CIRCULARITY OF THE NUMERICAL RANGE OF ISOMETRICALLY BOUNDED LINEAR OPERATORS ON A HILBERT SPACE.

J.S.Barasa[1]

B.N.Okello[2]

ABSTRACT

Circularity is the roundness or a measure of how closely the shape of an object approaches that of a circle. It is another very common shape factor which is at times called isoperimetric quotient .i.e. a function of the perimeter p and area A since $f_{circ.} = \frac{4\pi A}{p^2}$. The circularity of a circle is 1. The shape of the numerical range or more accurately its boundary is normally related to the combinatorial structure or the pattern of signs of entries and the pattern of nonzero entries. In this paper we characterize non negative matrices and the shape of their numerical ranges.

Key word s : Linear operator ,numerical range, connected graph, circularity, adjacency matrix, Hilbert space.

[1] Department of mathematics and physicals science, Pwani university, Kilifi, Kenya

[2] Department of mathematics and actuarial science, jaramogi oginga university

1 Introduction

Let A be an n-square matrix. The numerical range of A is the set denoted by $W(\mathrm{A}) = \{x^*\mathrm{A}\,x : x \in \mathbb{C}^n, x^*x = 1\}$.T he field of values, Wertovorrat , Hausdorff domain, range of values have been among others competing names for the same notion. The shape of the numerical range gives a lot of information or details about the individual operator .e.g. If the numerical range is real then as a standard result in Hilbert space theory asserts that the operator must be Hermitian. Isometries are the invariant mappings with respect to norms or distance like reflections, translations, rotations etc . we shall investigate what the shapes in terms of circularity say about the underlying Hilbert spaces.

2 PRELIMINARIES

2.1 NUMERICAL RANGE OF A MATRIX

The numerical range of a two by two matrix is

(a) A single point, if the operator is a scalar multiple of the identity.

(b) A line segment joining the eigenvalues, if the operator is normal with two distinct eigenvalues.

(c) An elliptical disc with foci at the eigenvalues , if the operator has distinct eigenvalues but not normal or not unitary diagonalizable.

(d) Suppose T is a two by two matrix with distinct eigenvalues λ_1 *and* λ_2 *with corresponding eigenvectors* x_1 *and* x_2 *and letting* $\mu = |\langle x_1, x_2\rangle|$. *Then*

(i) $W(\mathrm{T})$ *is an elliptical disc with foci at* λ_1 *and* λ_2.

(ii) The eccentricity of $W(\mathrm{T})$ *is* $1/\mu$

(iii) The major axis of $W(\mathrm{T})$ *has length* $\dfrac{2|\lambda_1 - \lambda_2|}{\sqrt{1-\mu^2}}$.

2.1.1 Toplitz Hausdorff Theorem

The numerical range of every bounded linear operator T on a hibert space is convex. $[7\,pg.10]$

2.1.1 Theorem

Suppose A is an n by n matrix with complete entries whose trace is zero , then A is unitarily equivalent to a matrix with diagonal identically zero. i.e . The trace of A is the sum of the eigenvalues of A. $\left[7\,pg.11\right]$

2.1.2 Theorem

The numerical range of a matrix is invariant under unitary transformations $\left[1\,pg.1\right]$

2.1.3. Theorem

Let

$A=\begin{bmatrix} B & 0 \\ C & \alpha I_{n-m} \end{bmatrix} \in M_n$ *with* $1\le m\le n$ *and let D be a circular disk centered at* α *incase m<n*

(a)if $W(A)\subseteq D$ *and the boundary of W(A) contains m+1 boundary points of D then W(A)=D.*

(b)If $D\subseteq W(A)$ *and the boundary of W(A) contains m+1 points of D, then the Boundary of W(A) contains a circular arc which is part of the boundary of D.*

(c) If the boundary of $W(A)$ *contains 2m+1 boundary points of D, then* $W(A)$ *contains D. Under any of the three conditions the center of D is an eigenvalue of B.*

2.1.4 . Ellipse lemma

Let T *be an operator on a two dimensional space. Then* $W(T)$ *is an ellipse whose foci are eigen values*

Proof

$$T-\frac{\lambda_1+\lambda_2}{2}=\begin{bmatrix} \frac{\lambda_1-\lambda_2}{2} & a \\ 0 & \frac{\lambda_2-\lambda_1}{2} \end{bmatrix}$$

$$e^{-i\theta}\left[\mathrm{T}-\frac{\lambda_1+\lambda_2}{2}\right]=\begin{bmatrix} r & ae^{-i\theta} \\ 0 & -r \end{bmatrix}=\mathrm{B} \text{ where } \quad \frac{\lambda_1-\lambda_2}{2}=re^{i\theta}.$$

$W(B)$ *is an ellipse with center (0,0) and minor axis* $|a|$ *and foci at (r,0) and (-r,0). Thus* $W(\mathrm{T})$ *is an ellipse with foci at* λ_1,λ_2 *and the major axis has an inclination of* θ *with the real axis.*

2.1.5 Schur decomposition Theorem

Any square matrix T may be transformed by unitarily similarity transformations to upper triangular form, with its eigenvalues on the diagonal and $W(\mathrm{T})$ *is invariant under unitary transformations .*

2.1.6 Theorem

If $W(\mathrm{T})$ *is a line segment, then T is normal.*

Proof

Let α *be the point on the line segment with inclination* θ*.Then* $W\left(e^{-i\theta}[\mathrm{T}-\alpha I]\right)$ *is contained in the real axis. Thus* $e^{-i\theta}[\mathrm{T}-\alpha I]$ *is self adjoint and so* T *is normal.*

2.1.7 Theorem

The closure of the numerical range of a normal operator is the convex hull of its spectrum.

Proof

Suppose that $a+ib\in W(\mathrm{T})$ *with* $a>0$ *and* $\langle \mathrm{T}x,x\rangle=a+ib, \|x\|=1.$

Let $\mathrm{T}x=(a+ib)x+y$ *where* $\langle x,y\rangle=0.$ *Let* $c\in\mathbb{R}, c>0$*.Then* $c\notin\sigma(\mathrm{T})$ *and we have*

$d(c,\sigma(\mathrm{T}))\le\|(\mathrm{T}-cI)x\|$ *.i.e.* $c^2\le\|(a-c+ib)x+y\|^2=(a-c)^2+b^2+\|y\|^2.$ *Hence* $2ac\le a^2+b^2$ *with* $a,c>0$*. Which is impossible since c is arbitrary .*

2.1.8 Theorem

Let A be an $n \times n$ marix with eigen values $\lambda_1, \ldots\ldots, \lambda_n$ and suppose that its associated curve c(A) consists of k ellipses , with minor axes of lengths $s_1, s_2, \ldots\ldots\ldots, s_k$ and n-2k points then

$$\sum_{i=1}^{k} s_i^2 = trace\left(\mathrm{A}^*\mathrm{A}\right) - \sum_{i=1}^{n} \left|\lambda_i\right|^2$$

3 Circularity

3.1 Convexity

Suppose T is a bounded linear operator on a Hilbert space , $\lambda_1$and λ_2 being distinct points of W(T).To show that the line segment $(\lambda_1 \lambda_2)$ lies entirely in W(T). Taking the unit vectors f and g we have $\lambda_1 = \langle \mathrm{T}f, f \rangle$ and $\lambda_2 = \langle \mathrm{T}g, g \rangle$ which are linearly independent hence they span a two dimensional closed subspace so it has a convex numerical range.

3.2 Unitary equivalent

By Schur theorem any square matrix is unitarily equivalent to one that is upper triangular. So zero is a linear combination of eigenvalues of the matrix where each lies in its numerical range. By Toeplitz Hausdorff Theorem $0 \in W(A)$ there is a unit vector

We can choose T as an upper triangular matrix .i.e. $\mathrm{T} = \begin{bmatrix} \lambda_1 & a \\ 0 & \lambda_2 \end{bmatrix}$ where λ_1 and λ_2 are eigenvalues of T. If $\lambda_1 = \lambda_2 = \lambda$, we have $\mathrm{T} - \lambda = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}, W(\mathrm{T} - \lambda) = \left\{ z : |z| \le \frac{|a|}{2} \right\}$ and the $W(\mathrm{T})$ is a circle

With center at λ and radius $\frac{|a|}{2}$.

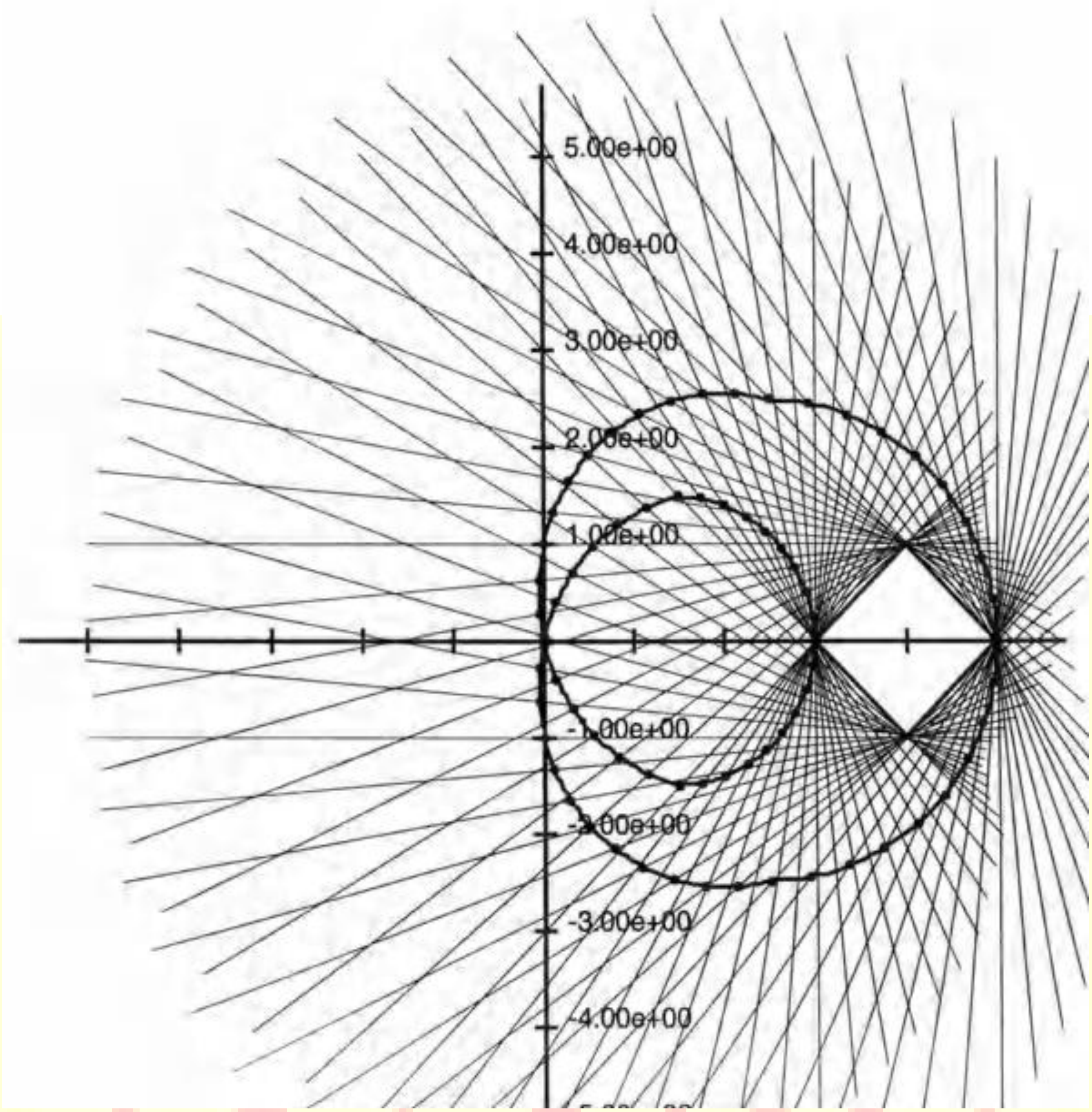
If $\lambda_1 \ne \lambda_2$ and a=0, we have $\mathrm{T} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$.If $x = (f, g), \langle \mathrm{T}x, x \rangle = \lambda_1 |f|^2 + \lambda_2 |g|^2 = t\lambda_1 + (1-t)\lambda_2$ where $t = |f|^2$ and $|f|^2 + |g|^2 = 1$. So $W(\mathrm{T})$ is the set of convex combinations of λ_1 and λ_2 and is the segment joining them.

From these results we may take the numerical range as the union of all of its two dimensional numerical ranges, which are ellipses and $W(\mathrm{T})$ *is just an interval in the real line* $\mathbb{R}$ *, so it does not need to possess an interior. The ellipse also allows straight line polygonal boundaries of* $W(\mathrm{T})$. $W(\mathrm{T})$ *is neither open nor closed. It is the continous image of two compact sets so it is compact.*

By Schur theorem then $W(\mathrm{T})$ *for a 2x2 matrix in this case by using inequalities since by unitary invariance and translation given* $\mathrm{A}=\begin{bmatrix} r & 2 \\ 0 & -r \end{bmatrix}$, *we know any unit vector can be written as*

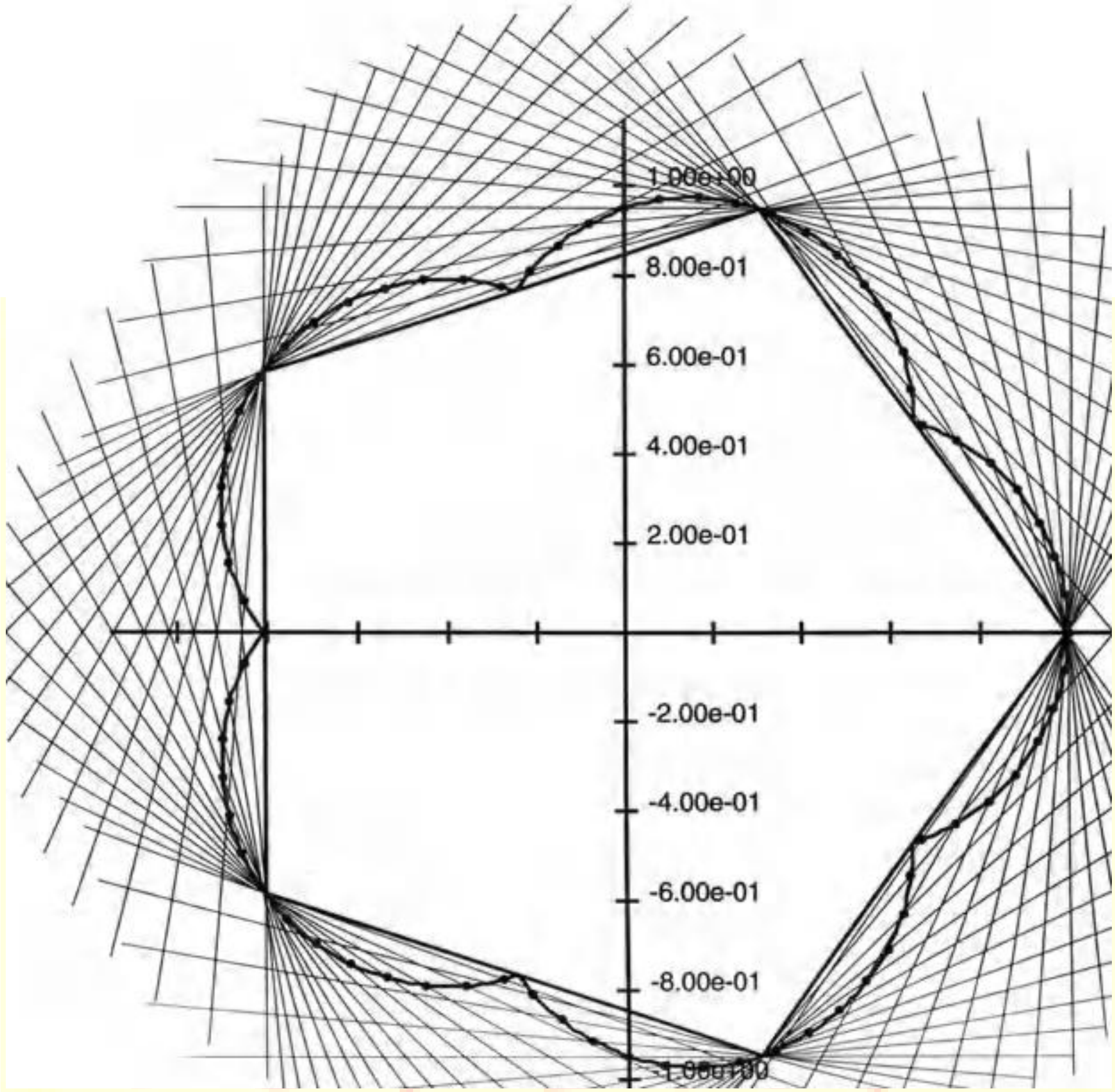
$f=e^{i\varphi}\left(\cos\theta, e^{i\phi}\sin\theta\right)$. *Then* $f^{*}\mathrm{A}f=\left(r\cos 2\theta+\cos\phi\sin 2\theta\right)+i\left(\sin\phi\sin 2\theta\right)\equiv x+iy$. *From which we can obtain* $\left|f^{*}\mathrm{A}f-r\cos 2\theta\right|^{2}=\sin^{2}2\theta=\left(x-r\cos 2\theta\right)^{2}+y^{2}$, *which clearly describes the equation of a circle centered at* $r\cos 2\theta$. *From* $\left(r^{2}+1\right)\cos^{2}2\theta-2xr\cos 2\theta+\left(x^{2}+y^{2}-1\right)=0$.

Using quadratic formula we get $\cos 2\theta=\dfrac{xr\pm\left(\left(r^{2}+1\right)-\left(x^{2}+y^{2}\right)\left(1+r^{2}\right)\right)^{1/2}}{r^{2}+1}$ *which yields the inequality* $\dfrac{x^{2}}{r^{2}+1}+y^{2}\leq 1$ *an ellipse with semi axes* $r^{2}+1$ *and* 1.

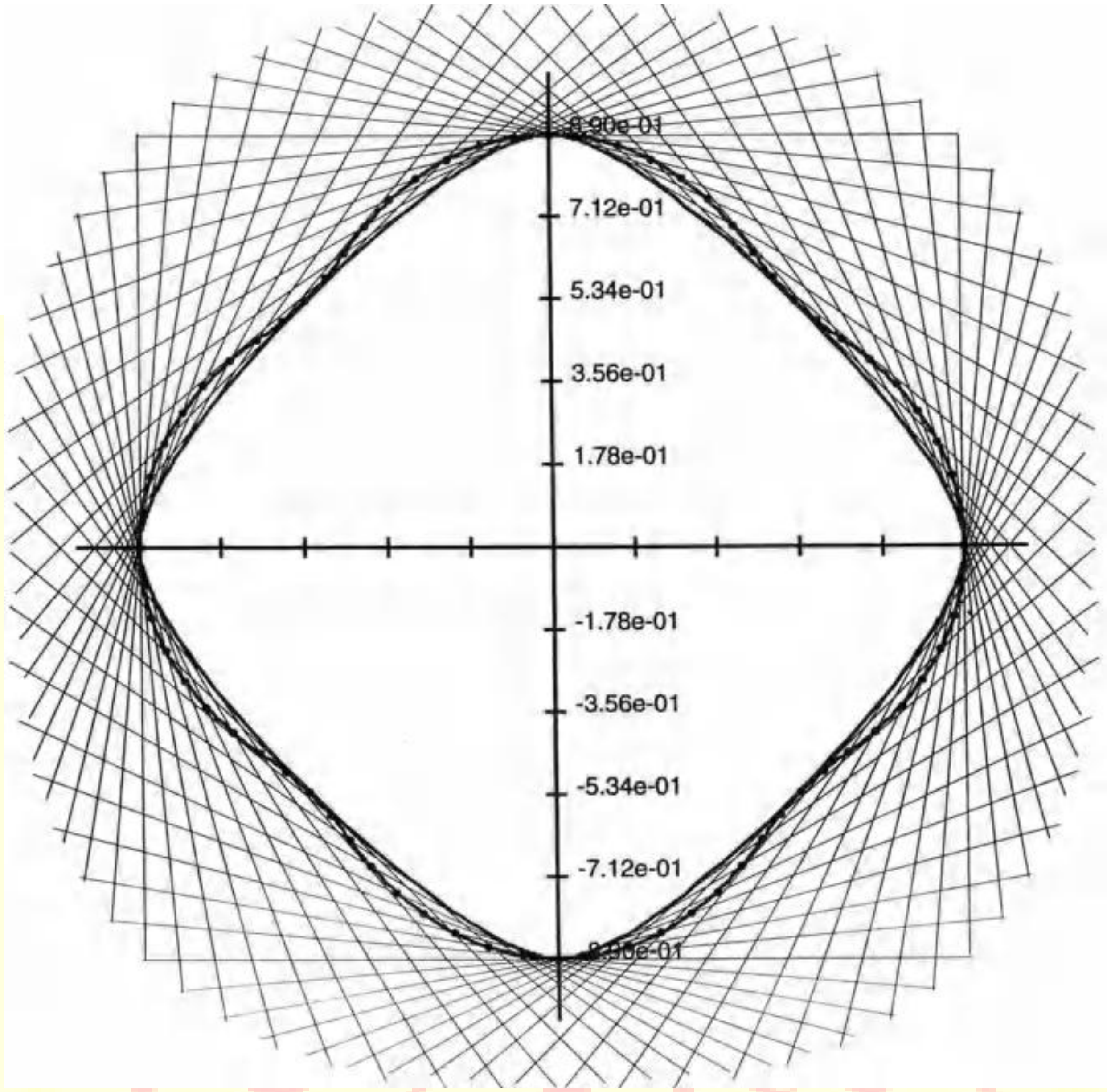


This is the shape of numerical range for

$$A = \begin{bmatrix} 4 & 0 & 0 & -1 \\ -1 & 4 & 0 & 0 \\ 0 & -1 & 4 & 0 \\ 0 & 0 & -1 & 4 \end{bmatrix}$$

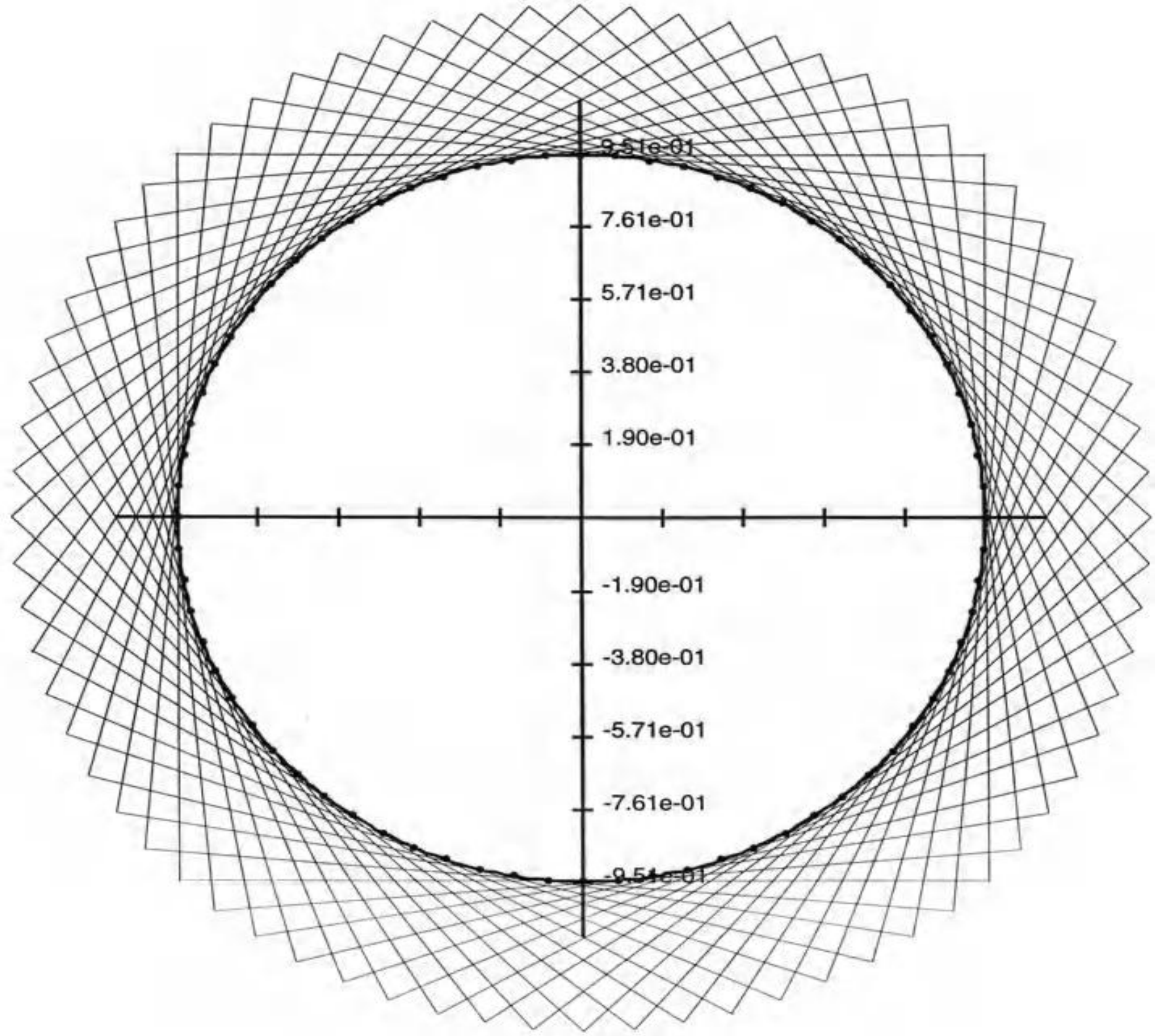


is is the numerical range of $A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

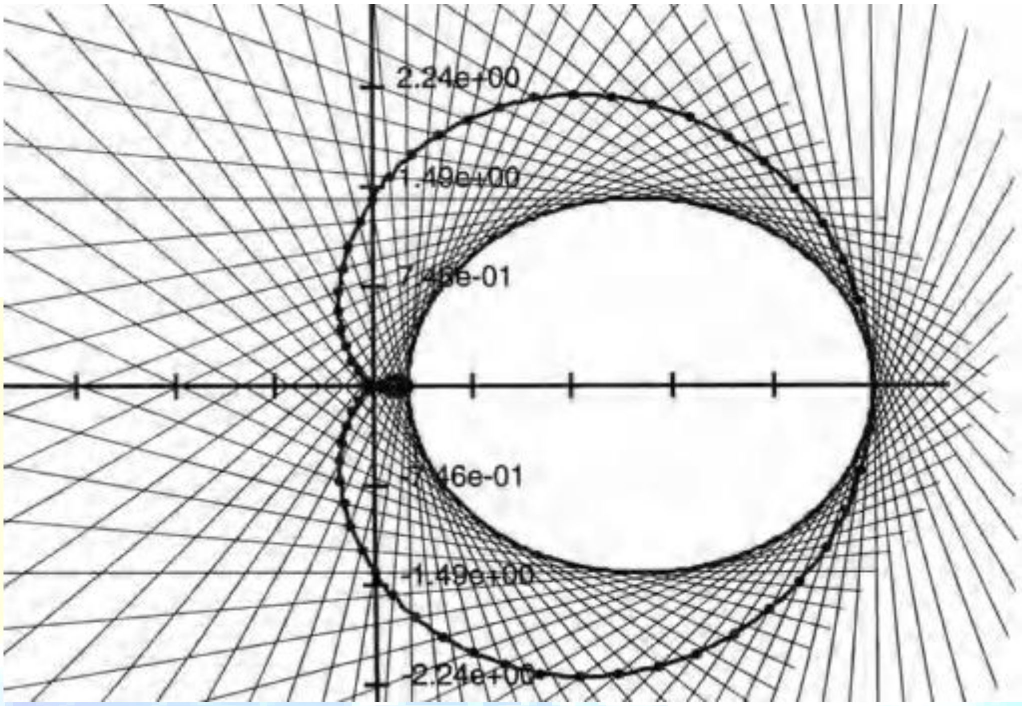


This is the numerical range of $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & 0 & 0 & 0 \end{bmatrix}$

July 2016 IJESR Volume 4, Issue 7 ISSN: 2347-6532



This is the numerical range of $A=\begin{bmatrix} 0&0&0&1&0&0&0&1&0\\ 0&0&0&0&1&0&0&0&1\\ 0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0 \end{bmatrix}$



$W(\mathrm{A})$ *for* $\mathrm{A} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$

(4) Conclusion

We are able to observe that the numerical range is the union of all its two dimensional ranges which are ellipses and $W(\mathrm{T})$ *is just an interval in the real line* $\mathbb{R}$. $W(\mathrm{T})$ *is neither open or closed but is a continuous image of two compact sets so it is compact . For* 2×2 *matrix of a translation we obtain the equation of a circle which when we equate with* $\cos 2\theta$, *it yields the equation of an ellipse with semi axes.Let A be an nxn matrix (0,1) matrix having at most 1 in each row and column. Let* $k_1.......,k_n$ *be the lengths of all circuits of G(A). Denote by v the longest length of a simple directed path in G(A) which is not part of a circuit of G(A). Then the numerical range of A is given by* $W(\mathrm{A}) = conv\left[D(A) \cup P(A)\right]$ *where D(A) is the circular disk centered at the origin with radius* $\cos\left(\pi/(v+2)\right)$ *and P(A) is the polygon in the complex plane.*

Furthermore, $W(A) = D(A)$ *if and only if* $G(A)$ *contains no circuits.*

If an adjacency complex matrix is such that its undirected graph is connected then the undirected graph is a tree and W(A) is a circular disk centered at the origin which by unique extension shows tractable equivalent conditions so that the numerical range of a matrix is a circular disk where the centre is not at origin.

REFERENCES

(1) Barasa J.S, Okello .B.N and etal, Circularity of numerical ranges of isometrically bounded linear operators on a Hilbert space.(2016)

(2) Chien M.T. and etal, Circularity of the numerical range, linear Algebra (1994)

(3) Hiroshi .N and etal, On the shape of numerical range of matrix polynomials.(2005)

(4) Fiedler .M. Geometry of the numerical range of matrices.(1981).

(5) Keeler.D.S, Rodman.l, numerical range of 3x3 matrices.

(6) Marcus and etal, computer generated numerical ranges.(1991)

(7) Marcus and etal, The numerical range of certain 0,1 matrices (1979)

(8) Shangjun.y, Shun Tam, on matrices whose numerical ranges have circular or weak circular Symmetry(2001).

(9) Shapiro and etal, notes on numerical range(2003)

(10) Tsatsomeros M.J, Numerical range in a matrix nutshell(2002)

(11) Wai-shun and etal, Elementary proofs for some results on the circular symmetry of the numerical range.(2012)