temperature phenomena or high-power radiation sources commonly encountered are in solar physics, in combustion applications such as fires, furnaces, IC engines, in nuclear reactions such as in the sun or nuclear explosions, in compressors in ships and gas flares from petrochemical industry. Korycki (2006) described radiative heat transfer as an important fundamental phenomena existing in practical engineering such as those found in solar radiation buildings, foundry engineering and solidification processes, dye forging, chemical engineering, composite structures applied in industry.Many of such processes in engineering occur at high temperature, and knowledge of radiation heat transfer becomes very important for the design of pertinent equipment. Nuclear power plants, gas turbines and the various propulsion devices for aircraft, missile, satellites and space vehicles are examples of such engineering areas. Hossain and Takhar (1996) analysed the effect of radiation using the Rosseland diffusion approximation that leads to non-similar boundary layer equations governing mixed convection flow of an optically dense viscous incompressible fluid past a heated vertical plate with a uniform free stream velocity and surface temperature. Mukhopadhyay (2009) studied the effect of thermal radiation on unsteady mixed convection flow and heat transfer over aporous stretching surface in porous medium. Anura (2010) investigated thermal boundary layer flow over a stretching sheet in a micro polar fluid with radiation effect. Hayat (2011) et al. considered thermal radiation flow and heat transfer characteristics for the boundary layer flow over a permeable stretching sheet in the presence of velocity and thermal slip conditions. Cess (1966), however, considered absorbing-emitting gray fluids with a black vertical plate. His solution was based on a perturbation technique and was applicable for the conduction-radiation interaction parameter. Arpaci (1968) applied the integral method to natural convection from a heated vertical plate and in the same work developed a thick gas model including the wall effect; the temperature profiles employed were those proposed by squire floor; the same problem in the absence of radiation and thus the effect of radiation was confined to the modification of the boundary layer thickness. Soundalgekar et al. (1988) studied radiation effects on free convection flow past a semi-infinite plate using the Colgey-Vincenti equilibrium model.Chung (2002) discussed thermal radiation boundary layer on a flat plate as a classical problem that has important applications. Mebine and Adigio (2009) discussed the effects of thermal radiation on unsteady free convection flow past a vertical porous plate with Newtonian heating obtaining analytical results by the use of Laplace transform technique.

The study of the hydrodynamic and thermal boundary layers over a flat plate has been of great interest, particularly using von Karman’s Integral Method (Ghoshdatidar, 2004). This method gives an approximate analysis and it involves the integration of the boundary layer momentum and energy partial differential equations over the thickness of the boundary layer. It must be noted that the integral equations themselves are exact within the boundary layer assumptions.It is the objective of this work to study thermal radiation effected boundary layer over an isothermal flat plate using von Karman's Integral Method. This study is intended to serve as a complement to previous studies that did not consider the incorporation of radiative heat transfer in the analysis of von Karman's Integral Method for hydrodynamic flow over a flat plate.

2.Mathematical Formulation

Consider a constant property fluid flowingover an infinite flat plate. The flow is laminar, the wall temperature T_w, is maintained constant, and at points far from the plate the fluid velocityand temperature have the uniform values u_∞ and T_∞, respectively. To simplify the problem and isolate theinfluence of radiation, the following assumptionsare made: the gas is considered such that the radiative flux reflects the notion of an optically thin environment, such as one would find in the intergalactic layers, where the plasma gas is assumed to be of low density andthat the penetration depth or absorption coefficient is less than one; radiationscattering, radiation pressure, and the contributionof radiation to internal energy are negligible;the effect of radiation is included to theenergy equation as a one-dimensional heat flux;non-equilibrium effects other than diffusionand radiation are negligible.

On this basis, the momentum, energy and the thermal radiative heat flux equations are, respectively:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}, \quad (1)$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha_d\frac{\partial^2 T}{\partial y^2} - \frac{1}{\rho c_p}\frac{\partial q}{\partial y}, \quad (2)$$

$$\frac{\partial q}{\partial y} = 4\sigma\alpha(T^4 - T_\infty^4). \quad (3)$$

Here $u, v, T, \nu, \alpha_d, \sigma, \alpha, \frac{\partial q}{\partial y}, \rho, c_p, x$ and y represents respectively, component of the fluid velocity along the longitudinal direction,x; component of the fluid velocity along the transverse or normal direction, y ; the fluid temperature; kinematic viscosity; thermal diffusivity $(\equiv \frac{k}{\rho c_p}; k,$ isothermal conductivity$)$; Stefan-Boltzmann constant; penetration or absorption of fluid depth; radiative heat flux; fluid density; heat capacity at constant pressure; variable along plate wall, and variable normal to plate wall.

The boundary and compatibility conditions are

$$y = 0: u = 0, v = 0, T = T_w, \tag{4}$$

$$y = 0: \frac{\partial^2 u}{\partial y^2} = 0, \alpha_d \frac{\partial^2 T}{\partial y^2} = \frac{1}{\rho c_p}\frac{\partial q}{\partial y}, \tag{5}$$

$$y \to \infty: u \to u_\infty, T \to T_\infty, \tag{6}$$

$$y \to \infty: \frac{\partial u}{\partial y} \to 0, \frac{\partial T}{\partial y} \to 0. \tag{7}$$

It is pertinent to note that the second (5)and the fourth (7) conditions came respectively from the application of the momentum and energy equations at the wall and far away from the wall. They are simply known as compatibility conditions.

The radiative flux equation (3) is highly nonlinear in T. However, when it isassumed that the temperature differences within the flow are sufficiently small, then thelinear differential approximation of Cogley-Vincenti-Gilles equilibrium model (1968) of the radiative flux becomes significant. In this caseT^4 could be expressed as a linear function oftemperature in Taylor series about T_∞ neglecting higher-order terms. Thus,

$$T^4 \cong 4T_\infty^3 T - 3T_\infty^4. \tag{8}$$

Therefore, the radiative flux equation (3) becomes

$$\frac{\partial q}{\partial y} = 16\sigma\alpha T_\infty^3 (T - T_\infty). \tag{9}$$

Equation (9) completes the formulation of the problem.

3.Integral Formulation and Solution

The usual integralformulation of the problem is modified toinclude the effect of radiation. Therefore, the integral equations of the momentum (1) and energy (2) with (9) are, respectively, given by

$$\frac{d}{dx}\int_0^\delta \rho u^2 dy = -\mu \left.\frac{\partial u}{\partial y}\right|_{y=0}, \tag{10}$$

$$\frac{d}{dx}\int_0^{\delta_t} u(T - T_\infty)dy + \frac{16\sigma\propto T_\infty^3}{\rho c_p}\int_0^{\delta_t}(T - T_\infty)\, dy = -\propto_d \left.\frac{\partial T}{\partial y}\right|_{y=0}, \tag{11}$$

where δ and δ_t are the respective hydrodynamic or momentum and thermal boundary layer thicknesses, which are functions of x.

We note that the functional forms of the velocity and temperature distributions must be obtained in order to arrive at the explicit solutions of the equations (10) and (11). Therefore, the velocity and temperature functions are assumed to have respective polynomial representations of the forms:

$$u = a_0 + a_1 y + a_2 y^2 + a_3 y^3, \tag{12}$$

$$T = b_0 + b_1 y + b_2 y^2 + b_3 y^3, \tag{13}$$

where the $a's$ and $b's$ may be functions of x. Now, the applications of the boundary and compatibility conditions result the velocity and temperature distributions as

$$\frac{u}{u_\infty} = \frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3, \tag{14}$$

$$\frac{T - T_w}{T_\infty - T_w} = \frac{\theta}{\theta_\infty} = \frac{1}{1+\gamma\left\{\frac{7}{4}\frac{y}{\delta_t} - \frac{1}{2}\left(\frac{y}{\delta_t}\right)^2\right\}}\left[\frac{3}{2}\frac{y}{\delta_t} - \frac{1}{2}\left(\frac{y}{\delta_t}\right)^3 + \gamma\left\{\frac{7}{4}\frac{y}{\delta_t} - \frac{1}{2}\left(\frac{y}{\delta_t}\right)^2\right\}\right], \tag{15}$$

where $\gamma = \frac{16\alpha\sigma T_\infty^3 \delta_t^2}{k}$. It is important to note here that the use of equations (14) and (15) on the integral energy equation (11) in order to obtain an explicit analytical result is not easily amenable; however, an approximate result is in order.

Now, using $N = \frac{16\nu\alpha\sigma T_\infty^3}{\rho c_p u_\infty^2}$ (Mebine and Adigio, 2009)to represent the thermal radiationparameter and forsufficiently high ambient velocity, the thermal radiation parameter may be considered to be small. Here the thermal radiation parameter is assumed to lie in the range $0 \le N \le 1$. Therefore, series expansion of the temperature profile (15) in N, and noting that $N >> N^2 >> N^3 \cdots$, we obtain

$$\frac{\theta}{\theta_\infty} = \frac{3}{2}\frac{y}{\delta_t} - \frac{1}{2}\left(\frac{y}{\delta_t}\right)^3 + \frac{u_\infty^2 \delta_t^2}{\alpha_d \nu} N\left\{\frac{7}{4}\frac{y}{\delta_t} - \frac{25}{8}\left(\frac{y}{\delta_t}\right)^2 + \frac{3}{4}\left(\frac{y}{\delta_t}\right)^3\right\}. \tag{16}$$

With the aid of equations (14) and (16), equation (11) reduces to the approximate first-order nonlinear ordinary differential equation:

$$\zeta^3 + 4\,x\,\zeta^2\frac{d\zeta}{dx} - \frac{55}{3}\,N\,Re_x\,\zeta^2 - \frac{13}{4\,Pr} = 0, \tag{17}$$

for $\zeta = \frac{\delta_t}{\delta} \ll 1$, where the hydrodynamic boundary layer thickness relations:

$$\delta\frac{d\delta}{dx} = \left(\frac{140}{13}\right)\frac{\nu}{u_\infty},\ \delta^2 = 2\left(\frac{140}{13}\right)\frac{\nu}{u_\infty}x \tag{18}$$

hold. As usual, $Pr = \frac{\nu}{\alpha_d}$, the Prandtl number and $Re_x = \frac{u_\infty x}{\nu}$, the local Reynolds number. Equation (17) is to be solved with the condition,

$$\zeta(L) = 0, \tag{19}$$

where L denotes an unheated or insulated section of the plate. In other words, it is assumed that the heating of the plate starts at $x = L$.

Equation (17) with the condition (19) has no exact solution to the best of the author's knowledge, except for some perturbation or iterative solution. Thus, since $N \leq 1$, singular perturbation (Hince, 1991) solution is herein advanced such that

$$\zeta = \zeta_0 + N\,\zeta_1. \tag{20}$$

Using this expansion in the equations (17) and (19), and equating the coefficients of N, we have the following respective equations:

$$N^0: \zeta_0^3 + 4\,x\,\zeta_0^2\frac{d\zeta_0}{dx} - \frac{13}{4\,Pr} = 0, \tag{21}$$

$$\zeta_0(L) = 0, \tag{22}$$

$$N: 3\,\zeta_0^2\zeta_1 + 8\,\zeta_0\zeta_1\frac{d\zeta_0}{dx} + 4\,x\,\zeta_0^2\frac{d\zeta_1}{dx} - \frac{55\,Re_x}{3}\zeta_0^2, \tag{23}$$

$$\zeta_1(L) = 0. \tag{24}$$

Solving the leading-order perturbation equations (21, 22),gives the exact solution:

$$\zeta_0 = \frac{1}{1.026\,Pr^{\frac{1}{3}}}\left\{1 - \left(\frac{L}{x}\right)^{\frac{3}{4}}\right\}^{\frac{1}{3}}. \tag{25}$$

Solving the first-order perturbation equations (23, 24), results the integral solution:

$$\zeta_1 = \int_L^x \left(\frac{55\,Re_x}{12}\frac{1}{z}e^{I(z)}\right)dz \;\times\; e^{-I(z)}, \tag{26}$$

where

$$I(z) = \frac{1}{4}\int_L^z \frac{L - 3\,z\left(\frac{L}{z}\right)^{\frac{1}{4}}}{z^2\left\{-1+\left(\frac{L}{z}\right)^{\frac{3}{4}}\right\}\left(\frac{L}{z}\right)^{\frac{1}{4}}}dz. \tag{27}$$

The integral solution can only be evaluated for some specific values of L, (where the heating started) such as $L = 1, 4, 5,$ etc. Therefore, the graphical illustrations presented below considered the case of $L = 1$.

In practical applications, the physical quantity of principal interest is the rate of heat flux, q_w, or the heat transfer coefficient, h, between the fluid and the wall of the plate. This is calculated from

$$h = \frac{-k}{T_w - T_\infty}\left(\frac{\partial T}{\partial y}\right)\Big|_{y=0} = \frac{k}{T_\infty - T_w}\left(\frac{\partial T}{\partial y}\right)\Big|_{y=0}. \quad (28)$$

By the scaling, $\frac{\theta}{\theta_\infty} = \frac{T-T_w}{T_\infty - T_w}$, and by the aid of equation (16), we have

$$\left(\frac{\partial \theta}{\partial y}\right)\Big|_{y=0} = \frac{3}{2\delta_t} + \frac{7}{4}\frac{u_\infty^2\, N\, \delta_t}{\nu\, \alpha_d} \quad (29)$$

such that

$$\frac{h\, x}{k} = Nu_x = \frac{3\, x}{2\,\zeta(x)\delta(x)} + \frac{7}{4}\frac{u_\infty^2\, x\, N\, \zeta(x)\delta(x)}{\nu\, \alpha_d}, \quad (30)$$

where Nu_x, the local Nusselt number. Substituting equations (18) and(20) into equation (30), we obtain

$$\frac{Nu_x}{Re_x^{\frac{1}{2}}} = 0.323\, \frac{1}{\zeta_0 + N\,\zeta_1} + 8.22\, Pe\, N\, \zeta_0, \quad (31)$$

where Pe is Peclet number and is equal to $\Pr Re_x$. The Pecletnumber signifies the ratio of the strength of convection to the strength of diffusion. It is important to note here that in the limit case as $N \to 0$, equations (29 – 31) reduce exactly to those obtained by Ghoshdatidar (2004).

4.Analyses of Results

The effect of thermal radiation on the problem of the growthof momentum and thermal boundary layers on a horizontal plate with an unheated initial section has been considered using von Karman's Integral Method. It is observed that the inclusion of the thermal radiation term presented considerablemathematical difficulties for the intended investigation, though second-order velocity and temperature distributions were applied.In view of this, the temperature distribution was approximated via series expansion of the thermal radiation parameter, N.

The effect of Prandtl number on the momentum-thermal boundary layer ratio, $\zeta$with or without thermal radiation is presented in Figure 1. Three representative Prandtl numbers: $Pr = 0.71$, which corresponds to Air at 20 degrees Celsius, $Pr = 7.2$, which corresponds to Seawater at 20 degrees Celsius, and$Pr = 13.4$, which corresponds to Seawater at 0 degrees Celsius are used such that increase in Prandtl number reduces thethermal-momentum boundary layer ratio without thermal radiation (Figure 1(a)). On the hand other, the inclusion of thermal radiation increases the momentum-thermal boundary layer ratio, ζ (Figure 1(b)). Physically, thermal radiation reduces the magnitude of the internal friction in the fluid thereby increasing thethermal-momentum boundary layer ratio.

Figures 2 – 4 pertain respectively to the investigations of the effect of thermal radiation on the thermal-momentum boundary layer ratio, ζ, thermal boundary layer thickness, δ_t and the local Nusselt number, Nu_x, based on theparticular value of the Prandtl number, $Pr = 0.71$.

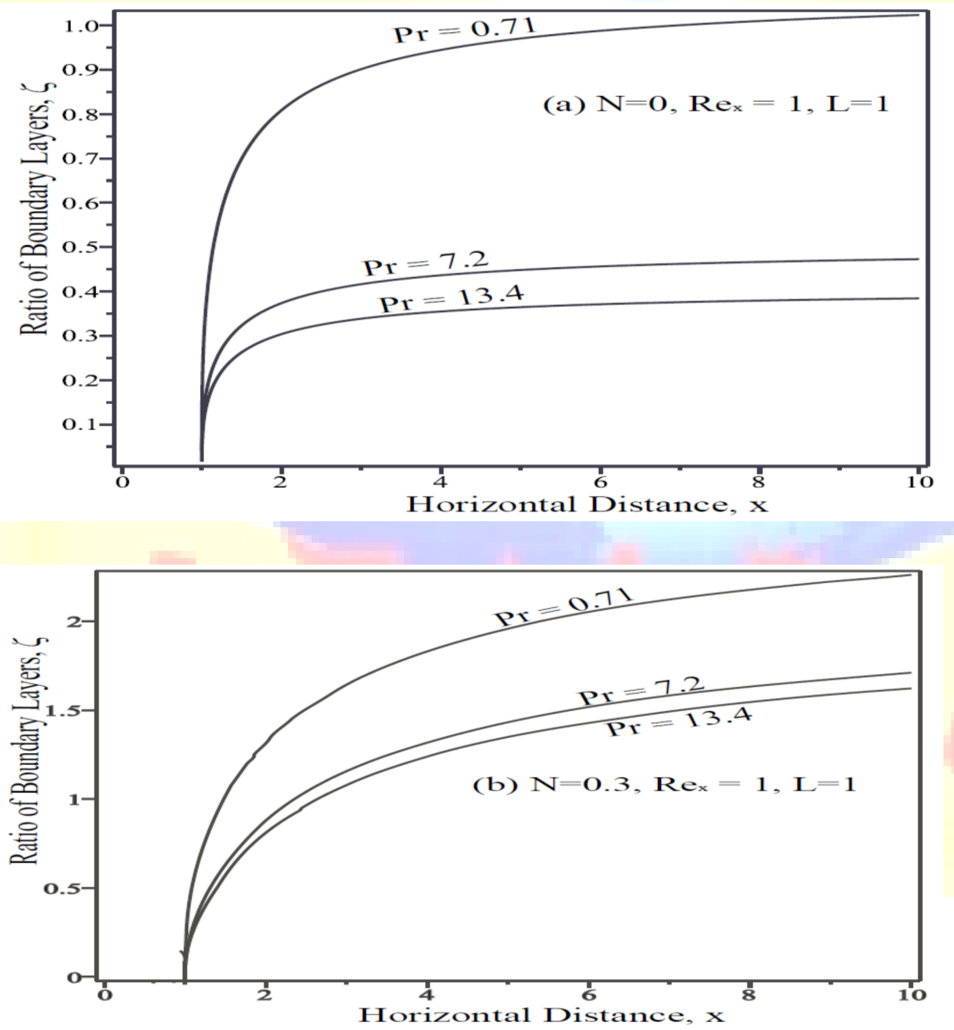

Figure 1: Ratio of Boundary Layers as a function of Horizontal Distance, x in variations of Prandtl number for (a) $N = 0$, (b) $N = 0.3$

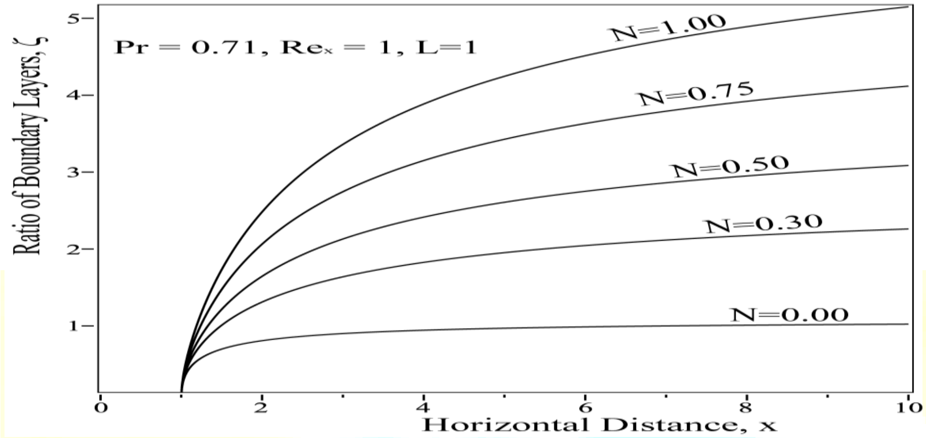

Figure 2: Ratio of Boundary Layers as a function of Horizontal Distance, x in variations of Thermal Radiation Parameter for Air at 20 degrees Celsius

Figure 2depicts the thermal-momentum boundary layer ratio of the flat plate as a function of the horizontal distance in variations of thermal radiation parameter for $Pr = 0.71$.As it is visibly seen, the thermal radiation increases the thermal-momentum boundary layer ratio.In other words, the thermal radiation gives significant increase to the thermal boundary layer thickness such that$\frac{\delta_t}{\delta} \gg 1$. This is clearly depicted in Figure 3. Physically, thermal radiation produces an increase in the diffusivity, and causes reduction in the viscosity. Consequently, the thermal boundary layer thickness is increased in the process.

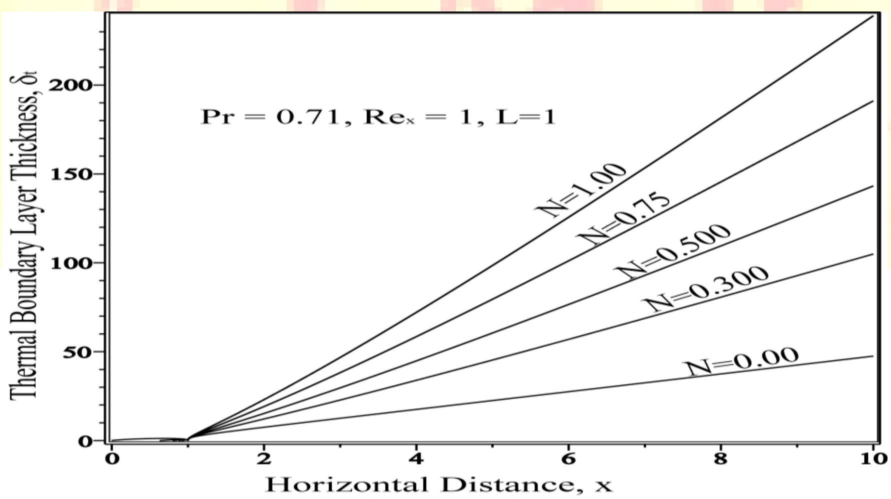

Figure 3: Thermal Boundary Layer Thicknesses as a function of Horizontal Distance, x in variations of Thermal Radiation Parameter

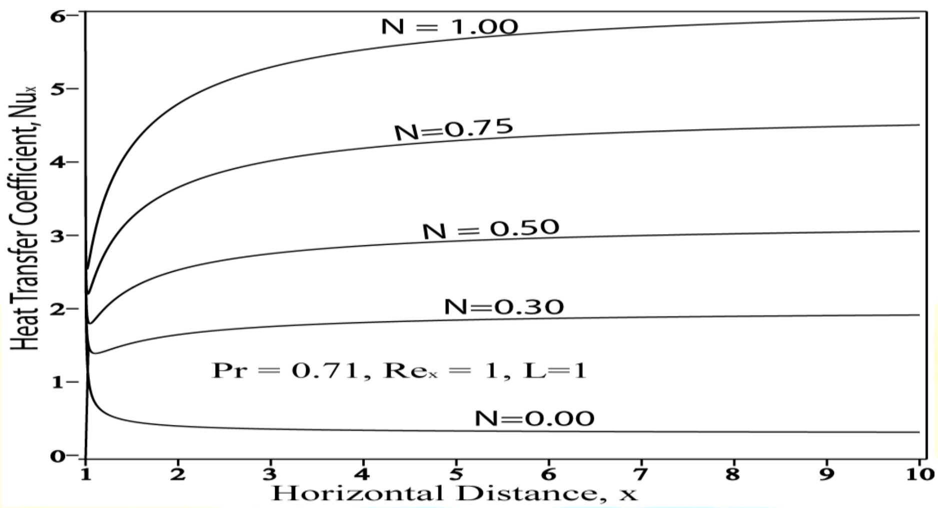

Figure 4: Heat Transfer Coefficient as a function of Horizontal Distance, x in variations of Thermal Radiation Parameter.

Figure 4 demonstrates the heat transfer coefficient as affected by the thermal radiation. It is observed that the heat transfer coefficient appears steep at the beginning of the section, where heating started and gradually becomes approximately linear within the boundary layer. Whereas the ratio of the boundary layers and the thermal boundary layer starts at the beginning of the plate where heating commenced and at zero, the heat transfer coefficient also starts at the beginning of the plate where heating commenced, but with a value way from zero. The overall testamentis that thermal radiation increases the heat transfer coefficient significantly.

It is emphasized here that since $\zeta \leq 1$ is taken, it is required that the Prandtl number, $\Pr \geq 1$. For small Prandtl numbers, $\Pr < 1$, equation (25) will not be valid. For $L = 0$ and $N = 0$, the results obtained in this work are the same with those obtained using similarity analysis (Ghoshdatidar, 2004).

5.Concluding Remarks

The purpose of this study was to incorporate Radiative heat transfer to the thermal boundary layer and to advance the solution via von Karman's Integral Method. Approximate analytical solutions were constructed using singular perturbation method. It is observed that the inclusion of the thermal radiation term presented considerable mathematical difficulties for the intended investigation, though second-order velocity and temperature distributions were applied.

From the approximate analytical results, the following main conclusions are made:
That the results satisfy the compatibility conditions and represent the characteristics of the problem.

1) That the thermal radiation increases the ratio of the boundary layer.
2) That the thermal radiation gives significant increase to the thermal boundary layer thickness such that $\frac{\delta_t}{\delta} \gg 1$.
3) That thermal radiation produces increase in diffusivity, and causes reduction in viscosity, and hence increases thermal boundary layer thickness.
4) Thatthermal radiation significantly affects the heat transfer coefficient.

References

1. **Anura, I.(2010);** Thermal boundary layer flow over a stretchingsheet in a micro polar fluid with radiation effect.Maccanica, Vol. 45, pp. 367-373.

2. **Arpaci,V. S. (1968);** Effect of thermal radiation on the laminar free convection from a heated vertical plate, Int. J. Heat Muss Transfer 11**,**871- 881.

3. **Cess, R.D.(1966);** The interaction of thermal radiation with Free convection Heat Transfer,

4. Int. J. Heat mass Trans.,9, pp 1268-1277.

5. **Chung, T. J. (2002);** *Computational Fluid Dynamics*, Cambridge University Press, Cambridge, UK.

6. **Cogley, A. C. L., Vincenti, W. G. and Gilles, E. S. (1968);** "Differential approximation for radiative heat transfer in a nonlinear equations-grey gas near equilibrium," American Institute of Aeronautics and Astronautics Journal, vol. 6, pp. 551 - 553.

7. **Ghoshdastidar, P. S. (2002);** *Heat Transfer*, Oxford University Press, 2004.

8. **Hayat, T., Qasim, M. and Mesloub, S. (2011);** MHD Flow and heat transfer permeable stretching sheet with slip conditions.

9. **Hinch,E.J. (1991);** Perturbation Methods, *Cambridge Text in applied mathematics*, Cambridge University Press, Cambridge, UK.

10. **Hossain, M. A. and Takhar, H. S., (1996);** Radiation Effect on Mixed Convection Along a Vertical Plate with Uniform Surface Temperature Heat and Mass Transfer 3, l (4), pp.243-248.

11. **Korycki, R. (2006);** Sensitivity analysis and shape optimization for transient heat conduction with radiation, Int. J. Heat Mass Transfer, 49, 2033 – 2045.

12. **Mebine, P. and Adigio, E. M.(2009);** "Unsteady free convection flow with thermal radiation past a vertical porous plate with Newtonian heating," Turkish Journal of Physics, vol. 33, no. 2, pp. 109-119.

13. **Mukhopadhyay, S. (2009);** Effect of thermal radiation on unsteady mixed convection flow and heat transfer over a porous stretching surface in porous medium. International Journal of heat and Transfer, Vol. 52, pp. 3261-3265.

14. **Rienstra, S. W.and Darau, M. (2011);** Boundary-layer thickness effectsof the hydrodynamic instability along an impedance wall, J. Fluid Mech., vol. 671, pp. 559 - 573

15. **Soundalgekar, V. M., Takhar, H. S., andVighnesam, N. V. (1988);** “Combined free and forced convection flow past a semi-infinite vertical plate with variable surface temperature,” Nuclear Engineering and Design, vol. 110, no. 1, pp. 95–98.

16. **Taitel, Y. and Hartnett, J.P. (1993);**Equilibrium Temperatures in a Boundary-Layer Flow over a Flat Plate of Absorbing-Emitting Gas. Paper 66-WA/HT-48.