PROPERTIES AND PARAMETER ESTIMATIONS FOR THE NEW EXPONENTIATED WEIBULL PARETO DISTRIBUTION

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Abstract

In this paper, a New Exponentiated Weibull Pareto Distribution NEWPD is discussed and studied. This new distribution extends Weibull Pareto distribution which presented by [10]. Limiting behavior for probability density function pdf, cumulative distribution function CDF and hazard rate function h(x) for the NEWPD are obtained. Some properties for the NEWPD are provided such moments, moment generating function, modes and order statistics. Estimation of parameters for the NEWPD using maximum likelihood estimation method are proposed. The results of real data for the NEWPD are compared with other known distribution results to illustrate the applications of the proposed distribution.

Keywords

Exponentiated Distribution, New Exponentiated Weibull Pareto Distribution, Moments, limiting behavior, Hazard Function.

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1. Introduction

There are special kinds of applied data in scientific fields such as medicine, engineering and finance, amongst others, need to be represented special lifetime distributions. So, variety distributions have been developed by many authors to become more flexible and more fitting to such data. These new statistical distributions have been used to describe and interpret the phenomena.

The idea of exponentiated distributions were utilized to create new distributions via various methods. A class of exponentiated distributions that based on cumulative distribution function have been introduced by [6] as follows:

$$G(x) = F(x)^{\gamma} \quad (1.1)$$

where $F(x)$ is the cumulative distribution function of the random variable X and γ is an additional shape parameter. Many known distributions as normal, Weibull, gamma, Gumbel, and inverse Gaussian distributions have been extended by [5]. They expressed the ordinary moments of these new family of generalized distributions as linear functions of probability weighted moments of the parent distribution. Weibull distribution have been extended by [9] to analyze bathtub failure data.

Another class of exponentiated distributions is T-X distribution. [2] proposed a new method for generating T -X family of distributions. The T -X family has a connection between the hazard functions and each generated distribution as a weighted hazard function of the random variable X. for example, four-parameter beta-Pareto distribution are suggested by [1]. They obtained various properties of the distribution and they used the method of maximum likelihood to estimate the parameters. Five general methods of combination and variations of two historical periods have been used by [7]. The objective of these combinations methods is to generate new statistical distributions such as adding parameters, beta generated, transformed-transformer and composite methods. [11] defines Generalized Weibull Exponential distribution which extends Weibull-Exponential distribution.. Different properties for the Generalized Weibull- Exponential distribution have been obtained such as moments, limiting behavior, quintile function, Shannon's entropy, skewness and kurtosis.

In reliability and survival studies, there are some widely used distributions such as Weibull and Pareto. This is due to their simplicity and easy mathematical manipulations. In additions, Weibull distribution has many advantages because its hazard rate. So, Weibull distribution is suitable model to applications in risk analysis and quality control.

The main aim of this paper is to provide a new form for Weibull Pareto distribution which presented by [10]. One shape parameter is added to their form to add new model that could offer a better fit to life time data. In addition, many structural properties of the NEWPD are studied.

The remainder of the paper is organized as follows. The NEWPD will be defined in section 2. Section 3 devoted to investigate the properties for the NEWPD such moments, moment generating function, modes and order statistics. Estimation of parameters for the NEWPD using maximum likelihood estimation method are proposed in section 4. Finally, analyses of real data is given to compare the results of the NEWPD with other selection distributions in section 5.

2. The NEWPD

[10] suggested the CDF and pdf for the Weibull Pareto distribution respectively as follows:

$$F(x) = \left[1 - e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right] \quad ;x>0 \text{ and } \delta,\beta,\theta>0 \qquad (2.1)$$

And,

$$f(x) = \frac{\delta\beta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1} e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}} \quad ;x>0 \text{ and } \delta,\beta,\theta>0 \qquad (2.2)$$

where $\delta=\lambda^{\alpha}$ and $\beta=\alpha k$.

By adding a shape parameter γ to (2.1), then, the CDF of the NEWPD can be written as:

$$G(x) = \left[1 - e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right]^{\gamma} \quad ;x>0 \text{ and } \delta,\beta,\theta,\gamma>0 \qquad (2.3)$$

Therefore , the pdf of the NEWPD is:

$$g(x) = \frac{\delta\beta\gamma}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1} e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\left[1 - e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right]^{\gamma-1} \quad ;x>0 \text{ and } \delta,\beta,\theta,\gamma>0 \qquad (2.4)$$

The survival function $R(x)$ and hazard rate function $h(x)$ of the NEWPD distribution respectively are given by:

$$R(x) = 1-\left[1 - e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right]^{\gamma} \quad ;x>0 \text{ and } \delta,\beta,\theta,\gamma>0 \qquad (2.5)$$

$$h(x) = \frac{\delta\beta\gamma}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1} e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}} \frac{\left[1 - e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right]^{\gamma-1}}{\left[1-\left[1 - e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right]^{\gamma}\right]} \quad ;x>0 \text{ and } \delta,\beta,\theta,\gamma>0 \qquad (2.6)$$

The limiting behavior of pdf , CDF and h(x) of the NEWPD can be studied through the next lemmas as follows:

Lemma (2.1): The limiting behavior of pdf as x goes to infinity is 0 and when x goes to zero is 0.

proof.

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} \frac{\delta\beta\gamma}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1} e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\left[1 - e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right]^{\gamma-1}$$
$$= \frac{\delta\beta\gamma}{\theta}\times\infty\times 0\times 1 = 0 \qquad (2.7)$$

and

$$\lim_{x\to 0} g(x) = \lim_{x\to 0} \frac{\delta\beta\gamma}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1} e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\left[1 - e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right]^{\gamma-1} \qquad (2.8)$$
$$= \frac{\delta\beta\gamma}{\theta}\times 0\times 1\times 0 = 0$$

Lemma (2.2): The limiting behavior of CDF as x goes to infinity is 1 and when x goes to zero is 0.

proof.

$$\lim_{x\to\infty} G(x) = \lim_{x\to\infty}\left[1-e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right]^{\gamma} = 1 \qquad (2.9)$$

and

$$\lim_{x\to 0} G(x) = \lim_{x\to 0}\left[1-e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right]^{\gamma} = 0 \qquad (2.10)$$

Lemma (2.3): the hazard is (increasing / decreasing) function of x if ($\beta\gamma \geq 1 / \beta\gamma < 1$). So that, the NEWPD is appropriate to modeling components that (wears faster / wears slower) with time.

Figure (1,2) are the plots of the pdf and figure (3) is the plot of the h(x) of the NEWPD for different values of the parameters of the NEWPD respectively.

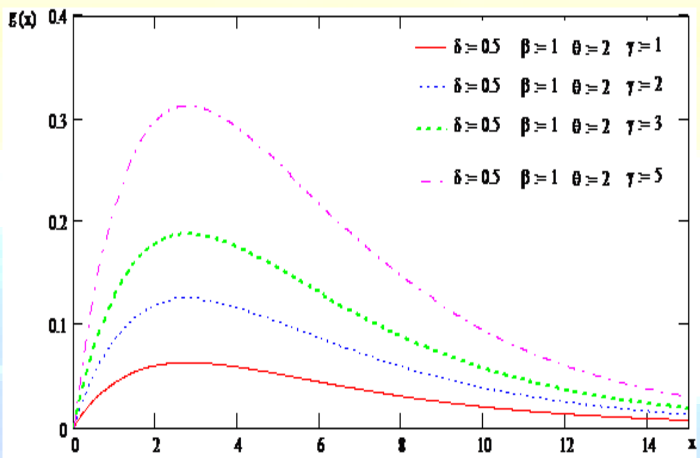

Figure (1), different forms for pdf of the NEWPD with various values of the parameters $\delta,\beta,\theta,\gamma$ respectively.

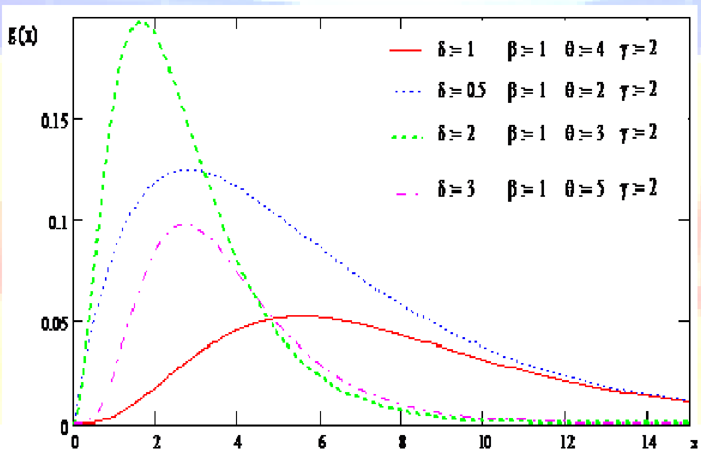

Figure (2), different forms for pdf of the NEWPD with various values of the parameters $\delta,\beta,\theta,\gamma$ respectively.

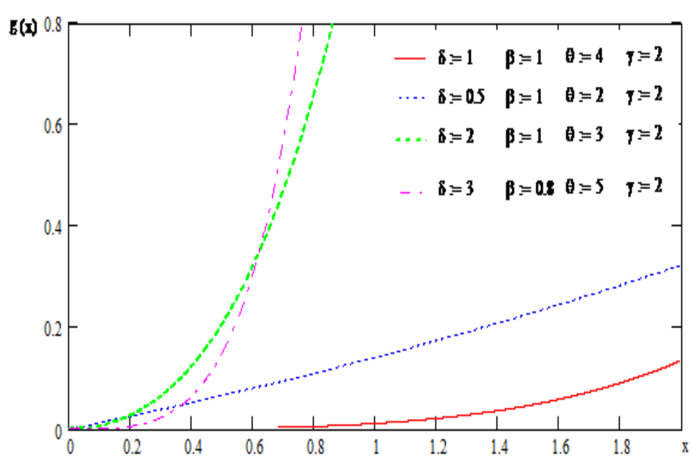

Figure (3), different forms for h(x) of the NEWPD with various values of the parameters $\delta, \beta, \theta, \gamma$ respectively.

There is a relation between the NEWPD and Exponentiated Exponential distribution. This relation can be obtained by using the transform of variables. Let Y be independent and identically random variable distributed as exponentiated exponential distribution with parameters δ, γ, then, using the transform $X = \theta e^{\frac{\ln(y)}{\beta}}$, the result is the pdf for the NEWPD. Also, if Y be an Exponentiated Weibull random variable with parameters δ, θ, γ, then, using $X = \theta Y$ as transformation, the result is the pdf for the NEWPD.

3. Properties of the NEWPD

3.1 Moments

If $X \approx NEWPD$ with parameters $\delta, \beta, \theta, \gamma$, then the r^{th} non-central moment of X is:

$$\mu_r' = E(x^r) = \frac{\gamma \theta^r}{\delta^{\frac{r}{\beta}}} \sum_{j=0}^{\infty} \binom{\gamma-1}{j} (-1)^j (1+j)^{-\frac{r}{\beta}} \Gamma(\frac{r}{\beta}+1) \qquad (3.1.1)$$

Proof.

$$\mu_r' = E(x^r) = \int_0^{\infty} X^r g(x)\, dx$$

$$= \int_0^{\infty} X^r \frac{\delta \beta \gamma}{\theta} \left(\frac{x}{\theta}\right)^{\beta-1} e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}} \left[1 - e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right]^{\gamma-1} dx \qquad (3.1.2)$$

Using the series expansion: $(1-s)^{\varepsilon-1} = \sum_{j=0}^{\infty} \binom{\varepsilon-1}{j} (-1)^j s^j$, then,

$\left[1 - e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right]^{\gamma-1} = \sum_{j=0}^{\infty} \binom{\gamma-1}{j} (-1)^j \left(e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right)^j$. So,

$$\mu_r' = \frac{\delta \beta \gamma}{\theta^{\beta}} \sum_{j=0}^{\infty} \binom{\gamma-1}{j} (-1)^j \int_0^{\infty} X^{r+\beta-1} e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}(1+j)} dx \qquad (3.1.3)$$

Now, let

$u = \delta\left(\frac{x}{\theta}\right)^{\beta}(1+j)$ then, $du = \frac{\delta \beta}{\theta^{\beta}} x^{\beta-1}(1+j)\, dx \Rightarrow dx = \frac{\theta^{\beta} du}{x^{\beta-1}(1+j)}$ and $x = \left(\frac{\theta^{\beta} u}{\delta(1+j)}\right)^{\frac{1}{\beta}}$.

$$\mu_r' = \frac{\gamma \theta^r}{\delta^{\frac{r}{\beta}}} \sum_{j=0}^{\infty} \binom{\gamma-1}{j} (-1)^j (1+j)^{-\frac{r}{\beta}} \int_0^{\infty} u^{\frac{r}{\beta}} e^{-u} du$$

$$= \frac{\gamma \theta^r}{\delta^{\frac{r}{\beta}}} \sum_{j=0}^{\infty} \binom{\gamma-1}{j} (-1)^j (1+j)^{-\frac{r}{\beta}} \Gamma(\frac{r}{\beta}+1). \qquad (3.1.4)$$

The first four non-central moments of the random variable X are:

$$\mu_1' = \frac{\gamma\theta}{\delta^{\frac{1}{\beta}}} \sum_{j=0}^{\infty} \binom{\gamma-1}{j} (-1)^j (1+j)^{-\frac{1}{\beta}} \Gamma\left(\frac{1}{\beta}+1\right). \tag{3.1.5}$$

$$\mu_2' = \frac{\gamma\theta}{\delta^{\frac{2}{\beta}}} \sum_{j=0}^{\infty} \binom{\gamma-1}{j} (-1)^j (1+j)^{-\frac{2}{\beta}} \Gamma\left(\frac{2}{\beta}+1\right). \tag{3.1.6}$$

$$\mu_3' = \frac{\gamma\theta}{\delta^{\frac{3}{\beta}}} \sum_{j=0}^{\infty} \binom{\gamma-1}{j} (-1)^j (1+j)^{-\frac{3}{\beta}} \Gamma\left(\frac{3}{\beta}+1\right). \tag{3.1.7}$$

$$\mu_4' = \frac{\gamma\theta}{\delta^{\frac{4}{\beta}}} \sum_{j=0}^{\infty} \binom{\gamma-1}{j} (-1)^j (1+j)^{-\frac{4}{\beta}} \Gamma\left(\frac{4}{\beta}+1\right). \tag{3.1.8}$$

So, the mean and variance of the NEWPD respectively are : $E(X) = \mu = \mu_1'$ and $\operatorname{var}(X) = \mu_2' - \mu^2$.

The co-efficient of variation C.V, co-efficient of skewness C.S and co-efficient of kurtosis C.K respectively are:

$$C.V = \frac{\sigma}{\mu}, \quad C.S = \frac{\mu_3' - 3\mu\mu_2' + 2\mu^3}{(\mu_2' - \mu^2)^{\frac{3}{2}}} \text{ and}$$

$$C.K = \frac{\mu_4' - 4\mu\mu_3' + 6\mu^2\mu_2' - 3\mu^4}{(\mu_2' - \mu^2)^2}. \tag{3.1.9}$$

3.2 Moment generating function

If $X \approx NEWPD$ with parameters $(\delta, \beta, \theta, \gamma)$ **, then, the moment generating function** $M_X(t)$ **is:**

$$M_X(t) = \sum_{c=0}^{\infty} \frac{t^c}{c!} \frac{\gamma\theta^c}{\delta^{\frac{c}{\beta}}} \sum_{j=0}^{\infty} \binom{\gamma-1}{j} (-1)^j (1+j)^{-\frac{c}{\beta}} \Gamma\left(\frac{c}{\beta}+1\right). \tag{3.2.1}$$

Proof.

$$M_X(t) = E(e^{tX}) = \int_0^{\infty} e^{tx} g(x)\, dx$$

$$= \int_0^{\infty} e^{tx} \frac{\delta\beta\gamma}{\theta} \left(\frac{x}{\theta}\right)^{\beta-1} e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}} \left[1 - e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right]^{\gamma-1} dx. \tag{3.2.2}$$

Taylor series can be used to find the value of the previous integral as follows:

$$M_X(t) = \int_0^{\infty} \left(1 + \frac{tx}{1!} + \frac{(tx)^2}{2!} + \cdots + \frac{(tx)^n}{n!} + \cdots\right) g(x)\, dx$$

$$= \sum_{c=0}^{\infty} \frac{t^c}{c!} E\left(X^c\right)$$

$$= \sum_{c=0}^{\infty} \frac{t^c}{c!} \frac{\gamma\theta^c}{\delta^{\frac{c}{\beta}}} \sum_{j=0}^{\infty} \binom{\gamma-1}{j} (-1)^j (1+j)^{-\frac{c}{\beta}} \Gamma\left(\frac{c}{\beta}+1\right). \tag{3.2.3}$$

3.3 Modes

To obtain the mode of the NEWPD, the equation $\frac{d\ln g(x)}{dx}=0$ can be solved for x. where:

$$\ln g(x) = \ln(\delta) + \ln(\beta) + \ln(\gamma) - \ln(\theta) + (\beta-1)\ln\left(\frac{x}{\theta}\right) - \delta\left(\frac{x}{\theta}\right)^{\beta} + (\gamma-1)\ln\left[1-e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right] \tag{3.3.1}$$

So that:

$$\frac{d\ln g(x)}{dx} = \frac{e^{\delta\left(\frac{x}{\theta}\right)^{\beta}}(\beta-1) - \delta\beta e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\left(\left(\frac{x}{\theta}\right)^{\beta} - \gamma\right) - \beta}{x\left[\left(\frac{x}{\theta}\right)^{\beta} - 1\right]} = 0 \tag{3.3.2}$$

3.4 order statistics

To discuss The distributions of the order statistics of the NEWPD, let $X_1, X_2, \ldots, X_n$ $NEWD$ with parameters $(\delta, \beta, \theta, \gamma)$, let $X_{(1)}$ be the 1st smallest of $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ and let $X_{(r)}$ be the r^{th} smallest of $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$. Then, the pdf of the r^{th} order statistics $X_{(r)}$ is:

$$g_X(x_r) = \frac{n!}{(r-1)!(n-r)!} g_X(x) \left[G_X(x)\right]^{r-1} \left[R_X(x)\right]^{n-r} \quad ; r = 1, 2, \ldots, n \tag{3.4.1}$$

$$= \frac{n!}{(r-1)!(n-r)!} \frac{\delta\beta\gamma}{\theta} \left(\frac{x}{\theta}\right)^{\beta-1} e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}} \left[1-e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right]^{\gamma-1} \times \left[1-e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right]^{\gamma(r-1)} \left[1-\left[1-e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right]^{\gamma}\right]^{n-r} \tag{3.4.2}$$

So, the pdf of the smallest order statistics $X_{(1)}$ and the largest order statistics $X_{(n)}$ respectively are:

$$g_X(x_1)=\frac{n!}{(r-1)!(n-r)}\frac{\delta\beta\gamma}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1}e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\left[1-e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right]^{\gamma-1}$$

$$\times\left[1-\left[1-e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right]^{\gamma}\right]^{n-1} \qquad (3.4.3)$$

$$g_X(x_n)=\frac{n!}{(r-1)!(n-r)}\frac{\delta\beta\gamma}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1}e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\left[1-e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right]^{\gamma-1}$$

$$\times\left[1-e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right]^{n(n-1)} \qquad (3.4.4)$$

4. Parameter Estimations of the NEWPD

4.1 Maximum Likelihood Estimation Method

The likelihood and log likelihood functions of the NEWPD based on the samples $X_1, X_2, ..., X_n$ respectively are:

$$L(X;\delta,\beta,\theta,\gamma)=\left(\frac{\delta\beta\gamma}{\theta}\right)^{n}\prod_{i=1}^{n}\left(\frac{x_i}{\theta}\right)^{\beta-1}\prod_{i=1}^{n}e^{-\delta(\frac{x_i}{\theta})^{\beta}}\prod_{i=1}^{n}\left[1-e^{-\delta(\frac{x_i}{\theta})^{\beta}}\right]^{\gamma-1} \qquad (4.1.1)$$

and

$$\ell = n\ln(\delta)+n\ln(\beta)+n\ln(\gamma)-n\ln(\theta)$$

$$+(\beta-1)\sum_{i=1}^{n}\ln(\frac{x_i}{\theta})-\delta\sum_{i=1}^{n}(\frac{x_i}{\theta})^{\beta}+(\gamma-1)\sum_{i=1}^{n}\ln\left[1-e^{-\delta(\frac{x_i}{\theta})^{\beta}}\right] \qquad (4.1.2)$$

The first derivatives of (4.1.2) with respect to the parameters $(\delta,\beta,\theta,\gamma)$ and equating them to zero respectively, are:

$$\frac{\partial\ell}{\partial\delta}=\frac{n}{\delta}-\sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right)^{\beta}-\gamma-1\sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right)^{\beta}Z=0 \qquad (4.1.3)$$

$$\frac{\partial\ell}{\partial\beta}=\frac{n}{\beta}-n\ln\theta-\sum_{i=1}^{n}\ln x_i-\delta\sum_{i=1}^{n}\left[\left(\frac{x_i}{\theta}\right)^{\beta}\ln\left(\frac{x_i}{\theta}\right)\right]-\gamma-1$$

$$\times\sum_{i=1}^{n}\left[\delta\left(\frac{x_i}{\theta}\right)^{\beta}\ln\left(\frac{x_i}{\theta}\right)Z\right]=0 \qquad (4.1.4)$$

$$\frac{\partial\ell}{\partial\theta}=-\frac{n}{\theta}\;2+\beta\;-\delta\sum_{i=1}^{n}\left[-\left(\frac{x_i}{\theta}\right)^{\beta}q\right]-\gamma-1\sum_{i=1}^{n}\left[-\delta\left(\frac{x_i}{\theta}\right)^{\beta}qZ\right]=0 \qquad (4.1.5)$$

$$\frac{\partial \ell}{\partial \gamma}=\frac{n}{\gamma}-\sum_{i=1}^{n}\ln\left[1-e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right]=0 \qquad (4.1.6)$$

where $Z=\dfrac{e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}}{\left[1-e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right]}$ and $q=\dfrac{\beta}{\theta}$.

The maximum likelihood estimators $(\hat{\delta},\hat{\beta},\hat{\theta},\hat{\gamma})$ for the parameters $(\delta,\beta,\theta,\gamma)$ can be obtained by solving the non-linear system equations from (4.1.3) to (4.1.6). Newton–Raphson method can be used to solve this system in numerical analysis.

4.2 Asymptotic Confidence Interval

Now, to construct approximate variance-covariance matrix which will be used to compute the standard errors and asymptotic confidence interval, the base of the sample approximation can be used. When $n\to\infty$,The maximum likelihood estimators $(\hat{\delta},\hat{\beta},\hat{\theta},\hat{\gamma})$ of the parameters $(\delta,\beta,\theta,\gamma)$ are asymptotically multivariate normal with mean $(\delta,\beta,\theta,\gamma)^{T}$ and variance $I(\Psi)$. Where:

$$I_{ij}(\Psi)=\begin{bmatrix} -E\left(\frac{\partial^2\ell}{\partial\delta^2}\right) & -E\left(\frac{\partial^2\ell}{\partial\delta\partial\beta}\right) & -E\left(\frac{\partial^2\ell}{\partial\delta\partial\theta}\right) & -E\left(\frac{\partial^2\ell}{\partial\delta\partial\gamma}\right)\\ & -E\left(\frac{\partial^2\ell}{\partial\beta^2}\right) & -E\left(\frac{\partial^2\ell}{\partial\beta\partial\theta}\right) & -E\left(\frac{\partial^2\ell}{\partial\beta\partial\gamma}\right)\\ & & -E\left(\frac{\partial^2\ell}{\partial\theta^2}\right) & -E\left(\frac{\partial^2\ell}{\partial\theta\partial\gamma}\right)\\ & & & -E\left(\frac{\partial^2\ell}{\partial\gamma^2}\right)\end{bmatrix}^{-1} \qquad (4.2.1)$$

where the diagonal elements in (4.2.1) represent the variances of the parameters $(\delta,\beta,\theta,\gamma)$ and the off-diagonal elements represent the covariance between the parameters $(\delta,\beta,\theta,\gamma)$.

The second derivative of (4.1.2) with respect to the parameters $(\delta,\beta,\theta,\gamma)$ respectively is given by:

$$\frac{\partial^2\ell}{\partial\delta^2}=-\frac{n}{\delta^2}-\ \gamma-1\ \sum_{i=1}^{n}\left[-\left[\left(\frac{x_i}{\theta}\right)^{\beta}\right]^{2}\left[Z-Z^{*}\right]\right] \qquad (4.2.2)$$

$$\frac{\partial^2\ell}{\partial\beta^2}=-\frac{n}{\beta^2}-\delta\sum_{i=1}^{n}\left[\left(\frac{x_i}{\theta}\right)^{\beta}\ln\left(\frac{x_i}{\theta}\right)^{2}\right]-\ \gamma-1$$
$$\times\sum_{i=1}^{n}\left[\delta\left(\frac{x_i}{\theta}\right)^{\beta}\ln\left(\frac{x_i}{\theta}\right)^{2}Z-\delta^{2}\left[\left(\frac{x_i}{\theta}\right)^{\beta}\right]^{2}\ln\left(\frac{x_i}{\theta}\right)^{2}\left[Z-Z^{*}\right]\right] \qquad (4.2.3)$$

$$\frac{\partial^2 \ell}{\partial \theta^2} = -\frac{n}{\theta^2}\ 2+\beta\ -\delta \sum_{i=1}^{n}\left[\left(\frac{x_i}{\theta}\right)^{\beta}\left(q^{*}+\frac{\beta}{\theta^2}\right)\right] - \ \gamma - 1$$

$$\times \sum_{i=1}^{n}\left[\delta\left(\frac{x_i}{\theta}\right)^{\beta} Z\left(q^{*}+\frac{\beta}{\theta^2}\right) - \delta^2 q^{*}\left[\left(\frac{x_i}{\theta}\right)^{\beta}\right]^2\left[Z - Z^{*}\right]\right] \qquad (4.2.4)$$

$$\frac{\partial^2 \ell}{\partial \gamma^2} = -\frac{n}{\gamma^2} \qquad (4.2.5)$$

$$\frac{\partial^2 \ell}{\partial \delta \partial \beta} = -\sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right)^{\beta} \ln\left(\frac{x_i}{\theta}\right)$$

$$\times \sum_{i=1}^{n}\left[\left(\frac{x_i}{\theta}\right)^{\beta} \ln\left(\frac{x_i}{\theta}\right) Z + \left[\left(\frac{x_i}{\theta}\right)^{\beta}\right]^2 \delta \ln\left(\frac{x_i}{\theta}\right)\left[Z - Z^{*}\right]\right]$$

$$(4.2.6)$$

$$\frac{\partial^2 \ell}{\partial \delta \partial \theta} = -\sum_{i=1}^{n}\left[-\left(\frac{x_i}{\theta}\right)^{\beta} q\right] - \ \gamma - 1$$

$$\times \sum_{i=1}^{n}\left[-\left(\frac{x_i}{\theta}\right)^{\beta} Z + \left[\left(\frac{x_i}{\theta}\right)^{\beta}\right]^2 \delta q\, Z + \left[\left(\frac{x_i}{\theta}\right)^{\beta}\right]^2 \delta q\, Z^{*}\right] \qquad (4.2.7)$$

$$\frac{\partial^2 \ell}{\partial \delta \partial \gamma} = -\sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right)^{\beta} Z \qquad (4.2.8)$$

$$\frac{\partial^2 \ell}{\partial \beta \partial \theta} = -\frac{n}{\theta} - \delta \sum_{i=1}^{n}\left[\left(\frac{x_i}{\theta}\right)^{\beta} q \ln\left(\frac{x_i}{\theta}\right) - \frac{\left(\frac{x_i}{\theta}\right)^{\beta}}{\theta}\right] - \ \gamma - 1$$

$$\times \sum_{i=1}^{n}\left[-\delta\left(\frac{x_i}{\theta}\right)^{\beta} q \ln\left(\frac{x_i}{\theta}\right) Z - \delta \frac{\left(\frac{x_i}{\theta}\right)^{\beta}}{\theta} Z + \delta^2\left[\left(\frac{x_i}{\theta}\right)^{\beta}\right]^2\right. \qquad (4.2.9)$$

$$\left. \times \ln\left(\frac{x_i}{\theta}\right) q\, Z + \delta^2\left[\left(\frac{x_i}{\theta}\right)^{\beta}\right]^2 \ln\left(\frac{x_i}{\theta}\right) Z^{*}\right]$$

$$\frac{\partial^2 \ell}{\partial \beta \partial \gamma} = -\sum_{i=1}^{n} \delta\left(\frac{x_i}{\theta}\right)^{\beta} \ln\left(\frac{x_i}{\theta}\right) Z \qquad (4.2.10)$$

$$\frac{\partial^2 \ell}{\partial \theta \partial \gamma} = -\sum_{i=1}^{n}\left[-\delta\left(\frac{x_i}{\theta}\right) q\, Z\right] \qquad (4.2.11)$$

where $Z^{*} = \frac{e^{\left(-\delta\left(\frac{x}{\theta}\right)^{\beta}\right)^{2}}}{\left[1-e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right]^{2}}$, $q = \frac{\beta}{\theta}$ and $q^{*} = \left(\frac{\beta}{\theta}\right)^{2}$.

The confidence interval for the parameters $(\delta, \beta, \theta, \gamma)$ with an approximate $100(1-\alpha)\%$ will be respectively:

$$\tilde{\delta} = \hat{\delta} \pm Z_{\alpha/2}\sqrt{-E\left[\frac{\partial^{2}\ell}{\partial\delta^{2}}\right]} \qquad (4.1.12)$$

$$\tilde{\beta} = \hat{\beta} \pm Z_{\alpha/2}\sqrt{-E\left[\frac{\partial^{2}\ell}{\partial\beta^{2}}\right]} \qquad (4.1.13)$$

$$\tilde{\theta} = \hat{\theta} \pm Z_{\alpha/2}\sqrt{-E\left[\frac{\partial^{2}\ell}{\partial\theta^{2}}\right]} \qquad (4.1.14)$$

$$\tilde{\gamma} = \hat{\gamma} \pm Z_{\alpha/2}\sqrt{-E\left[\frac{\partial^{2}\ell}{\partial\gamma^{2}}\right]} \qquad (4.1.15)$$

5. Applications

[10] used data in [5]. Recently, [3] analyzed these data using the Kumaraswamy Pareto KP distribution. These data represents the exceedances of flood peaks in (m^3/s) through the interval 1958-1984 of the Wheaton River near Carcross in Yukon Territory, Canada. In this section , Data set is used to compare the results of the proposed model with the other models. Also, Data set is applied to show that if the new distribution bitter fits data then the other or not. Table (1) shows the data while Table (2) provides the maximum likelihood ML and standard errors for the parameter estimators of the NEWPD and the two selection distributions (New Weibull Pareto NWP Distribution, Kumaraswamy Pareto KP distribution.

Table (1): exceedances of Wheaton River.

3.7	2.74	2.73	2.5	3.6	3.11	3.27	2.87	1.47	3.11	3.56
4.42	2.41	3.19	3.22	1.69	3.2	3.09	1.87	3.15	4.9	1.57
2.67	2.93	3.22	3.39	2.81	4.2	3.33	2.55	3.31	3.31	2.85
1.25	4.38	1.84	0.39	3.68	2.48	0.85	1.61	2.79	4.7	2.03
1.89	2.88	2.82	2.05	3.65	3.75	2.43	2.95	2.97	3.39	2.96
2.35	2.55	2.59	2.03	1.61	2.12	3.15	1.08	2.56	1.8	2.53

Table (2): Maximum likelihood estimates and standard errors for models.

Model	Parameter estimates	Standard error
NEWP	$\delta = 0.728$	0.0241
	$\beta = 0.525$	0.3105
	$\hat{\theta} = 0.1$	---
	$\gamma = 32.844$	0.0983
NWP	$\hat{\delta} = 11.7450$	2.8664
	$\hat{\beta} = 0.1999$	0.0263
	$\hat{\theta} = 0.1$	---

	$\delta = 0.0528$	0.0185
KP	$\hat{\beta} = 2.8553$	0.3371
	$\hat{\gamma} = 85.8468$	0.3371

Table (3) gives the values of criteria statistics like -2ℓ (the maximized value of the log likelihood for the estimated model), AIC (Akaike information criterion), AICC (corrected Akaike information criterion) of data. These statistics are considered to compare the results Based on the smaller values of -2ℓ, AIC and AICC . Then, the better distribution fits quite well to data than other.

$$AIC = -2\ell + 2k \text{ and } AICC = -2\ell + 2k + \frac{2k\ (k+1)}{n-k-1},$$

where k is the number of parameters to be estimated.

Table (3): Values of criteria statistics of data.

Model	-2ℓ	AIC	AICC
NEWP	75.125	83.125	74.168
NWP	158.3258	162.3258	162.6787
KP	382.956	386.956	383.942

Conclusion

The New Exponentiated Weibull Pareto Distribution NEWPD has been defined and studied. The limiting behavior for the CDF, pdf and h(x) have been obtained. various properties of the NEWPD have been investigated. Maximum likelihood estimation method had been suggested to estimation of parameters of the NEWPD. Maximum likelihood estimators have been used to construct approximate variance-covariance matrix which used to compute the standard errors and asymptotic confidence interval. Also, the smallest and largest order statistics for the NEWPD have been obtained. Results of application to real data sets have been compared with many known probability distributions. The results revealed that the proposed distribution is a good fit such data.

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